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A mathematical theory of the Feynman path integral for the generalized Pauli equations

By Wataru Ichinose

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Abstract. The definitions of the Feynman path integral for the Pauli equation and more general equations in configuration space and in phase space are proposed, probably for the first time. Then it is proved rigorously that the Feynman path integrals are well-defined and are the solutions to the corresponding equations. These Feynman path integrals are defined by the time-slicing method through broken line paths, which is familiar in physics. Our definitions of these Feynman path integrals and our results give the extension of ones for the Schrödinger equation.

1. Introduction.

We consider some charged non-relativistic particles in an electromagnetic field. For the sake of simplicity we suppose the charge and the mass of every particle to be q_c and $m > 0$, respectively. We consider $x = (x_1, \dots, x_n) \in R^n$ and $t \in [0, T]$, where $T > 0$ is an arbitrary constant. Let $E(t,x) = (E_1, \dots, E_n) \in R^n$ and $(B_{jk}(t,x))_{1 \le j < k \le n} \in R^{n(n-1)/2}$ denote electric strength and magnetic strength tensor, respectively, and $(V(t,x), A(t,x)) = (V, A_1, \dots, A_n) \in R^{n+1}$ an electromagnetic potential, i.e.,

$$E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x},$$
$$B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad (1 \le j < k \le n), \tag{1.1}$$

where $\partial V/\partial x = (\partial V/\partial x_1, \dots, \partial V/\partial x_n)$. Then the Lagrangian function $\mathscr{L}_0(t,x,\dot{x})$ ($\dot{x} \in R^n$) is given by

$$\mathscr{L}_0(t,x,\dot{x}) = \frac{m}{2}|\dot{x}|^2 + q_c \dot{x} \cdot A - q_c V. \tag{1.2}$$

The Hamiltonian function $\mathscr{H}_0(t,x,p)$ ($p \in R^n$) is defined through the Legendre transformation of $\mathscr{L}_0$ by

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$$\mathscr{H}_0(t,x,p) = \frac{1}{2m}|p - q_c A|^2 + q_c V. \tag{1.3}$$

Let $T^*R^n = R^n_x \times R^n_p$ denote the phase space, and $(R^n)^{[s,t]}$ and $(T^*R^n)^{[s,t]}$ the spaces of all paths $q : [s,t] \ni \theta \to q(\theta) \in R^n$ and $(q,p) : [s,t] \ni \theta \to (q(\theta), p(\theta)) \in T^*R^n$, respectively. The classical actions $S_c(t,s;q)$ for $q \in (R^n)^{[s,t]}$ in configuration space and $S(t,s;q,p)$ for $(q,p) \in (T^*R^n)^{[s,t]}$ in phase space are given by

$$S_c(t,s;q) = \int_s^t \mathscr{L}_0\big(\theta, q(\theta), \dot{q}(\theta)\big)d\theta, \quad \dot{q}(\theta) = \frac{dq}{d\theta}(\theta) \tag{1.4}$$

and

$$S(t,s;q,p) = \int_s^t \big\{p(\theta)\cdot\dot{q}(\theta) - \mathscr{H}_0\big(\theta, q(\theta), p(\theta)\big)\big\}d\theta, \tag{1.5}$$

respectively (cf. [**2**]).

In 1948 Feynman proposed an essentially different description of quantization from the Heisenberg and the Schrödinger ones in [**5**]. Let f be a probability amplitude given at the time 0. Then he claimed that the value of the probability amplitude at (T,x) can be described as the sum, in a sense, of $N^{-1}(\exp i\hbar^{-1}S_c(T,0;q))f(q(0))$ over all paths $q \in (R^n)^{[0,T]}$ satisfying $q(T) = x$ with a normalization factor N independent of q and x. This sum is called the Feynman path integral. In 1951 Feynman himself gave the description reformulated by means of the Feynman path integral in phase space in [**6**]. This Feynman path integral is called the phase space Feynman path integral. Now we know that his description is very useful and applied to wide areas in physics (cf. [**21**]).

The Feynman path integral and the phase space Feynman path integral were rigorously defined and proved to satisfy the Schödinger equation

$$i\hbar\frac{\partial}{\partial t}u(t) = H_0(t)u(t), \quad u(0) = f \tag{1.6}$$

in many papers, where

$$H_0(t) := \frac{1}{2m}\sum_{j=1}^n \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - q_c A_j\right)^2 + q_c V \tag{1.7}$$

and $\hbar$ is the Planck constant. For example, See [**1**], [**8**], [**14**], [**15**], [**18**], [**23**] and their references. See Theorem 4 in Section 5 of Chapter 4 in [**10**] for the non-existence of the measure defining the Feynman path integral.

Let ${}^t(a_1,\dots,a_n)$ denote the transposed of vector $(a_1,\dots,a_n)$, where a_j $(j = 1,2,\dots,n)$ are complex numbers. Let $n = 3, u(t) = {}^t(u_1(t), u_2(t)), I_2$ the identity matrix of degree 2, $B(t,x) = (B_{23}(t,x), -B_{13}(t,x), B_{12}(t,x)) = \nabla \times A(t,x)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where σ_j $(j = 1,2,3)$ are the Hermitian constant matrices of degree 2 called the Pauli matrices satisfying

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \ [\sigma_2, \sigma_3] = 2i\sigma_1, \ [\sigma_3, \sigma_1] = 2i\sigma_2. \tag{1.8}$$

Here $[\cdot, \cdot]$ denotes the commutator for matrices. Then we have the Pauli equation

$$i\hbar \frac{\partial}{\partial t} u(t) = H_e(t)u(t), \quad u(0) = {}^t(f_1, f_2)$$

representing the motion of a particle with spin 1/2, where

$$H_e(t) := H_0(t)I_2 - \frac{q_c \hbar}{2m} B(t) \cdot \sigma.$$

Feynman and Hibbs say in Section 12–10 of [**7**] that with regard to application to quantum mechanics, path integrals suffer most grievously from a serious defect, because they do not permit a discussion of particles with spin in a simple and lucid way. Also, Schulman says in Chapter 22 of [**22**] that it has been difficult to suggest a continuous path subsequently to be summed over so as to obtain the probability amplitude for the Pauli equation. The author does not know the mathematical references concerning the Feynman path integral for the Pauli equation. Let $\hbar = 1$, and $A(t, x) \in R^3$ and $V(t, x) \in R$ independent of $t \in [0, T]$. Then in [**9**] the formula of the Feynman-Kac type for the equation

$$\frac{\partial}{\partial t} u(t) = -H_e(t)u(t), \quad u(0) = {}^t(f_1, f_2)$$

was obtained. We note that in [**9**] the random variable matrix $M(t, \omega)$ $(\omega \in \Omega)$ of degree 2 defined by the solution to

$$\frac{d}{dt} M(t, \omega) = \frac{q_c}{2m} \big(B(b_\omega(t)) \cdot \sigma\big) M(t, \omega), \quad M(0, \omega) = I_2 \tag{1.9}$$

was used, where Ω is a basic space and $b_\omega(t)$ is the Brownian motion of dimension 3. As physical references the Feynman path integral on $R^3 \times SO(3)$ was studied in [**22**] and the Feynman path integral was studied by means of the introduction of new two variables for spin in [**4**]. See [**11**], [**22**] for the detailed physical references.

Let $u(t) = {}^t(u_1(t), \dots, u_l(t))$, $f = {}^t(f_1, \dots, f_l)$, I_l the identity matrix of degree l and $H_1(t, x) = (h_{1jk}(t, x))$ a Hermitian matrix of degree l. We consider the generalized Pauli equation

$$i\hbar \frac{\partial}{\partial t} u(t) = \big(H_0(t)I_l + \hbar H_1(t)\big)u(t), \quad u(t_0) = f. \tag{1.10}$$

Our aim in the present paper is to propose the definitions of the Feynman path integral and the phase space Feynman path integral for (1.10), probably for the first time. Then we prove rigorously that these Feynman path integrals are well-defined and satisfy (1.10). We note that our definitions of these Feynman path integrals and our results give the extension of ones for the Schrödinger equation in [**16**].

Assume that all $h_{1jk}(t,x)$ are continuous in $[0,T]\times R^n$. For a continuous path $q\in (R^n)^{[0,T]}$ let $\mathscr{F}(t,s;q)$ $(0\le t,s\le T)$ be the unitary matrix of degree l defined by the solution to

$$\frac{d}{dt}\mathscr{A}(t) = -iH_1(t,q(t))\mathscr{A}(t), \quad \mathscr{A}(s) = I_l. \tag{1.11}$$

We note that (1.11) is like (1.9). Let $f = {}^t(f_1,\dots,f_l)$ be a probability amplitude given at the time 0. Roughly speaking, we define the Feynman path integral and the phase space Feynman path integral for (1.10) with $t_0 = 0$ by the sums, in a sense, of $N^{-1}(\exp i\hbar^{-1}S_c(T,0;q))\mathscr{F}(T,0;q)f(q(0))$ over all continuous $q\in (R^n)^{[0,T]}$ satisfying $q(T) = x$ and of $N'^{-1}(\exp i\hbar^{-1}S(T,0;q,p))\mathscr{F}(T,0;q)\times f(q(0))$ over all continuous $q\in(R^n)^{[0,T]}$ and all $p\in (R^n)^{[0,T]}$ satisfying $q(T)=x$ respectively, where N' is also a normalization factor independent of q,p and x.

The outline of the proof is as follows. We define the Feynman path integral and the phase space Feynman path integral for (1.10) by the time-slicing method through broken line paths. This method is familiar in physics (cf. [**7**], [**21**]). We denote the space of all infinitely differentiable functions in R^n with compact support by $C_0^\infty(R^n)$. Let

$$q^{t,s}_{x,y}(\theta) := y + \frac{\theta - s}{t-s}(x-y), \quad 0\le s\le\theta\le t\le T \tag{1.12}$$

and set for $f = {}^t(f_1,\dots,f_l)\in C_0^\infty(R^n)^l$

$$\mathscr{C}(t,s)f := \begin{cases} \sqrt{m/(2\pi i\hbar(t-s))}^{\,n}\displaystyle\int \big(\exp i\hbar^{-1}S_c\big(t,s;q^{t,s}_{x,y}\big)\big) & s<t,\\ \quad\times\,\mathscr{F}\big(t,s;q^{t,s}_{x,y}\big)f(y)dy, & \\ f, & s=t. \end{cases} \tag{1.13}$$

Let $t-s>0$ be small. Then we can prove the so-called stability of $\mathscr{C}(t,s)$ and the so-called consistency of $\mathscr{C}(t,s)$ for (1.10) in Propositions 3.4 and 3.5 of the present paper by means of the theory of the oscillatory integral operators as in [**16**]. Then we can prove our results as in the same way as in [**16**].

The plan of the present paper is as follows. In section 2 we state our results and some remarks. Section 3 is devoted to the proof of the main theorem.

2. Results.

Let $\Delta : 0 = \tau_0 < \tau_1 < \cdots < \tau_\nu = T$ be a subdivision of the interval $[0,T]$. We set $|\Delta| := \max_{1\le j\le\nu}(\tau_j - \tau_{j-1})$. For $x^{(0)},\dots,x^{(\nu-1)}$ and x in R^n let's define $q_\Delta = q_\Delta(\theta;x^{(0)},\dots,x^{(\nu-1)},x)\in (R^n)^{[0,T]}$ by the broken line path joining points $x^{(j)}$ at τ_j $(j = 0,1,\dots,\nu, x^{(\nu)} = x)$ in order, i.e.

$$q_\Delta(\theta) = x^{(j-1)} + \frac{\theta - \tau_{j-1}}{\tau_j - \tau_{j-1}}\big(x^{(j)} - x^{(j-1)}\big), \quad \tau_{j-1}\le\theta\le\tau_j$$

for $j = 1, 2, \dots, \nu$. Then we have the following.

LEMMA 2.1. *Assume that all $h_{1jk}(t,x)$ are continuous in $[0,T] \times R^n$. Then we have*

$$\mathscr{F}(T,0;q_\Delta) = \mathscr{F}\big(T,\tau_{\nu-1};q^{T,\tau_{\nu-1}}_{x,x^{(\nu-1)}}\big)\mathscr{F}\big(\tau_{\nu-1},\tau_{\nu-2};q^{\tau_{\nu-1},\tau_{\nu-2}}_{x^{(\nu-1)},x^{(\nu-2)}}\big)\cdots\mathscr{F}\big(\tau_1,0;q^{\tau_1,0}_{x^{(1)},x^{(0)}}\big). \tag{2.1}$$

PROOF. For $\tau_k \le t \le \tau_{k+1}$ $(k = 0, 1, \dots, \nu-1)$ we have

$$\mathscr{F}(t,0;q_\Delta) = \mathscr{F}\big(t,\tau_k;q^{\tau_{k+1},\tau_k}_{x^{(k+1)},x^{(k)}}\big)\mathscr{F}(\tau_k,0;q_\Delta).$$

In fact both sides satisfy

$$\frac{d}{dt}\mathscr{A}(t) = -iH_1\big(t,q^{\tau_{k+1},\tau_k}_{x^{(k+1)},x^{(k)}}(t)\big)\mathscr{A}(t), \quad \mathscr{A}(\tau_k) = \mathscr{F}(\tau_k,0;q_\Delta).$$

So, (2.1) can be proved by induction. □

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $x \in R^n$ we write $|\alpha| = \sum_{j=1}^n \alpha_j$, $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ and $\langle x\rangle = \sqrt{1+|x|^2}$. Let $L^2 = L^2(R^n)$ be the space of all square integrable functions in R^n with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We introduce the weighted Sobolev spaces $B^a = B^a(\hbar) := \{f \in L^2;\ \|f\|_{B^a} := \|f\| + \sum_{|\alpha|=a}(\|x^\alpha f\| + \|(\hbar\partial_x)^\alpha f\|) < \infty\}$ $(a = 1, 2, \dots)$. We set $B^0 := L^2$. Let B^{-a} $(a = 1, 2, \dots)$ denote the dual space of B^a. Let $\|f\|_{B^a}$ for $f = {}^t(f_1, \dots, f_l) \in (B^a)^l$ denote the norm $(\sum_{j=1}^l \|f_j\|^2_{B^a})^{1/2}$.

Let $\chi \in C_0^\infty(R^n)$ such that $\chi(0) = 1$. For a function $g(x,y)$ on $R^n \times R^{n\nu}$ we define the oscillatory integral $\mathrm{Os} - \int g(\cdot,y)dy$ by $\lim_{\epsilon\to 0}\int\prod_{j=1}^\nu \chi(\epsilon y^{(j)})g(\cdot,y)dy$, where $y = (y^{(1)}, \dots, y^{(\nu)}) \in R^{n\nu}$ and the limit is taken in the topology of B^a different from that in [**19**]. Let $q_\Delta = q_\Delta(\theta; x^{(0)}, \dots, x^{(\nu-1)}, x)$. We define

$$\begin{aligned}&\int \big(\exp i\hbar^{-1}S_c(T,0;q_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}q_\Delta \\ &:= \Bigg(\prod_{j=1}^\nu \sqrt{\frac{m}{2\pi i\hbar(\tau_j-\tau_{j-1})}}^{\,n}\Bigg)\mathrm{Os} - \int\cdots\int \big(\exp i\hbar^{-1}S_c(T,0;q_\Delta)\big) \\ &\quad \times \mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))dx^{(0)}\cdots dx^{(\nu-1)}\end{aligned} \tag{2.2}$$

for $f \in C_0^\infty(R^n)^l$. Then from (1.4), (1.13) and Lemma 2.1 we can write

$$\begin{aligned}&\int \big(\exp i\hbar^{-1}S_c(T,0;q_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}q_\Delta \\ &= \lim_{\epsilon\to 0}\mathscr{C}(T,\tau_{\nu-1})\chi(\epsilon\cdot)\mathscr{C}(\tau_{\nu-1},\tau_{\nu-2})\cdots\chi(\epsilon\cdot)\mathscr{C}(\tau_1,0)\chi(\epsilon\cdot)f.\end{aligned} \tag{2.3}$$

As is seen from (2.3), Proposition 2.2 in [**13**] and Lemma 2.2 in [**3**], it seems to be difficult that the limit in (2.3) is taken pointwise in R^n. The existence of the limit (2.2) or (2.3) will be proved in Theorem 2.3 in the present paper. The Feynman path integral $\int(\exp i\hbar^{-1}S_c(T,0;q))\mathscr{F}(T,0;q)f(q(0))\mathscr{D}q$ in configuration space for the generalized Pauli equation (1.10) with $t_0=0$ is defined by

$$\lim_{|\Delta|\to 0}\int \big(\exp i\hbar^{-1}S_c(T,0;q_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}q_\Delta \tag{2.4}$$

as in [**7**], [**21**], where the limit in (2.4) is taken in the topology of $(B^a)^l$.

Let $0\le t_0\le t\le T$. We take $1\le \mu'<\mu\le\nu$ such that $\tau_{\mu'-1}\le t_0<\tau_{\mu'}$ and $\tau_{\mu-1}<t\le\tau_\mu$. Then we define for $f\in C_0^\infty(R^n)^l$

$$\mathscr{C}_\Delta(t,t_0)f:=\lim_{\epsilon\to 0}\mathscr{C}(t,\tau_{\mu-1})\chi(\epsilon\cdot)\mathscr{C}(\tau_{\mu-1},\tau_{\mu-2})\cdots\chi(\epsilon\cdot)\mathscr{C}(\tau_{\mu'},t_0)\chi(\epsilon\cdot)f. \tag{2.5}$$

Otherwise we set $\mathscr{C}_\Delta(t,t_0)f=\mathscr{C}(t,t_0)f$. Then from (2.3) we have

$$\mathscr{C}_\Delta(T,0)f=\int \big(\exp i\hbar^{-1}S_c(T,0;q_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}q_\Delta. \tag{2.6}$$

For $\Pi^{(j)}$ $(j=0,1,\dots,\nu-1)$ in kinetic momentum space R^n we define the path $\Pi_\Delta(\theta;\Pi^{(0)},\dots,\Pi^{(\nu-1)})\in(R^n)^{[0,T]}$ in kinetic momentum space by the piecewise constant path taking $\Pi^{(0)}$ at $\theta=0,\Pi^{(j)}$ for $\tau_j<\theta\le\tau_{j+1}$ $(j=0,1,\dots,\nu-1)$. Let $q_\Delta=q_\Delta(\theta;x^{(0)},\dots,x^{(\nu-1)},x)$ and set

$$p_\Delta(\theta):=\Pi_\Delta(\theta)+A(\theta,q_\Delta(\theta)). \tag{2.7}$$

Then we define for $f\in C_0^\infty(R^n)^l$

$$\begin{aligned}&\iint \big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta\\ &\quad:=(2\pi\hbar)^{-n\nu}\mathrm{Os}-\int\cdots\int\big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\\ &\qquad\times\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))d\Pi^{(0)}dx^{(0)}\cdots d\Pi^{(\nu-1)}dx^{(\nu-1)}.\end{aligned} \tag{2.8}$$

The existence of the limit (2.8) will be proved in Theorem 2.3 in the present paper. The Feynman path integral $\iint(\exp i\hbar^{-1}S(T,0;q,p))\mathscr{F}(T,0;q)f(q(0))\mathscr{D}\Pi\mathscr{D}q$ in phase space for the generalized Pauli equation (1.10) with $t_0=0$ is defined by

$$\lim_{|\Delta|\to 0}\iint \big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta \tag{2.9}$$

as in [**21**] in the topology of $(B^a)^l$.

Let

$$\zeta^{t,s}_{x,y,\Pi}(\theta) := \big(q^{t,s}_{x,y}(\theta), \Pi + A\big(\theta, q^{t,s}_{x,y}(\theta)\big)\big) \in (T^*R^n)^{[s,t]}, \tag{2.10}$$

and for $0 \le s \le t \le T$ and $f \in C_0^\infty(R^n)^l$ set

$$G_\epsilon(t,s)f := \begin{cases} (2\pi\hbar)^{-n} \displaystyle\iint \big(\exp i\hbar^{-1} S\big(t,s;\zeta^{t,s}_{x,y,\Pi}\big)\big) & s < t, \\ \quad \times \chi(\epsilon\Pi)\mathscr{F}\big(t,s;q^{t,s}_{x,y}\big) f(y) d\Pi dy, & \\ f, & s = t. \end{cases} \tag{2.11}$$

Let $0 \le t_0 \le t \le T$. If we take $1 \le \mu' < \mu \le \nu$ such that $\tau_{\mu'-1} \le t_0 < \tau_{\mu'}$ and $\tau_{\mu-1} < t \le \tau_\mu$, we define for $f \in C_0^\infty(R^n)^l$

$$G_\Delta(t,t_0)f := \lim_{\epsilon\to 0} G_\epsilon(t,\tau_{\mu-1})\chi(\epsilon\cdot)G_\epsilon(\tau_{\mu-1},\tau_{\mu-2})\cdots\chi(\epsilon\cdot)G_\epsilon(\tau_{\mu'},t_0)\chi(\epsilon\cdot)f. \tag{2.12}$$

Otherwise we set $G_\Delta(t,t_0)f = \lim_{\epsilon\to 0} G_\epsilon(t,t_0)f$. Then from (1.5), (2.7)–(2.8), (2.10)–(2.11) and Lemma 2.1 we have

$$G_\Delta(T,0)f = \iint \big(\exp i\hbar^{-1} S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,0;q_\Delta) f(q_\Delta(0)) \mathscr{D}\Pi_\Delta \mathscr{D}q_\Delta. \tag{2.13}$$

Let

$$\begin{aligned}\Phi(t,s;x,y,z) := z - \frac{x+y}{2} + \frac{t-s}{m}\int_0^1 A(s, y+\theta(x-y))d\theta \\ - \frac{t-s}{m}B'(t,s;x,y,z) - \frac{(t-s)^2}{m}E'(t,s;x,y,z) \in R^n,\end{aligned}$$

where $E' = (E'_1,\ldots,E'_n)$, $B' = (B'_1,\ldots,B'_n)$,

$$E'_j = \int_0^1\int_0^{\sigma_1} E_j\big(t-\sigma_1(t-s), z+\sigma_1(x-z)+\sigma_2(y-x)\big)d\sigma_2 d\sigma_1$$

and

$$B'_j = \sum_{k=1}^n (z_k - x_k)\int_0^1\int_0^{\sigma_1} B_{jk}\big(t-\sigma_1(t-s), z+\sigma_1(x-z)+\sigma_2(y-x)\big)d\sigma_2 d\sigma_1.$$

Then Lemma 3.2 in [**14**] says

Lemma 2.2. *Let $\partial_x^\alpha E_j(t,x)$ $(j=1,2,\ldots,n)$ and $\partial_x^\alpha B_{jk}(t,x)$ $(1 \le j < k \le n)$ be continuous in $[0,T]\times R^n$ for all α. We suppose*

$$\left|\partial_x^\alpha E_j(t,x)\right| \le C_\alpha,\ |\alpha| \ge 1, \quad \left|\partial_x^\alpha B_{jk}(t,x)\right| \le C_\alpha \langle x\rangle^{-(1+\delta)},\ |\alpha| \ge 1 \tag{2.14}$$

in $[0,T]\times R^n$, *where constants* $\delta = \delta_\alpha > 0$ *may depend on* α. *Then there exits a constant* $\rho^* > 0$ *such that the mapping*: $R^n \ni z \to \xi = \Phi \in R^n$ *is homeomorphic and* $\det \partial\Phi/\partial z \ge 1/2$ *for each fixed* $0 \le t - s \le \rho^*$, x *and* y.

We fix $\rho^* > 0$ determined in Lemma 2.2 through the present paper. The following is the main theorem in the present paper.

Theorem 2.3. *Besides the assumptions of Lemma* 2.2 *we suppose that* $\partial_x^\alpha A_j(t,x)$ $(j = 1,2,\dots,n)$ *and* $\partial_x^\alpha V(t,x)$ *are continuous in* $[0,T]\times R^n$ *for all* α *and that we have*

$$\left|\partial_x^\alpha A_j(t,x)\right| \le C_\alpha,\ |\alpha| \ge 1, \quad \left|\partial_x^\alpha V(t,x)\right| \le C_\alpha \langle x\rangle,\ |\alpha| \ge 1 \tag{2.15}$$

in $[0,T]\times R^n$. *We also assume that* $\partial_x^\alpha h_{1jk}(t,x)$ $(j,k = 1,2,\dots,l)$ *for all* α *are continuous in* $[0,T]\times R^n$ *and satisfy*

$$\left|\partial_x^\alpha h_{1jk}(t,x)\right| \le C_\alpha, \quad (t,x) \in [0,T]\times R^n. \tag{2.16}$$

Let $a = 0,1,\dots$ *and* $|\Delta| \le \rho^*$. *Then we have*: (1) *Both of* $\mathscr{C}_\Delta(t,t_0)$ *and* $G_\Delta(t,t_0)$ *are well-defined on* $C_0^\infty(R^n)^l$ *independently of the choice of* χ *and can be extended to bounded operators on* $(B^a)^l$. *In addition, they are equal to one another and are continuous in* $0 \le t_0 \le t \le T$ *as* $(B^a)^l$*-valued functions.* (2) *As* $|\Delta| \to 0$, $\mathscr{C}_\Delta(t,t_0)f$ *for* $f \in (B^a)^l$ *converges in* $(B^a)^l$ *uniformly in* $0 \le t_0 \le t \le T$ *and this limit satisfies the generalized Pauli equation* (1.10).

We have together with (2.6) and (2.13)

Corollary 2.4. *Let* $a = 0,1,\dots$ *and* $f \in (B^a)^l$. *Under the assumptions of Theorem* 2.3 *there exist the Feynman path integrals* $\int(\exp i\hbar^{-1}S_c(t,0;q))\mathscr{F}(t,0;q)f(q(0))\mathscr{D}q$ *and* $\iint(\exp i\hbar^{-1}S(t,0;q,p))\mathscr{F}(t,0;q)f(q(0))\mathscr{D}\Pi\mathscr{D}q$ *in configuration space and in phase space for* $0 \le t \le T$, *which are equal to one another, are continuous in* $0 \le t \le T$ *as* $(B^a)^l$*-valued functions and satisfy the generalized Pauli equation* (1.10) *with* $t_0 = 0$.

Remark 2.1. We have $\mathscr{F}(t,s;q) = \exp(-i\int_s^t H_1(\theta,q(\theta))d\theta)$ in case that $H_1(t,x)$ is a diagonal matrix. So,

$$\begin{aligned}
&\iint \big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,0;q_\Delta)f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta \\
&= \iint \left(\exp i\hbar^{-1}\int_0^T \big\{p_\Delta(\theta)\cdot\dot q_\Delta(\theta)I_l - \mathscr{H}_0(\theta,q_\Delta(\theta),p_\Delta(\theta))I_l - \hbar H_1(\theta,q_\Delta(\theta))\big\}d\theta\right) \\
&\quad \times f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta.
\end{aligned}$$

We note that $\int_0^T\{p(\theta)\cdot\dot q(\theta)I_l - \mathscr{H}_0(\theta,q(\theta),p(\theta))I_l - \hbar H_1(\theta,q(\theta))\}d\theta$ is the classical action in phase space corresponding to the quantum equation (1.10).

Remark 2.2. Let $\mathscr{E}^0_{t,s}([0,T];(B^{a+2})^l)\bigcap\mathscr{E}^1_{t,s}([0,T];(B^a)^l)$ denote the space of all $(B^{a+2})^l$-valued continuous and $(B^a)^l$-valued continuously differentiable functions in $0\le s\le t\le T$. We know from the second step of the proof of Theorem in [**12**] that the solutions to (1.10) are unique in $\bigcup_{a=-\infty}^{\infty}(\mathscr{E}^0_{t,s}([0,T];(B^{a+2})^l)\bigcap\mathscr{E}^1_{t,s}([0,T];(B^a)^l))$.

Remark 2.3. If $H_1(t,x)=0$ identically in $[0,T]\times R^n$, then we have $\mathscr{F}(t,s)=I_l$ from (1.11). Hence Theorem 2.3 and Corollary 2.4 in the present paper state the same results as in Theorems 1 and 2 in [**16**].

Remark 2.4. If we suppose, besides the assumptions of Lemma 2.2 that $\partial_t B_{jk}(t,x)$ $(1\le j<k\le n)$ are continuous in $[0,T]\times R^n$, then we can get the same assertions as in Theorem 2.3 and Corollary 2.4 in the present paper where $a=0$. The proof can be given by virtue of the gauge transformation as in the proof of Theorem in [**14**].

Remark 2.5. Let $M\ge 0$ be an integer and $z_{jk}(x,\Pi)$ $(j,k=1,2,\dots,l)$ infinitely differentiable scalar functions in R^{2n}. We suppose

$$\left|\partial_\Pi^\alpha\partial_x^\beta z_{jk}(x,\Pi)\right|\le C_{\alpha,\beta}(1+|x|+|\Pi|)^M,\quad (x,\Pi)\in R^{2n}$$

for all α and β. We set the matrix $z(x,\Pi):=(z_{jk}(x,\Pi))$ of degree l. Then we define for a $t_0\in[0,T]$

$$\iint\big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,t_0;q_\Delta)z(q_\Delta(t_0),\Pi_\Delta(t_0))$$
$$\times\mathscr{F}(t_0,0;q_\Delta)f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta$$

as in the same way of defining (2.8). From (1.11) we note

$$\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big)-I_l=-i\int_{s'}^{t'}H_1\big(\theta,q^{t,s}_{x,y}(\theta)\big)\mathscr{F}\big(\theta,s';q^{t,s}_{x,y}\big)d\theta$$

for $s\le s'\le t'\le t$. Assume

$$\sum_{i=1}^n\frac{\partial^2 z_{jk}}{\partial x_i\partial\Pi_i}(x,\Pi)=0$$

identically in R^{2n} for $j,k=1,2,\dots,l$. We take $\mathscr{F}(t,t_0;q^{t,s}_{x,y})z(q^{t,s}_{x,y}(t_0),\Pi)\mathscr{F}(t_0,s;q^{t,s}_{x,y})$ for $s\le t_0\le t$ as $\omega(x,\Pi,y)$ in Proposition 5.1 and Theorem 5.2 in [**17**]. Then from Corollary 4.3 in [**17**], (2.1) and Lemma 3.1 in the present paper we can prove as in the proof of Theorem 2.2 in [**17**] under the assumptions of Theorem 2.3 that for $f\in(B^{a+M})^l$ $(a=0,1,\dots)$ there exists

$$\lim_{|\Delta|\to 0}\iint\big(\exp i\hbar^{-1}S(T,0;q_\Delta,p_\Delta)\big)\mathscr{F}(T,t_0;q_\Delta)z(q_\Delta(t_0),\Pi_\Delta(t_0))$$
$$\times\mathscr{F}(t_0,0;q_\Delta)f(q_\Delta(0))\mathscr{D}\Pi_\Delta\mathscr{D}q_\Delta$$

in $(B^a)^l$, which is equal to $U(T,t_0)Z(t_0)U(t_0,0)f$. Here $U(t,t_0)f$ denotes the solution to the generalized Pauli equation (1.10) and $Z(t_0)$ the pseudo-differential operator $Z(X,\hbar D_x - \int_0^1 A(t_0, X+\theta(X'-X))d\theta)$ with double symbol (cf. [**19**]). One can prove the same assertions as in Theorems 2.2 and 2.3 in [**17**], which give the generalization of the results for the Schödinger equation in [**16**], [**17**], in the same way that the above was proved. The detailed proof will be published elsewhere.

Remark 2.6. We consider the Pauli equation. Then since the trace of σ_j $(j = 1,2,3)$ is equal to zero, the trace of $H_1(t,x) = (-q_c/2m)B(t,x)\cdot\sigma$ is also zero. So, we have $\det \mathscr{F}(t,s;q) = 1$. Consequently, we see $\mathscr{F}(t,s;q) \in SU(l)$.

3. Proof of Theorem 2.3.

For the sake of simplicity we set $\hbar = 1$ and $q_c = 1$.

Lemma 3.1. *Assume that $\partial_x^\alpha h_{1jk}(t,x)$ $(j,k = 1,2,\dots,l)$ are continuous for all α in $[0,T]\times R^n$ and satisfy*

$$\left|\partial_x^\alpha h_{1jk}(t,x)\right| \le C_\alpha, \quad |\alpha| \ge 1, \quad (t,x) \in [0,T]\times R^n. \tag{3.1}$$

Let $0 \le s < t \le T$. Then $\mathscr{F}(t',s';q^{t,s}_{x,y})$ for $s \le s'$, $t' \le t$ is a unitary matrix. We also see that for all α and β $\partial_x^\alpha \partial_y^\beta \mathscr{F}(t',s';q^{t,s}_{x,y})$ exist, are continuous and satisfy

$$\left|\partial_x^\alpha \partial_y^\beta \mathscr{F}(t',s';q^{t,s}_{x,y})\right| \le C_{\alpha,\beta} \tag{3.2}$$

in $0 \le s \le s'$, $t' \le t \le T$, $x \in R^n$ and $y \in R^n$.

Proof. We see from (1.11) that $\mathscr{F}(t',s';q^{t,s}_{x,y})$ is a unitary matrix. So we have (3.2) for $\alpha = \beta = 0$. We also know from the theory of the ordinary differential equations that for all α and β $\partial_x^\alpha \partial_y^\beta \mathscr{F}(t',s';q^{t,s}_{x,y})$ exist and are continuous in $0 \le s \le s'$, $t' \le t \le T$, $x \in R^n$ and $y \in R^n$. We have

$$\begin{aligned}
&\frac{d}{dt'}\frac{\partial}{\partial x_j}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) \\
&\quad = -iH_1\big(t',q^{t,s}_{x,y}(t')\big)\frac{\partial}{\partial x_j}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) - i\left\{\frac{\partial}{\partial x_j}H_1\big(t',q^{t,s}_{x,y}(t')\big)\right\}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big), \\
&\frac{\partial}{\partial x_j}\mathscr{F}\big(s',s';q^{t,s}_{x,y}\big) = 0.
\end{aligned}$$

Consequently we have

$$\frac{\partial}{\partial x_j}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) = -i\int_{s'}^{t'} \mathscr{F}\big(t',\theta;q^{t,s}_{x,y}\big)\left\{\frac{\partial}{\partial x_j}H_1\big(\theta,q^{t,s}_{x,y}(\theta)\big)\right\}\mathscr{F}\big(\theta,s';q^{t,s}_{x,y}\big)d\theta. \tag{3.3}$$

In the same way we can prove (3.2) from (3.1) for $|\alpha+\beta| = 1$. In the same way we can prove (3.2) for all α and β by induction. □

Let $p(x,w) = (p_{jk}(x,w))$ be a matrix valued function of degree l. We define the operator $P_{jk}(t,s)$ $(j,k = 1,2,\dots,l)$ on $C_0^\infty(R^n)$ by

$$P_{jk}(t,s)f = \begin{cases} \sqrt{m/(2\pi i(t-s))}^{\,n} \displaystyle\int \big(\exp iS_c\big(t,s;q^{t,s}_{x,y}\big)\big) & s<t, \\ \quad \times p_{jk}(x,(x-y)/\sqrt{t-s})f(y)dy, & \\ \sqrt{m/(2\pi i)}^{\,n} \mathrm{Os}-\displaystyle\int (\exp im|w|^2/2) & s=t \\ \quad \times p_{jk}(x,w)dw f(x), & \end{cases} \tag{3.4}$$

as in [**13**], [**14**], where the oscillatry integral in (3.4) is taken pointwise in the usual sense as in [**19**]. We set for $f = {}^t(f_1,\dots,f_l) \in C_0^\infty(R^n)^l$

$$P(t,s)f := \big(P_{jk}(t,s)\big)f. \tag{3.5}$$

Let $\mathscr{C}(t,s)$ be the operator on $C_0^\infty(R^n)^l$ defined by (1.13).

Lemma 3.2. *Let $M_1 \ge 0$ and assume*

$$\big|\partial_x^\alpha V(t,x)\big| + \sum_{j=1}^n \big|\partial_x^\alpha A_j(t,x)\big| \le C_\alpha \langle x\rangle^{M_1}, \quad (t,x)\in[0,T]\times R^n \tag{3.6}$$

for all α. Let $f \in C_0^\infty(R^n)^l$. Then we have: (1) *Let $M_2 \ge 0$ and suppose*

$$\big|\partial_w^\alpha \partial_x^\beta p_{jk}(x,w)\big| \le C_{\alpha,\beta}(1+|x|+|w|)^{M_2}, \quad (x,w)\in R^{2n} \tag{3.7}$$

for all j,k,α and β. Then, $\partial_x^\alpha(P(t,s)f)(x)$ exist for all α and are continuous in $0\le s\le t\le T$ and $x\in R^n$. (2) *We see under the assumptions of Lemma* 3.1 *that $\partial_x^\alpha(\mathscr{C}(t,s)f)(x)$ exist for all α and are continuous in $0\le s\le t\le T$ and $x \in R^n$.*

Proof. The assertion (1) follows from Lemma 2.1 in [**14**].
From (1.11) we have

$$\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) - I_l = -i\int_{s'}^{t'} H_1\big(\theta, q^{t,s}_{x,y}(\theta)\big)\mathscr{F}\big(\theta,s';q^{t,s}_{x,y}\big)d\theta. \tag{3.8}$$

So from Lemma 3.1 we get

$$\big|\partial_x^\alpha\partial_y^\beta\big\{\mathscr{F}\big(t,s;q^{t,s}_{x,y}\big) - I_l\big\}\big| \le C_{\alpha,\beta}(\langle x\rangle + \langle x-y\rangle)|t-s|, \quad (x,y)\in R^{2n}$$

for $0\le s<t\le T$, all α and β. Consequently, the assertion (2) can be proved as in the proof of Lemma 2.1 in [**14**]. □

Proposition 3.3. *Under the assumptions of Theorem* 2.3 *except for the assumptions concerning* $H_1(t,x)$ *we have for* $a = 0, 1, \dots$: (1) *There exists a constant* $K_a \geq 0$ *such that*

$$\left\| \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int \left(\exp iS_c\left(t,s;q^{t,s}_{\cdot,y}\right)\right) f(y)dy \right\|_{B^a} \\ \leq \mathrm{e}^{K_a(t-s)} \|f\|_{B^a}, \quad 0 \leq t-s \leq \rho^* \tag{3.9}$$

for $f \in B^a$. (2) *Let* $q(x,w)$ *be a scalar function satisfying* (3.7) *and let* $Q(t,s)$ *denote the operator defined by* (3.4). *Set for* $k = 0, 1, \dots$

$$|q|_k = \max_{|\alpha+\beta|\leq k} \sup_{x,w} (1+|x|+|w|)^{-M_2} \left|\partial_w^\alpha \partial_x^\beta q(x,w)\right|. \tag{3.10}$$

Then there exists a $k = k(a, M_2)$ *such that we have*

$$\|Q(t,s)f\|_{B^a} \leq C_a |q|_k \|f\|_{B^{a+M_2}} \tag{3.11}$$

for $f \in B^{a+M_2}$.

Proof. The assertion (1) follows from Proposition 3.4 in [**16**]. The assertion (2) follows from Theorem 4.4 in [**14**]. □

Proposition 3.4. *Under the assumptions of Theorem* 2.3 *there exist constants* $K'_a \geq 0$ $(a = 0, 1, \dots)$ *such that we have*

$$\|\mathscr{C}(t,s)f\|_{B^a} \leq \mathrm{e}^{K'_a(t-s)} \|f\|_{B^a}, \quad 0 \leq t-s \leq \rho^* \tag{3.12}$$

for $f \in (B^a)^l$.

Proof. The inequality (3.12) is clear for $t = s$. Let $0 < t-s \leq \rho^*$. From the assumptions (2.16) for all α, (3.8) and Lemma 3.1 we have

$$\left|\partial_x^\alpha \partial_y^\beta \left\{\mathscr{F}\left(t,s;q^{t,s}_{x,y}\right) - I_l\right\}\right| \leq C_{\alpha,\beta}(t-s) \tag{3.13}$$

for all α and β. We write from (1.13)

$$\mathscr{C}(t,s)f = \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int \left(\exp iS_c\left(t,s;q^{t,s}_{x,y}\right)\right) I_l f(y)dy \\ + \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int \left(\exp iS_c\left(t,s;q^{t,s}_{x,y}\right)\right) \left\{\mathscr{F}\left(t,s;q^{t,s}_{x,y}\right) - I_l\right\} f(y)dy. \tag{3.14}$$

Hence, applying Proposition 3.3 to (3.14), we get

$$\|\mathscr{C}(t,s)f\|_{B^a} \le \mathrm{e}^{K_a(t-s)}\|f\|_{B^a} + \mathrm{Const.}(t-s)\|f\|_{B^a}$$
$$\le \mathrm{e}^{K'_a(t-s)}\|f\|_{B^a}, \quad 0 < t-s \le \rho^*,$$

which completes the proof. □

Proposition 3.5. *We write*

$$H(t) := H_0(t)I_l + H_1(t). \tag{3.15}$$

Besides the assumptions of Lemmas 3.1 *and* 3.2 *we suppose* $|\partial_x^\alpha \partial_t A_j(t,x)| \le C_\alpha \langle x \rangle^{M_1}$ *in* $[0,T]\times R^n$ *for* $j = 1,2,\dots,n$ *and all* α. *Then, there exist an integer* $M_2 \ge 0, r_{jk}(t,s;x,w)$ *and* $r'_{jk}(t,s;x,w)$ $(j,k = 1,2,\dots,l)$ *satisfying* (3.7) *for all* $0 \le s \le t \le T$ *such that*

$$i\frac{\partial}{\partial t}\mathscr{C}(t,s)f - H(t)\mathscr{C}(t,s)f = \sqrt{t-s}R(t,s)f \tag{3.16}$$

and

$$i\frac{\partial}{\partial s}\mathscr{C}(t,s)f + \mathscr{C}(t,s)H(s)f = \sqrt{t-s}R'(t,s)f \tag{3.17}$$

for $f \in C_0^\infty(R^n)^l$, *where* $R(t,s)$ *and* $R'(t,s)$ *are the operators defined by* (3.5) *with* $p_{jk}(x,w) = r_{jk}(t,s;x,w)$ *and* $p_{jk}(x,w) = r'_{jk}(t,s;x,w)$, *respectively.*

Proof. Let $0 \le s < t \le T$. For $0 \le s \le s' \le t' \le t \le T$ and $f \in C_0^\infty(R^n)^l$ we write

$$\tilde{\mathscr{C}}(t',s';t,s)f := \begin{cases} \sqrt{m/(2\pi i(t'-s'))}^{\,n}\displaystyle\int \big(\exp iS_c\big(t',s';q^{t',s'}_{x,y}\big)\big) & s' < t', \\ \quad \times \mathscr{F}\big(t',s';q^{t,s}_{x,y}\big)f(y)dy, & \\ f, & s' = t'. \end{cases} \tag{3.18}$$

In this proof we often write $S_c(t,s;q^{t,s}_{x,y})$ as $S(q^{t,s}_{x,y})$ for the sake of simplicity. It follows from (1.2), (1.4) and (1.12) that we have

$$\begin{aligned} S_c\big(t,s;q^{t,s}_{x,y}\big) &= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot\int_0^1 A(t-\theta(t-s), x-\theta(x-y))d\theta \\ &\quad - \int_s^t V\left(\theta, x - \frac{t-\theta}{t-s}(x-y)\right)d\theta \\ &= \frac{m|x-y|^2}{2(t-s)} + (x-y)\cdot\int_0^1 A(t-\theta(t-s), x-\theta(x-y))d\theta \\ &\quad - (t-s)\int_0^1 V(t-\theta(t-s), x-\theta(x-y))d\theta. \end{aligned} \tag{3.19}$$

From (1.11) and (3.18) as in the proof of Lemma 4.1 in [**16**] we write

$$
\begin{aligned}
& i\frac{\partial}{\partial t'}\tilde{\mathscr{C}}(t',s';t,s)f - H(t')\tilde{\mathscr{C}}(t',s';t,s)f \\
&= -\sqrt{\frac{m}{2\pi i(t'-s')}}^{\,n}\int e^{iS(q^{t',s'}_{x,y})}\Big(r_1(t',s';t,s,x,y) + \frac{i}{2m}r_2(t',s';t,s,x,y)\Big)f(y)dy,
\end{aligned}
\tag{3.20}
$$

where

$$
r_1 = \Big\{\partial_{t'}S\big(q^{t',s'}_{x,y}\big) + \frac{1}{2m}\sum_{j=1}^{n}\big(\partial_{x_j}S\big(q^{t',s'}_{x,y}\big) - A_j(t',x)\big)^2 + V(t',x)\Big\}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big),
\tag{3.21}
$$

$$
\begin{aligned}
r_2 = & \Big\{\frac{mn}{t'-s'} - \Delta_x S\big(q^{t',s'}_{x,y}\big) + \sum_{j=1}^{n}(\partial_{x_j}A_j)(t',x)\Big\}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) \\
& - 2\sum_{j=1}^{n}\big(\partial_{x_j}S(q^{t',s'}_{x,y}) - A_j(t',x)\big)\partial_{x_j}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) + i\Delta_x\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) \\
& + 2mi\big\{H_1\big(t',q^{t,s}_{x,y}(t')\big) - H_1(t',x)\big\}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big).
\end{aligned}
\tag{3.22}
$$

We can see from Lemma 4.1 in [**16**] and Lemma 3.1 in the present paper that setting $t'=t$ and $s'=s$, r_1 in (3.21) and the first term on the right-hand side of (3.22) are of the form of the right-hand side of (3.16). We see from (3.19)

$$
\partial_{x_j}S\big(q^{t,s}_{x,y}\big) - A_j(t,x) = \frac{m(x_j-y_j)}{t-s} + \sqrt{t-s}\,p_j\Big(t,s;x,\frac{x-y}{\sqrt{t-s}}\Big)
\tag{3.23}
$$

for $j=1,2,\dots,n$, where $p_j(t,s;x,w)$ satisfy (3.7). Consequently, we see together with (3.3) from Lemma 3.1 that setting $t'=t$ and $s'=s$, the second term on the right-hand side of (3.22) is of the form of the right-hand side of (3.16). In the same way we can see that setting $t'=t$ and $s'=s$, r_2 is of the form of the right-hand side of (3.16). As in the proof of (3.3) we have

$$
\frac{\partial}{\partial t}\mathscr{F}\big(t',s';q^{t,s}_{x,y}\big) = -i\int_{s'}^{t'}\mathscr{F}\big(t',\theta;q^{t,s}_{x,y}\big)\Big\{\frac{\partial}{\partial t}H_1\big(\theta,q^{t,s}_{x,y}(\theta)\big)\Big\}\mathscr{F}\big(\theta,s';q^{t,s}_{x,y}\big)d\theta.
\tag{3.24}
$$

Hence, setting $t'=t$ and $s'=s$, $\partial_t\tilde{\mathscr{C}}(t',s';t,s)f$ is written of the form of the right-hand side of (3.16). Thus we can prove (3.16), noting $\tilde{\mathscr{C}}(t,s;t,s)f = \mathscr{C}(t,s)f$.

Let $q \in (R^n)^{[0,T]}$ be continuous. Let $0 \le t,\theta,s \le T$. Then from (1.11) as in the proof of (2.1) we can easily see

$$
\mathscr{F}(t,\theta;q)\mathscr{F}(\theta,s;q) = \mathscr{F}(t,s;q).
$$

So from the unitarity of $\mathscr{F}(t,s;q)$ we have

$$\mathscr{F}(t,s;q) = \overline{{}^t\mathscr{F}(s,t;q)} \tag{3.25}$$

for $0 \le t, s \le T$, where the right-hand side above denotes the complex conjugate of the transposed of the matrix $\mathscr{F}(s,t;q)$. Consequently, we have

$$\frac{\partial}{\partial s}\mathscr{F}(t,s;q) = i\mathscr{F}(t,s;q)H_1(s,q(s)). \tag{3.26}$$

We will prove (3.17). From (3.18) and (3.26) as in the proof of Lemma 4.1 in [**16**] we write

$$\begin{aligned}
&i\frac{\partial}{\partial s'}\tilde{\mathscr{C}}(t',s';t,s)f + \tilde{\mathscr{C}}(t',s';t,s)H(s')f \\
&\quad = -\sqrt{\frac{m}{2\pi i(t'-s')}}^{\,n}\int e^{iS(q_{x,y}^{t',s'})}\left(r_1'(t',s';t,s,x,y) + \frac{i}{2m}r_2'(t',s';t,s,x,y)\right)f(y)dy,
\end{aligned} \tag{3.27}$$

where

$$r_1' = \left\{\partial_{s'}S\big(q_{x,y}^{t',s'}\big) - \frac{1}{2m}\sum_{j=1}^{n}\big(\partial_{y_j}S(q_{x,y}^{t',s'}) + A_j(s',y)\big)^2 - V(s',y)\right\}\mathscr{F}\big(t',s';q_{x,y}^{t,s}\big), \tag{3.28}$$

$$\begin{aligned}
r_2' = {} & \left\{-\frac{mn}{t'-s'} + \Delta_y S\big(q_{x,y}^{t',s'}\big) + \sum_{j=1}^{n}(\partial_{x_j}A_j)(s',y)\right\}\mathscr{F}\big(t',s';q_{x,y}^{t,s}\big) \\
& + 2\sum_{j=1}^{n}\big(\partial_{y_j}S(q_{x,y}^{t',s'}) + A_j(s',y)\big)\partial_{y_j}\mathscr{F}\big(t',s';q_{x,y}^{t,s}\big) - i\Delta_y\mathscr{F}\big(t',s';q_{x,y}^{t,s}\big) \\
& + 2mi\mathscr{F}\big(t',s';q_{x,y}^{t,s}\big)\big\{H_1(s',y) - H_1\big(s',q_{x,y}^{t,s}(s')\big)\big\}.
\end{aligned} \tag{3.29}$$

So, we can prove (3.17) as in the proof of (3.16). □

Proof of Theorem 2.3. Let K_j and K_j' $(j = 1,2,\ldots,\nu)$ be bounded operators on $(B^a)^l$ and $f \in C_0^\infty(R^n)^l$. Then, it holds that

$$\begin{aligned}
&K_\nu\chi(\epsilon\cdot)K_{\nu-1}\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)K_1 f - K_\nu'K_{\nu-1}'\cdots K_1'f \\
&\quad = \sum_{j=1}^{\nu}K_\nu\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)K_{j+1}\chi(\epsilon\cdot)\big(K_j - K_j'\big)K_{j-1}'\cdots K_1'f \\
&\qquad + \sum_{j=1}^{\nu-1}K_\nu\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)K_{j+1}(\chi(\epsilon\cdot)-1)K_j'\cdots K_1'f.
\end{aligned} \tag{3.30}$$

Let $|\Delta| \le \rho^*$ and $a = 0, 1, \dots$. We can easily see

$$\sup_{0<\epsilon\le 1} \|\chi(\epsilon\cdot)f\|_{B^a} \le \text{Const.}\|f\|_{B^a}$$

and

$$\lim_{\epsilon\to 0} \big\|(\chi(\epsilon\cdot) - 1)f\big\|_{B^a} = 0$$

for $f \in (B^a)^l$. So, applying Proposition 3.4 to (2.5), then from (3.30) for $f \in (B^a)^l$ we have

$$\mathscr{C}_\Delta(t, t_0)f = \mathscr{C}(t, \tau_{\mu-1})\mathscr{C}(\tau_{\mu-1}, \tau_{\mu-2}) \cdots \mathscr{C}(\tau_{\mu'}, t_0)f \tag{3.31}$$

in $(B^a)^l$. It follows from Proposition 3.4 that

$$\|\mathscr{C}_\Delta(t, t_0)f\|_{B^a} \le e^{K_a(t-t_0)}\|f\|_{B^a}, \quad 0 \le t_0 \le t \le T \tag{3.32}$$

for $f \in (B^a)^l$.

From (1.3), (1.5), (1.12), (2.10) and (3.19) we can easily see

$$S\big(t, s; \zeta^{t,s}_{x,y,\Pi}\big) = -\frac{(t-s)}{2m}\left|\Pi - \frac{m(x-y)}{t-s}\right|^2 + S_c\big(t, s; q^{t,s}_{x,y}\big). \tag{3.33}$$

Consequently, from (2.11) we get

$$\begin{aligned} G_\epsilon(t, s)f = \sqrt{\frac{m}{2\pi i(t-s)}}^{\,n} \int \big(\exp iS_c\big(t, s; q^{t,s}_{x,y}\big)\big)\mathscr{F}\big(t, s; q^{t,s}_{x,y}\big)f(y)dy\ \sqrt{i/2\pi}^{\,n} \\ \times \int (\exp -i|z|^2/2)\chi\big(\epsilon\sqrt{m/(t-s)}z + \epsilon m(x-y)/(t-s)\big)dz. \end{aligned} \tag{3.34}$$

Let $0 < t - s \le \rho^*$. Applying Lemma 6.2 in [**16**] to (3.34), we have from Lemma 3.1 in the present paper

$$\lim_{\epsilon\to 0} G_\epsilon(t, s)f = \mathscr{C}(t, s)f \tag{3.35}$$

in $(B^a)^l$ for $f \in (B^a)^l$. From (2) of Proposition 3.3 we also have

$$\sup_{0<\epsilon\le 1} \big\|G_\epsilon(t, s)f\big\|_{B^a} \le C_a\|f\|_{B^a}. \tag{3.36}$$

Hence, applying (3.30) to (2.12) and (3.31), then from (3.12), (3.35) and (3.36) we have

$$G_\Delta(t, t_0)f = \mathscr{C}_\Delta(t, t_0)f \tag{3.37}$$

in $(B^a)^l$ for $f \in (B^a)^l$ as in the proof of (3.31), which completes the proof of (1) of Theorem 2.3.

We can prove (2) of Theorem 2.3 as in the proof of Theorem 1 in [**16**]. So we shall give a rough sketch of the proof. See [**16**] for the detailed proof. We have $|\partial_x^\alpha \partial_t A_j(t,x)| \le C_\alpha \langle x \rangle$ in $[0,T] \times R^n$ for $j = 1, 2, \dots, n$ and all α from the assumptions and (1.1). We first note that from Lemma 3.2 and Proposition 3.5 we have

$$i\big(\mathscr{C}(t_1,s)f - \mathscr{C}(t_2,s)f\big) = \int_{t_1}^{t_2} \big(H(\theta)\mathscr{C}(\theta,s)f + \sqrt{\theta - s}R(\theta,s)f\big)d\theta \tag{3.38}$$

for $f \in C_0^\infty(R^n)^l$ and $0 \le s \le t_1, t_2 \le T$.

We take an arbitrary t' such that $0 \le t_0 \le t' \le T$ and a $1 \le k \le \nu$ such that $\tau_{k-1} < t' \le \tau_k$. Let $|\Delta| \le \rho^*$. Suppose $\mu = k$ and $t_0 \le \tau_{\mu-1}$. Then from (3.31) and (3.38) we have

$$\begin{aligned} &i\big(\mathscr{C}_\Delta(t,t_0) - \mathscr{C}_\Delta(t',t_0)\big) \\ &\quad = i\big(\mathscr{C}(t,\tau_{\mu-1}) - \mathscr{C}(t',\tau_{\mu-1})\big)\mathscr{C}_\Delta(\tau_{\mu-1},t_0) \\ &\quad = \int_{t'}^{t} H(\theta)\mathscr{C}_\Delta(\theta,t_0)d\theta + \int_{t'}^{t} \sqrt{\theta - \tau_{\mu-1}}R(\theta,\tau_{\mu-1})d\theta\, \mathscr{C}_\Delta(\tau_{\mu-1},t_0). \end{aligned} \tag{3.39}$$

Suppose $\mu > k$ and $t_0 \le \tau_{k-1}$. Then

$$\begin{aligned} &\mathscr{C}_\Delta(t,t_0) - \mathscr{C}_\Delta(t',t_0) \\ &\quad = \mathscr{C}_\Delta(t,t_0) - \mathscr{C}_\Delta(\tau_{\mu-1},t_0) + \sum_{l=1}^{\mu-k-1} \big(\mathscr{C}_\Delta(\tau_{\mu-l},t_0) - \mathscr{C}_\Delta(\tau_{\mu-l-1},t_0)\big) \\ &\qquad + \mathscr{C}_\Delta(\tau_k,t_0) - \mathscr{C}_\Delta(t',t_0). \end{aligned}$$

So, as in the proof of (3.39) we have

$$\begin{aligned} &i\big(\mathscr{C}_\Delta(t,t_0) - \mathscr{C}_\Delta(t',t_0)\big) \\ &\quad = \int_{t'}^{t} H(\theta)\mathscr{C}_\Delta(\theta,t_0)d\theta + \int_{\tau_{\mu-1}}^{t} \sqrt{\theta - \tau_{\mu-1}}R(\theta,\tau_{\mu-1})d\theta\, \mathscr{C}_\Delta(\tau_{\mu-1},t_0) \\ &\qquad + \sum_{l=1}^{\mu-k-1} \int_{\tau_{\mu-l-1}}^{\tau_{\mu-l}} \sqrt{\theta - \tau_{\mu-l-1}}R(\theta,\tau_{\mu-l-1})d\theta\, \mathscr{C}_\Delta(\tau_{\mu-l-1},t_0) \\ &\qquad + \int_{t'}^{\tau_k} \sqrt{\theta - \tau_{k-1}}R(\theta,\tau_{k-1})d\theta\, \mathscr{C}_\Delta(\tau_{k-1},t_0). \end{aligned} \tag{3.40}$$

Set $M = max(M_2, 2)$. Then, applying (2) of Proposition 3.3 and (3.32) to (3.40), we get

$$
\begin{aligned}
&\big\|\mathscr{C}_\Delta(t,t_0)f - \mathscr{C}_\Delta(t',t_0)f\big\|_{B^a} \\
&\quad \le \mathrm{Const.}\bigg(\int_{t'}^{t} \big\|\mathscr{C}_\Delta(\theta,t_0)f\big\|_{B^{a+M}}d\theta + \sqrt{|\Delta|}\int_{\tau_{\mu-1}}^{t} d\theta\,\big\|\mathscr{C}_\Delta(\tau_{\mu-1},t_0)f\big\|_{B^{a+M}} \\
&\qquad\qquad + \sum_{l=1}^{\mu-k-1}\sqrt{|\Delta|}\int_{\tau_{\mu-l-1}}^{\tau_{\mu-l}} d\theta\,\big\|\mathscr{C}_\Delta(\tau_{\mu-l-1},t_0)f\big\|_{B^{a+M}} \\
&\qquad\qquad + \sqrt{|\Delta|}\int_{t'}^{\tau_k} d\theta\,\big\|\mathscr{C}_\Delta(\tau_{k-1},t_0)f\big\|_{B^{a+M}}\bigg) \\
&\quad \le \mathrm{Const.}\, e^{K_{a+M}T}(1+\sqrt{\rho^*})|t-t'|\|f\|_{B^{a+M}}.
\end{aligned}
$$

Hence, we obtain

$$
\big\|\mathscr{C}_\Delta(t,t_0)f - \mathscr{C}_\Delta(t',t_0)f\big\|_{B^a} \le \mathrm{Const.}\, e^{K_{a+M}T}(1+\sqrt{\rho^*})|t-t'|\|f\|_{B^{a+M}} \tag{3.41}
$$

when $\mu > k$ and $t_0 \le \tau_{k-1}$. It is easy to see that (3.41) is valid in general for $0 \le t_0 \le t$, $t' \le T$.

As in the proof of (3.41) we have

$$
\big\|\mathscr{C}_\Delta(t,t_0)f - \mathscr{C}_\Delta(t,t_0')f\big\|_{B^a} \le \mathrm{Const.}\, e^{K_{a+M}T}(1+\sqrt{\rho^*})|t_0-t_0'|\|f\|_{B^{a+M}}
$$

for $0 \le t_0$, $t_0' \le t \le T$. Thus we obtain

$$
\begin{aligned}
&\big\|\mathscr{C}_\Delta(t,t_0)f - \mathscr{C}_\Delta(t',t_0')f\big\|_{B^a} \\
&\quad \le \mathrm{Const.}\, e^{K_{a+M}T}\big(1+\sqrt{\rho^*}\big)\big(|t-t'|+|t_0-t_0'|\big)\|f\|_{B^{a+M}}
\end{aligned} \tag{3.42}
$$

in general for $0 \le t_0 \le t \le T$ and $0 \le t_0' \le t' \le T$.

Let $\{\Delta_j\}_{j=1}^{\infty}$ be a family of subdivisions of $[0,T]$ such that $|\Delta_j| \le \rho^*$ and $\lim_{j\to\infty}|\Delta_j| = 0$. Take an arbitrary $f \in (B^{a+2M})^l$ $(a = 0,1,\dots)$. Then, we see from (3.32) and (3.42) that $\{\mathscr{C}_{\Delta_j}(t,t_0)f\}_{j=1}^{\infty}$ is uniformly bounded as a family of $(B^{a+2M})^l$-valued functions and equicontinuous as a family of $(B^{a+M})^l$-valued functions in $0 \le t_0 \le t \le T$. We note from the Rellich criterion (cf. [**20**]) that the embedding map from B^{a+2M} into B^{a+M} is compact. So, from the Ascoli-Arzelà theorem we can find a subsequence $\{\Delta_{j_k}\}_{k=1}^{\infty}$, which may depend on f, such that $\mathscr{C}_{\Delta_{j_k}}(t,t_0)f$ converges in $(B^{a+M})^l$ uniformly in $0 \le t_0 \le t \le T$ as $k \to \infty$. It follows from Lemma 3.2 and (3.40) with $t' = t_0$ that $\lim_{k\to\infty}\mathscr{C}_{\Delta_{j_k}}(t,t_0)f \in \mathscr{E}^0_{t,t_0}([0,T];(B^{a+M})^l)\bigcap\mathscr{E}^1_{t,t_0}([0,T];(B^a)^l)$ satisfies the generalized Pauli equation (1.10). As was noted in Remark 2.2, the solutions to (1.10) are unique. Therefore, $\mathscr{C}_\Delta(t,t_0)f$ converges to the solution to (1.10) in $(B^{a+M})^l$ uniformly in $0 \le t_0 \le t \le T$ as $|\Delta| \to 0$.

Take an arbitrary $f \in (B^a)^l$. Let Δ and Δ' be subdivisions such that $|\Delta| \le \rho^*$ and $|\Delta'| \le \rho^*$. Then for any $\epsilon > 0$ we can take a $g \in (B^{a+2M})^l$ such that $\|g-f\|_{B^a} < \epsilon$. From (3.32) we have

$$\begin{aligned}
&\left\|\mathscr{C}_\Delta(t,t_0)f-\mathscr{C}_{\Delta'}(t,t_0)f\right\|_{B^a}\\
&\le \left\|\mathscr{C}_\Delta(t,t_0)g-\mathscr{C}_{\Delta'}(t,t_0)g\right\|_{B^a}+\left\|\mathscr{C}_\Delta(t,t_0)(f-g)\right\|_{B^a}+\left\|\mathscr{C}_{\Delta'}(t,t_0)(f-g)\right\|_{B^a}\\
&\le \left\|\mathscr{C}_\Delta(t,t_0)g-\mathscr{C}_{\Delta'}(t,t_0)g\right\|_{B^{a+M}}+2\mathrm{e}^{K_aT}\epsilon.
\end{aligned}$$

So,

$$\overline{\lim}_{|\Delta|,|\Delta'|\to 0}\max_{0\le t_0\le t\le T}\left\|\mathscr{C}_\Delta(t,t_0)f-\mathscr{C}_{\Delta'}(t,t_0)f\right\|_{B^a}\le 2\mathrm{e}^{K_aT}\epsilon.$$

Hence, we can see that $\mathscr{C}_\Delta(t,t_0)f$ converges in $(B^a)^l$ uniformly in $0\le t_0\le t\le T$ as $|\Delta|\to 0$. It follows from Lemma 3.2, (3.40) with $t'=t_0$ and Lemma 2.5 in [**12**] that $\lim_{|\Delta|\to 0}\mathscr{C}_\Delta(t,t_0)f$ belongs to $\mathscr{E}^0_{t,t_0}([0,T];(B^a)^l)\bigcap\mathscr{E}^1_{t,t_0}([0,T];(B^{a-M})^l)$ and satisfies the generalized Pauli equation (1.10). Thus, we could complete the proof of Theorem 2.3. □

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Wataru ICHINOSE
Department of Mathematical Science
Shinshu University
Matsumoto 390-8621, Japan
E-mail: ichinose@math.shinshu-u.ac.jp