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Totally geodesic boundaries are dense in the moduli space

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Let F be a closed, oriented surface such that the genus of each component of F is greater than 1. In this paper, we will study the subset $\mathcal{R}(F)$ of the moduli space $\mathcal{M}(F)$ such that a hyperbolic structure $s \in \mathcal{M}(F)$ is an element of $\mathcal{R}(F)$ if there exists a compact, connected, oriented hyperbolic 3-manifold M with totally geodesic boundary and admitting an orientation-preserving isometry $\varphi: \partial M \to F(s)$, where ∂M is assumed to have the orientation induced naturally from that on M. Note that $\mathcal{R}(F)$ is a countable subset of $\mathcal{M}(F)$.

First, consider the special case where F consists of two components each of which is homeomorphic to a given closed surface Σ of genus >1. In Fujii [3], it is implicitly seen that, for any $s \in \mathcal{M}(\Sigma)$, one can construct a compact, connected, oriented, hyperbolic 3-manifold M with totally geodesic, two-component boundary such that one component is arbitrarily close to $\Sigma(s)$ in $\mathcal{M}(\Sigma)$ and the other is to $\Sigma(\bar{s})$ (see Lemma 1 in §2 for the explicit proof based on the circle-packing argument in Brooks [2]). Here, $\bar{s} \in \mathcal{M}(\Sigma)$ denotes the hyperbolic structure on Σ admitting an orientation-reversing isometry $\varphi: \Sigma(s) \to \Sigma(\bar{s})$. This implies that the closure of $\mathcal{R}(F)$ in $\mathcal{M}(F)$ contains the skew diagonal $\Delta_{\mathrm{skew}}(\Sigma) = \{(s, \bar{s}); s \in \mathcal{M}(\Sigma)\}$ of $\mathcal{M}(F) = \mathcal{M}(\Sigma) \times \mathcal{M}(\Sigma)$.

In this paper, we will consider a more general case and prove the following theorem.

THEOREM. *Suppose that $F = \Sigma_1 \sqcup \cdots \sqcup \Sigma_t$ is any closed, oriented surface such that the genus of each component Σ_i is greater than* 1. *Then, $\mathcal{R}(F)$ is dense in $\mathcal{M}(F) = \mathcal{M}(\Sigma_1) \times \cdots \times \mathcal{M}(\Sigma_t)$.*

McMullen's results in [9], [10] play important roles in our proof of Theorem. Especially, the argument in [10] for skinning maps is well applicable to construct a compact, connected, hyperbolic 3-manifold M by joining "long" hyperbolic 3-manifolds associated to any $s_i \in \mathcal{M}(\Sigma_i)$ $(i=1, \cdots, t)$ so that ∂M is totally geodesic and arbitrarily close to $\Sigma_1(s_1) \sqcup \cdots \sqcup \Sigma_t(s_t)$ in $\mathcal{M}(F)$.

We would like to thank W. Thurston for his suggestion which directs our attention to McMullen's works.

§ 1. Preliminaries.

In this section, we will review the fundamental notation and definitions needed in later sections, and refer to Hempel [5], Jaco [7] for more details on 3-manifold topology and to Gardiner [4], Imayoshi-Taniguchi [6] on Teichmüller spaces.

A *Haken manifold* is a compact, connected, irreducible, oriented 3-manifold containing an incompressible surface. A Haken manifold M with incompressible boundary ∂M is called *boundary-irreducible*. A Haken manifold M is *atoroidal* (resp. *acylindrical*) if any π_1-injective map $\varphi: T^2 \to M$ is homotopic (resp. any π_1-injective, proper map $\varphi:(A, \partial A) \to (M, \partial M)$ is homotopic rel. ∂A) into ∂M, where T^2 is a torus and A is an annulus.

Orientation-preserving (resp. orientation-reversing) homeomorphisms are, for short, called o.p.-(resp. o.r.-)homeomorphisms. For any oriented surface or 3-manifold N, an orientation-reversed copy of N is denoted by $\bar{N}$. Let M be an oriented 3-manifold whose boundary consists of two components which are o.p.-homeomorphic to each other. If Σ is one component of ∂M, then the other is denoted by Σ_-. We always assume that Σ_- has a marking induced from that on Σ by an o.p.-homeomorphism $\varphi: \Sigma \to \Sigma_-$ (do not confuse Σ_- with $\bar{\Sigma}$).

Let $F=\Sigma_1 \sqcup \cdots \sqcup \Sigma_t$ be a closed, oriented surface such that the genus of each component Σ_i is greater than 1. The *Teichmüller space* $\mathcal{T}(F)$ of F is the set of equivalence classes of hyperbolic structures on F, where two hyperbolic structures s_1, s_2 on F are *equivalent* to each other if there exists an o.p.-isometry $\varphi: F(s_1) \to F(s_2)$ homotopic to the identity $\mathrm{id}_F: F \to F$. We denote by $[F(s)]$ (or simply by s) the element of $\mathcal{T}(F)$ represented by $F(s)$. The *Teichmüller distance* $d_F(s_1, s_2)$ between two elements s_1, s_2 of $\mathcal{T}(F)$ is given by

$$d_F(s_1, s_2) = \frac{1}{2}\inf_f \{\log K_f(s_1, s_2)\},$$

where f ranges over all quasiconformal homeomorphisms from $F(s_1)$ to $F(s_2)$ homotopic to the identity id_F, and $K_f(s_1, s_2)$ is the maximal dilatation of f. It is well known that the i-th factor $\mathcal{T}(\Sigma_i)$ of the metric space $\mathcal{T}(F)$ is homeomorphic to $\boldsymbol{R}^{6g_i-6}$, where $g_i=\mathrm{genus}(\Sigma_i)$. So, $\mathcal{T}(F)$ is homeomorphic to the $6(g_1+\cdots+g_t-t)$-dimensional Euclidean space. For an $s \in \mathcal{T}(F)$ and $r>0$, we denote by $B_F(s, r)$ the closed r-neighborhood of s in $\mathcal{T}(F)$, that is,

$$B_F(s, r) = \{s' \in \mathcal{T}(F)\,;\, d_F(s, s') \leqq r\}.$$

The *moduli space* $\mathcal{M}(F)$ of F is the quotient space of $\mathcal{T}(F)$ such that two elements s_1, s_2 of $\mathcal{T}(F)$ represent the same element of $\mathcal{M}(F)$ if there exists an o.p.-isometry $\varphi: F(s_1) \to F(s_2)$ with $\varphi(\Sigma_i)=\Sigma_i$ for $i=1, \cdots, t$. Roughly, an element of $\mathcal{T}(F)$ is a hyperbolic structure on F respecting markings and an element of $\mathcal{M}(F)$ is one neglecting markings.

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\boldsymbol{C})$, the group of all o.p.-isometries on the hyperbolic 3-space $\boldsymbol{H}^3$. This group Γ acts conformally on the sphere $S^2_\infty=\boldsymbol{C}\cup\{\infty\}$ at infinity. We denote the region of discontinuity and the limit set of Γ respectively by $\Omega(\Gamma)$ and $\Lambda(\Gamma)$. A Kleinian group Γ is *elementary* if Γ contains an abelian group of finite index. In this paper, we only consider the case where Γ is finitely generated, torsion free, and non-elementary. We fix an orientation on $\boldsymbol{H}^3$. Then, $N=\boldsymbol{H}^3/\Gamma$ is an oriented hyperbolic 3-manifold and the quotient map $p:\boldsymbol{H}^3\to N$ is the universal covering. Furthermore, the convex hull $H(\Gamma)$ of $\Lambda(\Gamma)$ in $\boldsymbol{H}^3$ is non-empty, and the image $C(\Gamma)=p(H(\Gamma))$ is the smallest closed, convex core of N. The *Kleinian manifold* for Γ is $O(\Gamma)=(\boldsymbol{H}^3\cup\Omega(\Gamma))/\Gamma$, see [**14**, DEFINITION 8.3.5]. We have obviously $\partial O(\Gamma)=\Omega(\Gamma)/\Gamma$ and $\operatorname{int} O(\Gamma)=N$. A Kleinian group Γ is called *geometrically finite* if the volume of the ε-neighborhood $C_\varepsilon(\Gamma)$ of $C(\Gamma)$ in N is finite for some $\varepsilon>0$. According to Thurston's Uniformization Theorem [**15**] (a special case), for any boundary-irreducible, atoroidal Haken manifold M containing a closed, incompressible surface of genus >1, there exists a geometrically finite Kleinian group Γ such that $C_\varepsilon(\Gamma)$ is homeomorphic to $M-\partial_T M$, where $\partial_T M$ is the union of torus components of ∂M.

Let M be a boundary-irreducible, atoroidal, Haken manifold with nonempty boundary and such that the genus of each component of ∂M is greater than 1. Let $QH_0(M)$ be the set of equivalence classes of pairs (N,φ) such that $N=\boldsymbol{H}^3/\Gamma$ is an oriented hyperbolic 3-manifold and $\varphi: M\to O(\Gamma)$ is an o.p.-homeomorphism. Here, two elements (N_1,φ_1), (N_2,φ_2) with $N_1=\boldsymbol{H}^3/\Gamma_1$, $N_2=\boldsymbol{H}^3/\Gamma_2$ are *equivalent* to each other if there exists an o.p.-homeomorphism $\psi: O(\Gamma_1)\to O(\Gamma_2)$ isotopic to $\varphi_2\circ\varphi_1^{-1}$ such that the restriction $\psi|_{N_1}: N_1\to N_2$ is isometric. Since M is compact, Γ is geometrically finite. We endow $QH_0(M)$ with the *quasi-isometric topology* so that (N_1,φ_1) and (N_2,φ_2) are close to each other if there exists an o.p.-homeomorphism $\psi': O(\Gamma_1)\to O(\Gamma_2)$ isotopic to $\varphi_2\circ\varphi_1^{-1}$ such that the derivative of $\psi'|_{N_1}: N_1\to N_2$ is uniformly close to being an o.p.-isometry. Consider the correspondence

$$\mathrm{conf}: QH_0(M)\longrightarrow \mathcal{T}(\partial M) \tag{1.1}$$

such that $\mathrm{conf}(\boldsymbol{H}^3/\Gamma,\varphi)$ is the element of $\mathcal{T}(\partial M)$ conformally equivalent to $\varphi^*([\partial O(\Gamma)])$. By works of several people including Ahlfors, Bers, Kra and Marden, it is shown that this correspondence is a well-defined homeomorphism.

We suppose further that M is not a deformation retract of a closed surface, and Σ_i $(i=1,\cdots,t)$ are the components of ∂M. For any $(\boldsymbol{H}^3/\Gamma,\varphi)=\mathrm{conf}^{-1}(s_1,\cdots,s_t)\in QH_0(M)$, let $p_i:\widetilde{O(\Gamma)}_i\to O(\Gamma)$ be the covering associated to

$\Gamma_i=\varphi_*(\pi_1(\Sigma_i))\subset\pi_1(O(\Gamma))=\Gamma$. Since the Kleinian manifold $O(\Gamma_i)$ is homeomorphic to $\Sigma_i\times[0, 1]$, $\partial O(\Gamma_i)$ consists of two components each of which is homeomorphic to Σ_i. One of them coincides with the compact component of $\partial\widetilde{O(\Gamma)}_i$. We can regard the conformal structure on the other component as representing the element on $\mathcal{T}(\bar{\Sigma}_i)$, denoted by $\sigma_i(s_1, \cdots, s_t)$. Then, the *skinning map* $\sigma_M: \mathcal{T}(\partial M)\to\mathcal{T}(\overline{\partial M})$ is defined by

$$\sigma_M(s_1, \cdots, s_t)=(\sigma_1(s_1, \cdots, s_t), \cdots, \sigma_t(s_1, \cdots, s_t)).$$

Let $W=M_1\sqcup\cdots\sqcup M_n$ be the disjoint union of the M_j's each of which satisfies the same conditions as the above M does. Then, the skinning map

$$\sigma_W: \mathcal{T}(\partial W)=\mathcal{T}(\partial M_1)\times\cdots\times\mathcal{T}(\partial M_n)\longrightarrow\mathcal{T}(\overline{\partial W})=\mathcal{T}(\overline{\partial M_1})\times\cdots\times\mathcal{T}(\overline{\partial M_n})$$

is given by $\sigma_W=(\sigma_{M_1}, \cdots, \sigma_{M_n})$. Consider the case where W is divided into two families W_1, W_2 admitting an o.r.-homeomorphism $\gamma: \partial W_2\to\partial W_1$. Then, γ and its inverse $\gamma^{-1}: \partial W_1\to\partial W_2$ determine the o.r.-involution $\tau: \partial W\to\partial W$. The map $\tau_*: \mathcal{T}(\overline{\partial W})\to\mathcal{T}(\partial W)$ induced by τ is an isometry. By Maskit's Combination Theorem [8], a fixed point $(s_1, s_2)\in\mathcal{T}(\partial W)=\mathcal{T}(\partial W_1)\times\mathcal{T}(\partial W_2)$ of the composition

$$\tau_*\circ\sigma_W: \mathcal{T}(\partial W)\longrightarrow\mathcal{T}(\partial W),$$

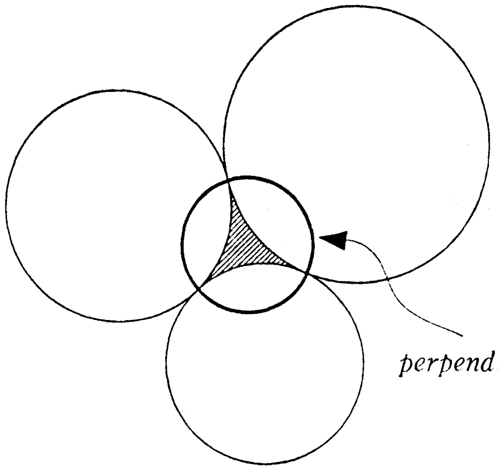
called a solution to the *gluing problem* for (W_1, W_2), determines a hyperbolic structure on $W_1\cup_\gamma W_2$. This is just Thurston's formulation for the proof of his Uniformization Theorem, see [11] for more details.

§ 2. Construction of manifolds with totally geodesic boundary.

Let Σ be a closed, connected, oriented surface with genus >1, and let s be a hyperbolic structure on Σ. A *circle* on $\Sigma(s)$ is a simple, closed curve which bounds a metric disk in $\Sigma(s)$. A configuration of circles on $\Sigma(s)$ is a collection C of a finite number of circles on $\Sigma(s)$, such that the interiors of all disks bounded by them are mutually disjoint. A configuration of circles on $\Sigma(s)$ is said to be a *circle packing*, if the complement of the interiors of the disks consists only of curvilinear triangles. Such a curvilinear triangle is bounded by three mutually tangent circles. Then, there exists a unique circle on $\Sigma(s)$, called the *perpendicular circle* for the curvilinear triangle, which meets each of the three circles perpendicularly, see Figure 1. A point s in the Teichmüller space $\mathcal{T}(\Sigma)$ is said to be a *circle packing point*, if there exists a circle packing on the hyperbolic surface $\Sigma(s)$.

First of all, we will prove the following lemma.

LEMMA 1. *For any* $s\in\mathcal{T}(\Sigma)$ *and* $\varepsilon>0$, *there exists an* $s'\in\mathcal{T}(\Sigma)$ *with* $d_\Sigma(s, s')<\varepsilon$ *and a compact, connected, oriented, hyperbolic* 3-*manifold* M *with totally*



The shaded area is a curvilinear triangle.

Fig. 1.

geodesic, two-component boundary o.p.-isometric to $\Sigma(s')\sqcup\Sigma(\bar{s}')$. Moreover, M admits an isometric o.r.-involution exchanging the components of ∂M.

To prove Lemma 1, we need the following result due to Brooks [**2**] (see also Bowers-Stephenson [**1**]).

THEOREM [(**Brooks**)]. *The set of circle packing points forms a dense subset of $\mathcal{T}(\Sigma)$.*

PROOF OF LEMMA 1. By Brooks' theorem, there exists a circle packing point $s_0\in\mathcal{T}(\Sigma)$ with $d_\Sigma(s, s_0)<\varepsilon/2$. First, we will construct a cusped hyperbolic 3-manifold N with totally geodesic boundary ∂N o.p.-isometric to $\Sigma(s_0)\sqcup\Sigma(\bar{s}_0)$. In order to explain this construction, we will use the Poincaré model of the hyperbolic 3-space $\boldsymbol{H}^3$. Namely, let $\boldsymbol{H}^3$ be the space $\{\boldsymbol{x}=(x_1, x_2, x_3)\in\boldsymbol{R}^3;\ |\boldsymbol{x}|<1\}$ endowed with the Riemannian metric ds given by $ds=2|d\boldsymbol{x}|/(1-|\boldsymbol{x}|^2)$, and let $\boldsymbol{H}^2$ be the totally geodesic plane $\{\boldsymbol{x}=(x_1, x_2, x_3)\in\boldsymbol{H}^3;\ x_3=0\}$ in $\boldsymbol{H}^3$. We set $\boldsymbol{H}^3_+=\{\boldsymbol{x}=(x_1, x_2, x_3)\in\boldsymbol{H}^3;\ x_3\geqq 0\}$ and $U_\infty=\{\boldsymbol{x}=(x_1, x_2, x_3)\in S^2_\infty;\ x_3>0\}$. Consider the orthogonal projection $\mathrm{proj}:\boldsymbol{H}^2\to U_\infty$ along geodesics in $\boldsymbol{H}^3_+$ each of which starts from $\boldsymbol{H}^2$ in the orthogonal direction, see Figure 2. Let Γ_0 be a Fuchsian group corresponding to $\Sigma(s_0)$, i.e., $\boldsymbol{H}^2/\Gamma_0=\Sigma(s_0)$, and let $p:\boldsymbol{H}^2\to\Sigma(s_0)$ be the universal covering. Since s_0 is a circle packing point, we have a circle packing $\mathcal{C}$ on $\Sigma(s_0)$. Let $\mathcal{P}$ be the set of perpendicular circles for all curvilinear triangles which are complementary to $\mathcal{C}$. The hyperbolic 2-space $\boldsymbol{H}^2$ is packed by the set $\tilde{\mathcal{C}}$ of circles C in $\boldsymbol{H}^2$ with $p(C)\in\mathcal{C}$. The set $\tilde{\mathcal{P}}$ of circles C' in $\boldsymbol{H}^2$ with $p(C')\in\mathcal{P}$ consists of circles perpendicular to curvilinear triangles complementary to $\tilde{\mathcal{C}}$. The projection $\mathrm{proj}:\boldsymbol{H}^2\to U_\infty$ maps $\tilde{\mathcal{C}}$, $\tilde{\mathcal{P}}$ to sets of circles in U_∞, denoted

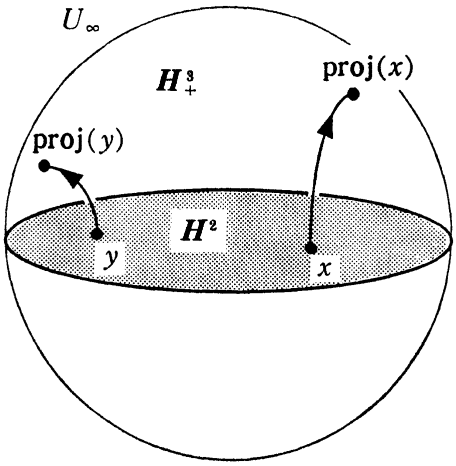
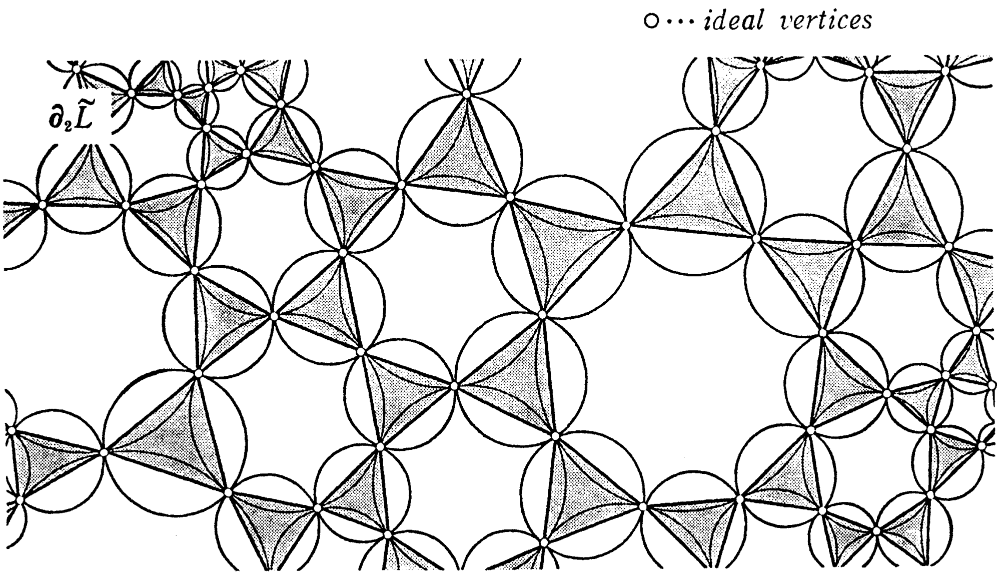


Fig. 2.

respectively by $\hat{\mathcal{C}}$, $\hat{\mathcal{P}}$. Let $\tilde{L}$ be the subspace of $\boldsymbol{H}^3_+$ obtained as follows: consider the regions interior to the hemispheres in $\boldsymbol{H}^3_+$ lying on circles in $\hat{\mathcal{C}} \sqcup \hat{\mathcal{P}}$, and then, obtain $\tilde{L}$ by removing these regions from $\boldsymbol{H}^3_+$. The space $\tilde{L}$ is a geodesic polyhedron with ideal vertices and has two boundary components $\partial_1 \tilde{L}$, $\partial_2 \tilde{L}$, where $\partial_1 \tilde{L} = \boldsymbol{H}^2$ and $\partial_2 \tilde{L}$ is the face obtained by carving $\boldsymbol{H}^3_+$ along these hemispheres. Thus, $\partial_2 \tilde{L}$ consists of infinitely many ideal, totally geodesic polygons which meet each other at the right-angle. Note that, for any hemisphere H lying on a circle in $\hat{\mathcal{P}}$, $H \cap \tilde{L} = H \cap \partial_2 \tilde{L}$ is an ideal triangle in $\partial_2 \tilde{L}$. We denote the union of such triangular faces of $\partial_2 \tilde{L}$ by $\hat{T}$, see Figure 3. The Fuchsian group Γ_0 acts on U_∞ conformally, and both $\hat{\mathcal{C}}$, $\hat{\mathcal{P}}$ are invariant under the Γ_0-action. The quotient map $q : \tilde{L} \to L = \tilde{L}/\Gamma_0$ is the universal covering which is an extension of $p : \boldsymbol{H}^2 \to \Sigma(s_0)$ with $\Sigma(s_0) = q(\partial_1 \tilde{L})$. Set $\partial_2 L = q(\partial_2 \tilde{L})$ and $T = q(\hat{T})$.

Now, take the double $d(L)$ of L along T. Then, $\partial d(L)$ contains a closed, two-component surface A o.p.-isometric to $\Sigma(s_0) \sqcup \Sigma(\bar{s}_0)$. Since every component Δ of $B = \partial_2 L - \operatorname{int} T$ intersects T along the edges of Δ at the right-angle in L and since $\Delta \cap (B - \Delta) = \emptyset$, each component of $\partial d(L) - A$ is a totally geodesic, punctured surface which is the double of some component Δ of B. Again, take the double $dd(L)$ of $d(L)$ along $\partial d(L) - A$. Then, the boundary $\partial dd(L)$ of $dd(L)$ consists of four components. Denote by $\partial_1 dd(L)$ and $\partial_3 dd(L)$ the components each of which is o.p.-isometric to $\Sigma(s_0)$, and by $\partial_2 dd(L)$ and $\partial_4 dd(L)$ the components each of which is o.p.-isometric to $\Sigma(\bar{s}_0)$. It is easily seen that each end of $dd(L)$ is a torus cusp. Let N be the hyperbolic 3-manifold obtained from $dd(L)$ by identifying $\partial_3 dd(L)$ with $\partial_4 dd(L)$ via an o.r.-isometry. In this way, we have obtained a connected, oriented, cusped hyperbolic 3-manifold N with totally geodesic boundary $\partial_1 dd(L) \sqcup \partial_2 dd(L)$ o.p.-isometric $\Sigma(s_0) \sqcup \Sigma(\bar{s}_0)$.



The union of shaded areas is $\hat{T}$.

Fig. 3.

To construct a manifold M satisfying the conditions of Lemma 1, we will double N two more times. The first doubling is done so that there is an isometric o.r.-involution of M. The second is a temporary doubling to show that the compactified manifold M by Dehn surgery still has a totally geodesic boundary. Let $d(N)$ be the double of N along $\partial_2 dd(L)$, and let $dd(N)$ be the double of $d(N)$ along $\partial d(N)$. The resulting manifold $dd(N)$ is a complete, connected, hyperbolic 3-manifold without boundary such that each end of $dd(N)$ is a torus cusp, and $dd(N)$ admits the isometric o.r.-involutions Φ_1, Φ_2 with $\mathrm{Fix}(\Phi_1)=\partial d(N)$, $\mathrm{Fix}(\Phi_2|_{d(N)})=\partial_2 dd(L)$. These involutions generate the isometric $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$-action on $dd(N)$. By Hyperbolic-Dehn-Surgery Theorem [**14**, THEOREM 5.9], there exists a compact hyperbolic 3-manifold M' obtained by a $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$-equivariant Dehn surgery along the torus cusps of $dd(N)$. Then, Φ_1, Φ_2 are naturally extended to involutions of M', still denoted by Φ_1, Φ_2. By Mostow's Rigidity Theorem [**12**], Φ_1, Φ_2 can be assumed to be isometric also in the new hyperbolic 3-manifold M'. This shows that $\mathrm{Fix}(\Phi_1)=\partial d(N)$ is totally geodesic in M'. Let M be the half of M' with $\partial M=\mathrm{Fix}(\Phi_1)$ and containing $d(N)$. The restriction $\Phi=\Phi_2|_M$ is an isometric o.r.-involution of M exchanging the components of ∂M. Let $\varphi:\partial M=\partial d(N)\to\Sigma\sqcup\Sigma_-$ be an o.p.-diffeomorphism with $\varphi_*([\partial d(N)])=(s_0, \bar{s}_0)\in\mathcal{T}(\Sigma\sqcup\Sigma_-)$, where Σ_- is a copy of Σ. Set $\varphi_*([\partial M])=(s', \bar{s}')$. According to the proof of [**14**, THEOREM 5.9], we can choose our Dehn surgery so that the inclusion $d(N)\subset M$ is nearly isometric except in small neighborhoods of cusps. So, we may assume that $d_\Sigma(s', s_0)<\varepsilon/2$, and so that

$$d_{\Sigma}(s, s') \leqq d_{\Sigma}(s, s_0) + d_{\Sigma}(s_0, s') < \varepsilon .$$

Thus, M is our desired manifold. □

LEMMA 2. *Suppose that F is a closed, oriented surface such that the genus of each component of F is greater than* 1. *Then, there exists a compact, connected, oriented, hyperbolic* 3-*manifold M_0 with totally geodesic boundary o.r.-homeomorphic to F.*

PROOF. Let M be any compact, connected, oriented 3-manifold such that ∂M is o.r.-homeomorphic to F. We set $\partial M = \bar{F}$. By Myers [**13**, THEOREM 6.1], M contains a knot K such that $R = M - \operatorname{int} \mathcal{N}(K)$ is a boundary-irreducible, atoroidal and acylindrical, Haken manifold, where $\mathcal{N}(K)$ is a tubular neighborhood of K in M. Consider the double $d(R)$ of R along $\bar{F}$. The manifold $d(R)$ is an atoroidal, Haken manifold admitting an o.r.-involution $\Phi : d(R) \to d(R)$ with $\mathrm{Fix}(\Phi) = \bar{F}$. Note that $\partial(d(R))$ consists of two tori. By Thurston's Uniformization Theorem, $\operatorname{int} d(R)$ has a complete hyperbolic structure of finite volume. Again, by Hyperbolic-Dehn-Surgery Theorem and Mostow's Rigidity Theorem, there exists a compact, hyperbolic 3-manifold M_0' obtained from $d(R)$ by a Φ-equivariant Dehn surgery along $\partial(d(R))$ so that $\bar{F}$ is totally geodesic in M_0'. Cut M_0' along $\bar{F}$ into two parts, and let M_0 be one of the parts which includes R. M_0 is our desired manifold. □

§ 3. Proof of Theorem.

For any $s \in \mathcal{T}(F)$, let $Q(F(s))$ be the Banach space of integrable, holomorphic, quadratic differentials $\varphi = \varphi(z) dz^2$ on $F(s)$ with the norm

$$\|\varphi\| = \int_F |\varphi(z)| dx dy ,$$

where we regard $F(s)$ as a Riemann surface conformally equivalent to the hyperbolic surface $F(s)$. Note that $Q(F(s))$ is naturally identified with the cotangent space $T_s(\mathcal{T}(F))^*$ of $\mathcal{T}(F)$ at s, see [**4**], [**10**]. For a covering $p : Y \to X$ over a closed, connected, oriented surface X of genus >1, let $p^* : \mathcal{T}(X) \to \mathcal{T}(Y)$ be the induced map so that, for any $s \in \mathcal{T}(X)$, $\tilde{s} = p^*(s)$ is the pull-backed metric on Y. As was pointed out in [**10**], the dual of the derivative dp^* of p^* at $s \in \mathcal{T}(X)$;

$$(dp^*|_s)^* : T_{\tilde{s}}(\mathcal{T}(Y))^* \longrightarrow T_s(\mathcal{T}(X))^*,$$

coincides with the Poincaré series (or the push-forward operation)

$$\Theta_{Y/X} : Q(Y(\tilde{s})) \longrightarrow Q(X(s))$$

under the identifications of $T_{\bar{s}}(\mathcal{T}(Y))^*=Q(Y(\bar{s}))$, $T_s(\mathcal{T}(X))^*=Q(X(s))$.

PROOF OF THEOREM. Let $F=\Sigma_1\sqcup\cdots\sqcup\Sigma_t$ be a closed, oriented surface with genus$(\Sigma_i)>1$ $(i=1, \cdots, t)$. Take an arbitrary element $s_F=(s_1, \cdots, s_t)$ of $\mathcal{M}(F)=\mathcal{M}(\Sigma_1)\times\cdots\times\mathcal{M}(\Sigma_t)$. For convenience, fix markings on $\Sigma_1, \cdots, \Sigma_t$ and regard s_F as an element of the Teichmüller space $\mathcal{T}(F)=\mathcal{T}(\Sigma_1)\times\cdots\times\mathcal{T}(\Sigma_t)$. Similarly, $\bar{s}_F=(\bar{s}_1, \cdots, \bar{s}_t)$ can be regarded as an element of $\mathcal{T}(F_-)=\mathcal{T}(\Sigma_{1,-})\times\cdots\times\mathcal{T}(\Sigma_{t,-})$, where each $\Sigma_{i,-}$ is a copy of Σ_i and $F_-=\Sigma_{1,-}\sqcup\cdots\sqcup\Sigma_{t,-}$. By Lemma 1, for any $\varepsilon>0$, there exist compact, connected, oriented, hyperbolic 3-manifolds M_i $(i=1, \cdots, t)$ with totally geodesic, two-component boundary o.p.-homeomorphic to $\Sigma_i\sqcup\Sigma_{i,-}$ and with $d_{\Sigma_i}(s_i, s_i')<\varepsilon$, $d_{\Sigma_{i,-}}(\bar{s}_i, \bar{s}_i')<\varepsilon$, where $(s_i', \bar{s}_i')=[\partial M_i]\in\mathcal{T}(\Sigma_i\sqcup\Sigma_{i,-})=\mathcal{T}(\Sigma_i)\times\mathcal{T}(\Sigma_{i,-})$ under a suitable identification ∂M_i with $\Sigma_i\sqcup\Sigma_{i,-}$. This implies that

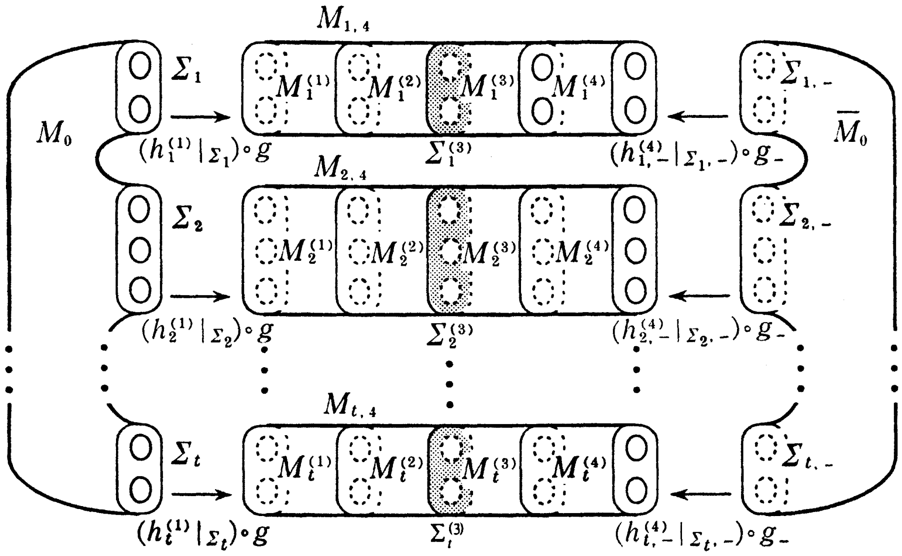
$$d_F(s_F, s_F')<\varepsilon,\quad d_{F_-}(\bar{s}_F, \bar{s}_F')<\varepsilon, \tag{3.1}$$

for $s_F'=(s_1', \cdots, s_t')\in\mathcal{T}(F)$, $\bar{s}_F'=(\bar{s}_1', \cdots, \bar{s}_t')\in\mathcal{T}(F_-)$. Let Φ_i be the isometric o.r.-involution of M_i given in Lemma 1 exchanging Σ_i with $\Sigma_{i,-}$. For any $n\in\boldsymbol{N}$, let $M_i^{(1)}, \cdots, M_i^{(2n)}$ be $2n$ copies of M_i with identification maps $h_i^{(j)}: M_i\to M_i^{(j)}$ $(j=1, \cdots, 2n)$. Consider the hyperbolic 3-manifold $M_{i,2n}$ obtained from $M_i^{(1)}, \cdots, M_i^{(2n)}$ by connecting $M_i^{(j)}$ with $M_i^{(j+1)}$ via the o.r.-isometry $h_i^{(j+1)}\circ\Phi_i\circ(h_i^{(j)})^{-1}|_{\Sigma_{i,-}^{(j)}}: \Sigma_{i,-}^{(j)}\to\Sigma_i^{(j+1)}$ $(j=1, 2, \cdots, 2n-1)$, where $\Sigma_i^{(j)}, \Sigma_{i,-}^{(j)}$ are the components of $\partial M_i^{(j)}$ corresponding to Σ_i, $\Sigma_{i,-}$ of ∂M_i. Note that $M_{i,2n}$ admits the o.r.-isometric involution $\Phi_{i,2n}$ exchanging the two components $\Sigma_i^{(1)}, \Sigma_{i,-}^{(2n)}$ of $\partial M_{i,2n}$ and with $\mathrm{Fix}(\Phi_{i,2n})=\Sigma_{i,-}^{(n)}=\Sigma_i^{(n+1)}$.

For any $j=1, \cdots, n$, consider the compact submanifold $M_{i,2j}$ of $M_{i,2n}$ with $\partial M_{i,2j}=\Sigma_i^{(n-j+1)}\sqcup\Sigma_{i,-}^{(n+j)}$. From now on, we identify $\partial M_{i,2j}$ with $\Sigma_i\sqcup\Sigma_{i,-}$ via the o.p.-isometries $h_i^{(n-j+1)}|_{\Sigma_i}: \Sigma_i\to\Sigma_i^{(n-j+1)}$, $h_i^{(n+j)}|_{\Sigma_{i,-}}: \Sigma_{i,-}\to\Sigma_i^{(n+j)}$. Then, $[\partial M_{i,2j}]\in\mathcal{T}(\partial M_{i,2j})$ coincides with $(s_i', \bar{s}_i')\in\mathcal{T}(\Sigma_i\sqcup\Sigma_{i,-})$. Suppose that

$$\beta_{i,2j}=\mathrm{conf}: QH_0(M_{i,2j})\longrightarrow\mathcal{T}(\Sigma_i\sqcup\Sigma_{i,-})$$

is the homeomorphism for $M_{i,2j}$ given in (1.1). For $j=1, \cdots, n$, let $N_{i,2j}=\boldsymbol{H}^3/\Gamma_{i,2j}$ be the hyperbolic 3-manifold containing $M_{i,2j}$ as a convex core. Since $\partial M_{i,2j}=\Sigma_i\sqcup\Sigma_{i,-}$ is totally geodesic in $N_{i,2j}$, the subgroups $\pi_1(\Sigma_i)$, $\pi_1(\Sigma_{i,-})$ of $\Gamma_{i,2j}$ are Fuchsian. This implies that $\beta_{i,2j}([N_{i,2j}])=(s_i', \bar{s}_i')$. By Lemma 2, there exists a compact, connected, oriented, hyperbolic 3-manifold M_0 with totally geodesic boundary admitting an o.r.-homeomorphism $g: \partial M_0\to F$. Fixing an o.r.-isometry $\alpha: M_0\to\overline{M}_0$, the o.r.-homeomorphism $g_-: \partial\overline{M}_0\to F_-$ is given by $\Phi_i\circ g\circ\alpha^{-1}(x)$ if $x\in\alpha(g^{-1}(\Sigma_i))$. Set $W_0=M_0\sqcup\overline{M}_0$, $Y_{2n}=M_{1,2n}\sqcup\cdots\sqcup M_{t,2n}$ and define the o.r.-homeomorphism $\gamma_{2n}: \partial W_0\to\partial Y_{2n}=F\sqcup F_-$ by $\gamma_{2n}(x)=(h_i^{(1)}|_{\Sigma_i})\circ g(x)$ if $x\in\partial M_0$, $g(x)\in\Sigma_i$, and $\gamma_{2n}(x)=(h_i^{(2n)}|_{\Sigma_{i,-}})\circ g_-(x)$ if $x\in\partial\overline{M}_0$, $g_-(x)\in\Sigma_{i,-}$, see Figure 4.



The case of $n=2$

Fig. 4.

Since $[\partial M_{i,2j}]=(s_i', \bar{s}_i')$ for any $i\in\{1, \cdots, t\}$,

$$[\partial Y_{2j}] = (s_1', \cdots, s_t', \bar{s}_1', \cdots, \bar{s}_t') = (s_F', \bar{s}_F')\in\mathcal{T}(F\sqcup F_-). \tag{3.2}$$

Since Y_{2n} and W_0 are atoroidal, acylindrical and Haken, $Y_{2n}\cup_{\gamma_{2n}}W_0$ is a closed, atoroidal, Haken manifold. Then, by Thurston's Uniformization Theorem, it has a hyperbolic structure. The o.r.-homeomorphisms α, α^{-1}, $\Phi_{1,2n}$, $\cdots$, $\Phi_{t,2n}$ determine the involution Φ on $Y_{2n}\cup_{\gamma_{2n}}W_0$ with $\mathrm{Fix}(\Phi)=F^{(n+1)}=\Sigma_1^{(n+1)}\sqcup\cdots\sqcup\Sigma_t^{(n+1)}$. By Mostow's Rigidity Theorem, we may assume that $F^{(n+1)}$ is totally geodesic also in the new hyperbolic 3-manifold $Y_{2n}\cup_{\gamma_{2n}}W_0$. The o.r.-involution τ_{2n} : $\partial(Y_{2n}\sqcup W_0)\to\partial(Y_{2n}\sqcup W_0)$ determined by $\gamma_{2n}:\partial W_0\to F\sqcup F_-$ and $(\gamma_{2n})^{-1}: F\sqcup F_-\to\partial W_0$ induces an isometry

$$(\tau_{2n})_*:\mathcal{T}(\bar{F}\sqcup\bar{F}_-)\times\mathcal{T}(\overline{\partial W_0})\longrightarrow\mathcal{T}(F\sqcup F_-)\times\mathcal{T}(\partial W_0).$$

Under our identification of $F^{(1)}=F$, $F_-^{(2n)}=F_-$, we have $\gamma_{2n}(x)=g(x)$ if $x\in\partial M_0$ and $\gamma_{2n}(x)=g_-(x)$ if $x\in\partial\overline{M}_0$. Thus, $(\tau_{2n})_*$ is independent of n. We set $[\partial W_0]=s_W\in\mathcal{T}(\partial W_0)$. Since the topological type of Y_{2n} depends on n, the skinning map

$$\sigma_{2n}:\mathcal{T}(F\sqcup F_-)\times\mathcal{T}(\partial W_0)\longrightarrow\mathcal{T}(\bar{F}\sqcup\bar{F}_-)\times\mathcal{T}(\overline{\partial W_0})$$

also does. However, for the $s'=(s_F', \bar{s}_F', s_W)\in\mathcal{T}(F\sqcup F_-)\times\mathcal{T}(\partial W_0)$, $\sigma_{2n}(s')$ is independent of n. In fact, each component X of $\partial M_{i,2n}=\Sigma_i\sqcup\Sigma_{i,-}$ is totally geodesic in $N_{i,2n}$, the covering of $N_{i,2n}$ corresponding to $\pi_1(X)\subset\Gamma_{i,2n}$ is determined only by the hyperbolic structure on X and independent of the topological type of $M_{i,2n}$. Since the similar fact holds on each component of ∂W_0, we have the desired independence. In particular, the Teichmüller distance $d(s', (\tau_{2n})_* \circ \sigma_{2n}(s'))=L$ in $\mathcal{T}(F \sqcup F_-) \times \mathcal{T}(\partial W_0)$ is independent of n. According to McMullen [**10**], the solution $s''_{2n} \in \mathcal{T}(F \sqcup F_-) \times \mathcal{T}(\partial W_0)$ to the gluing problem for (Y_{2n}, W_0) is contained in $B_{F \sqcup F_- \sqcup \partial W_0}(s', L/(1-c_0))$, where c_0, $0<c_0<1$, is the constant depending only on the topological type of $F \sqcup F_- \sqcup \partial W_0$ and hence independent of n. The $(F \sqcup F_-)$-entry $(s''^{(1)}, \bar{s}''^{(2n)}) \in \mathcal{T}(F \sqcup F_-)$ of s''_{2n} is contained in $B_{F \sqcup F_-}((s'_F, \bar{s}'_F), L/(1-c_0))$. Let $p_{i,2j}: N_{i,2j-2} \to N_{i,2j}$ $(j=2, \cdots, n)$ (resp. $p_{i,2}: N_{i,0} \to N_{i,2}$) be the covering associated to $\pi_1(M_{i,2j-2}) \subset \pi_1(M_{i,2j})=\pi_1(N_{i,2j})$ (resp. $\pi_1(\Sigma_i^{(n+1)}) \subset \pi_1(M_{i,2})=\pi_1(N_{i,2})$). Each $p_{i,2j}$ induces the pull-back $\delta_{i,2j}: QH_0(M_{i,2j}) \to QH_0(M_{i,2j-2})$, where $M_{i,0}=\Sigma_i^{(n+1)} \times [0, 1]$. Consider the map

$$\eta_{2j}: \mathcal{T}(F \sqcup F_-) \longrightarrow \mathcal{T}(F \sqcup F_-)$$

defined by

$$\eta_{2j}|_{\mathcal{T}(\Sigma_i \sqcup \Sigma_{i,-})}: \mathcal{T}(\Sigma_i \sqcup \Sigma_{i,-}) \xrightarrow{(\beta_{i,2j})^{-1}} QH_0(M_{i,2j}) \xrightarrow{\delta_{i,2j}}$$
$$QH_0(M_{i,2j-2}) \xrightarrow{\beta_{i,2j-2}} \mathcal{T}(\Sigma_i \sqcup \Sigma_{i,-}).$$

By (3.2), for any $j \in \{1, \cdots, n\}$, we have $\eta_{2j}(s'_F, \bar{s}'_F)=(s'_F, \bar{s}'_F)$. We set inductively

$$\eta_{2n}(s''^{(1)}, \bar{s}''^{(2n)}) = (s''^{(2)}, \bar{s}''^{(2n-1)}), \ \eta_{2n-2}(s''^{(2)}, \bar{s}''^{(2n-1)}) = (s''^{(3)}, \bar{s}''^{(2n-2)}), \ \cdots,$$
$$\eta_2(s''^{(n)}, \bar{s}''^{(n+1)}) = (s''^{(n+1)}, \bar{s}''^{(n)}).$$

Let $O_{i,2n}=(\boldsymbol{H}^3 \cup \Omega(\Gamma_{i,2n}))/\Gamma_{i,2n}$ be the Kleinian manifold, and let $q_{i,2n}: \tilde{O}_{i,2n} \to O_{i,2n}$ be the covering associated to $\pi_1(M_{i,2n-2}) \subset \pi_1(O_{i,2n})$. Note that $N_{i,2n-2} \subset \tilde{O}_{i,2n} \subset O_{i,2n-2}$, $q_{i,2n}|_{N_{i,2n-2}}=p_{i,2n}$ and $\partial(\tilde{O}_{i,2n})$ is a full-measure, open subset of $\partial(O_{i,2n-2})$ such that each component U of $\partial(\tilde{O}_{i,2n})$, called a *spot* by McMullen [**10**], is homeomorphic to an open disk.

It is easily seen that McMullen's argument [**10**] for skinning maps is applicable also to η_{2n}. We will review that briefly. The dual of the derivative $d\eta_{2n}$ of η_{2n} at $v \in \mathcal{T}(F \sqcup F_-)$ is given by

$$(d\eta_{2n}|_v)^* = \sum_U \Theta_{U/X}: Q(F(\hat{v}) \sqcup F_-(\hat{v})) \longrightarrow Q(F(v) \sqcup F_-(v)),$$

where U ranges over all spots in $\partial(\tilde{O}_{1,2n}) \sqcup \cdots \sqcup \partial(\tilde{O}_{t,2n})$, $X=q_{i,2n}(U) \subset \partial(O_{i,2n})$ and $\hat{v}=\eta_{2n}(v)$. Here, we set $\Theta_{U/X}(\varphi)=\Theta_{U/X}(\varphi|_U)$ for $\varphi \in Q(F(\hat{v}) \sqcup F_-(\hat{v}))$. By [**9**, Theorem 10.3], there exists a continuous map $c: \mathcal{M}(X) \to \boldsymbol{R}$ with $\|\Theta_{U/X}\| \leqq c([X])<1$. Since $B_{F \sqcup F_-}((s'_F, \bar{s}'_F), L/(1-c_0))$ is compact, there exists a positive constant $c_1<1$, depending only on $(s'_F, \bar{s}'_F)$ and $L/(1-c_0)$, such that, for any $v \in B_{F \sqcup F_-}((s'_F, \bar{s}'_F), L/(1-c_0))$ and all spots U, $\|\Theta_{U/X}\| \leqq c_1$. Thus, we have

$$\|d\eta_{2n}|_v\| = \|(d\eta_{2n}|_v)^*\| \leqq \sup_U \|\Theta_{U/X}\| \leqq c_1. \tag{3.3}$$

Now, since $\eta_{2n}(s'_F, \bar{s}'_F)=(s'_F, \bar{s}'_F)$, the inequality (3.3) implies that

$$\eta_{2n}\Big(B_{F\sqcup F_-}\Big((s'_F, \bar{s}'_F), \frac{L}{1-c_0}\Big)\Big) \subset B_{F\sqcup F_-}\Big((s'_F, \bar{s}'_F), \frac{Lc_1}{1-c_0}\Big).$$

Since $B_{F\sqcup F_-}((s'_F, \bar{s}'_F), Lc_1/(1-c_0)) \subset B_{F\sqcup F_-}((s'_F, \bar{s}'_F), L/(1-c_0))$, the same constant c_1 works for

$$\eta_{2n-2}\colon \mathcal{T}(F\sqcup F_-) \longrightarrow \mathcal{T}(F\sqcup F_-)$$

in $B_{F\sqcup F_-}((s'_F, \bar{s}'_F), Lc_1/(1-c_0))$. This shows that

$$\eta_{2n-2}\Big(B_{F\sqcup F_-}\Big((s'_F, \bar{s}'_F), \frac{Lc_1}{1-c_0}\Big)\Big) \subset B_{F\sqcup F_-}\Big((s'_F, \bar{s}'_F), \frac{Lc_1^2}{1-c_0}\Big).$$

Since $(s''^{(1)}, \bar{s}''^{(2n)}) \in B_{F\sqcup F_-}((s'_F, \bar{s}'_F), L/(1-c_0))$, by repeating the same process n times, we have

$$(s''^{(n+1)}, \bar{s}''^{(n)}) \in B_{F\sqcup F_-}\Big((s'_F, \bar{s}'_F), \frac{Lc_1^n}{1-c_0}\Big). \tag{3.4}$$

Let Z_{2n} be the half of $Y_{2n}\cup_{\gamma_{2n}} W_0$ with $\partial Z_{2n}=F^{(n+1)}$ and $Z_{2n}\supset \bar{M}_0$. Since $\partial Z_{2n}=F^{(n+1)}$ is totally geodesic in $Y_{2n}\cup_{\gamma_{2n}} W_0$, we have $[\partial Z_{2n}]=s''^{(n+1)}$. Since Z_{2n} is a compact, connected, oriented, hyperbolic 3-manifold with totally geodesic boundary, ∂Z_{2n} represents an element of $\mathcal{R}(F)$. If we choose $n\in \boldsymbol{N}$ so large that $Lc_1^n/(1-c_0)<\varepsilon$, then by (3.1) and (3.4),

$$d_F(s_F, [\partial Z_{2n}]) \leqq d_F(s_F, s'_F)+d_F(s'_F, s''^{(n+1)}) < 2\varepsilon.$$

Thus, $\mathcal{R}(F)$ is dense in $\mathcal{M}(F)$. □

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