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A generating function of strict Gelfand patterns and some formulas on characters of general linear groups

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Introduction.

Gelfand patterns and strict Gelfand patterns are triangular arrays of non-negative integers satisfying certain conditions. I. M. Gelfand and M. L. Zetlin used Gelfand patterns for the parametrization of the weight vectors of representation spaces of general linear groups [4].

Since then, many mathematicians and physicists have used them in order to study representations of the classical groups and corresponding particles. Still, strict Gelfand patterns have mainly been studied from the combinatorial point of view. However, R. P. Stanley has demonstrated an interesting relation between a generating function of strict Gelfand patterns and the 'most singular' values of the Hall-Littlewood polynomials [14], [10].

Since the Hall-Littlewood polynomial is a fundamental tool for investigating the representations of general linear groups over finite fields and local fields, there must be a strong connection between the Gelfand-Zetlin parametrization and the formula of Stanley.

The initial motivation of this paper was to find a natural deformation of Stanley's formula so that it involves the Gelfand-Zetlin parametrization as a specialization. In the course this searching, we encountered a more important formula, Weyl's character formula, as another specialization of our formula. The following is Weyl's character formula for $GL(n, \boldsymbol{C})$.

$$S_{\lambda}(z_1, z_2, \cdots, z_{n-1}, z_n)=\frac{V_{\lambda+\delta}}{\prod_{j>i}(z_i-z_j)}. \tag{3.2.1*}$$

In the above formula, δ is "the half of sum of positive roots", and V_{β} denotes the "Vandermonde-type determinant" of type β.

The left side of (3.2.1*) is called the Schur function associated with the highest weight λ, which is the character of an irreducible representation of $GL(n, \boldsymbol{C})$. Its actual definition will be shown in Section 2. We often denote

U_α for S_λ when $\alpha=\lambda+\delta$.

Our formula is the following:

THEOREM 2.1.

$$J(\alpha;t;z_1,z_2,\cdots,z_{n-1},z_n)=\{\prod_{j>i}(z_i+z_jt)\}\ U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n)$$

where J is the generating function of strict Gelfand patterns defined in 1.3, *and the product is taken over all pairs (i, j) satisfying $n\geqq j>i\geqq 1$.*

From the specializations substituting 0, 1, -1 for the parameter t in our theorem, we get three classical formulas mentioned above: Gelfand's parametrization, Stanley's formula, and Weyl's character formula.

Besides, as a corollary, we have a neat generalization of Weyl's denominator formula for general linear groups.

COROLLARY 3.4.-(*).

$$\prod_{\alpha\in\Delta_+}(1+te^{\alpha})=\sum_{X\in Alt_n}\left(1+\frac{1}{t}\right)^{s(X)}t^{i(X)}e^{\delta-X\delta}.$$

Δ_+ is the positive root system of type A_{n-1} (cf. [2]), *Alt_n is the set of all alternating sign matrices (see Definition* 3.4.3) *of size n, $s(X)$ and $i(X)$ are the "number of special elements" and "number of inversions" of X respectively, and δ is the half sum of all the positive roots of the root system of type A_{n-1}.*

One of the motivations of this article is the work of Mills, Robbins, and Rumsey Jr. [9]. The notations required in the corollary above had been given there.

1. Combinatorial notations.

A partition is a weakly decreasing finite sequence of nonnegative integers. A distinct partition is a strictly decreasing sequence of nonnegative integers (its last entry may be a zero).

For a partition (or distinct partition) $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$, $|\lambda|=\sum_{i=1}^n\lambda_i$.

DEFINITION 1.1. A Gelfand pattern $T(a_{i,j})$ of size n is a triangular array

$$\begin{array}{ccccccc}
a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\
& a_{22} & a_{23} & \cdots\cdots & a_{2n} \\
& & a_{33} & a_{34} \cdots & a_{3n} \\
& & \cdots & & \\
& & & \cdots & \\
& & & & a_{nn}
\end{array}$$

of nonnegative integers which is weakly decreasing in each row and satisfying

$$a_{i-1,j-1} \geqq a_{i,j} \geqq a_{i-1,j} \qquad \text{for any indices } i \text{ and } j\,.$$

For a partition λ, $G(\lambda)$ is the set of all Gelfand patterns having λ as their top rows.

A Gelfand pattern is called strict if each row of it is a strictly decreasing sequence. For a distinct partition α, $SG(\alpha)$ is the set of all strict Gelfand patterns having α as their top rows.

$$SG((3,2,1)) = \left\{\begin{matrix}3\,2\,1\\2\,1\\1\end{matrix}\quad\begin{matrix}3\,2\,1\\2\,1\\2\end{matrix}\quad\begin{matrix}3\,2\,1\\3\,1\\1\end{matrix}\quad\begin{matrix}3\,2\,1\\3\,1\\2\end{matrix}\quad\begin{matrix}3\,2\,1\\3\,1\\3\end{matrix}\quad\begin{matrix}3\,2\,1\\3\,2\\2\end{matrix}\quad\begin{matrix}3\,2\,1\\3\,2\\3\end{matrix}\right\}$$

Figure 1. Example of strict Gelfand patterns: This is the set of all strict Gelfand patterns having (3,2,1) as their top rows.

An entry $a_{i,j}$ of a strict Gelfand pattern T is called "special" if

$$2 \leqq i \leqq j \qquad \text{and} \qquad a_{i-1,j-1} > a_{i,j} > a_{i-1,j}\,.$$

An entry $a_{i,j}$ of a strict Gelfand pattern T is called "lefty" if

$$2 \leqq i \leqq j \qquad \text{and} \qquad a_{i,j} = a_{i-1,j-1}\,.$$

$s(T)$ (resp. $l(T)$) is the number of special (resp. lefty) entries in the strict Gelfand pattern T.

$d_i(T)$ denotes i-th row sum, the summation of the entries in the i-th row of T. We set

$$m_i(T) = d_i(T) - d_{i+1}(T) \quad \textit{for } i=1, 2, \cdots, n-1 \quad \textit{and} \quad m_n(T) = d_n(T)\,.$$

EXAMPLE 1. If $T = \begin{matrix}3\ 1\\2\end{matrix}$, $s(T)=1$, $l(T)=0$, and $(m_1(T), m_2(T))=(2, 2)$. If $T = \begin{matrix}3\ 1\\3\end{matrix}$, $s(T)=0$, $l(T)=1$ and $(m_1(T), m_2(T))=(1, 3)$.

DEFINITION 1.2. Let α be a distinct partition of length n. Then we define the generating function H of $SG(\alpha)$ by

$$H(\alpha\,;x\,;y\,;z_1, z_2, \cdots, z_{n-1}, z_n) \overset{\text{def}}{=} \sum_{T\in SG(\alpha)} x^{s(T)} y^{l(T)} Z_n^{M(T)}\,.$$

In the above, $Z_n^{M(T)} = \prod_{i=1}^{n}(z_i^{m_i(T)})$.

We shall study it in the special case of $x=y+1$.

DEFINITION 1.3. $J(\alpha\,;t\,;z_1, z_2, \cdots, z_{n-1}, z_n) \overset{\text{def}}{=} H(\alpha\,;t+1\,;t\,;z_1, z_2, \cdots, z_{n-1}, z_n)$.

NOTE. $H(\alpha)$ and $J(\alpha)$ should be regarded as generalizations of the more simple generating function $f_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n)=\sum_{T\in SG(\alpha)} Z_n^{M(T)}$.

EXAMPLE 2. (1) $H((2, 0); x; y; z_1, z_2) = z_1^2+xz_1z_2+yz_2^2$, and

$$J((2, 0); t; z_1, z_2) = (z_1+z_2t)(z_1+z_2).$$

(2) $H((3, 2, 1); x; y; z_1, z_2, z_3) = z_1^3z_2^2z_3+yz_1^3z_2z_3^2+yz_1^2z_2^3z_3+y^2z_1^2z_2z_3^3+y^2z_1z_2^3z_3^2$
$+y^3z_1z_2^2z_3^3+xyz_1^2z_2^2z_3^2$, and

$$J((3, 2, 1); t; z_1, z_2, z_3) = (z_1+z_2t)(z_1+z_3t)(z_2+z_3t)z_1z_2z_3\,.$$

2. Schur functions and theorem.

Let us recall some basic notations in the representation theory of general linear groups. $GL(n, \boldsymbol{C})$ is the general linear group of rank n over the complex number field $\boldsymbol{C}$. The equivalence classes of irreducible finite dimensional polynomial representations of $GL(n, \boldsymbol{C})$ are usually parametrized by means of partitions of length n. Here 'length' of a partition λ means the length of the sequence λ regarded as a sequence of nonnegative integers. More precisely, there is a canonical one-to-one correspondence between the set of partitions of length n and the set of equivalence classes of irreducible (finite dimensional polynomial) representations of $GL(n, \boldsymbol{C})$. Refer [**12**] and [**13**] for details.

We write ρ_λ for the irreducible representation of $GL(n, \boldsymbol{C})$ corresponding to the partition λ of length n. The character S_λ of ρ_λ is called the Schur function associated with λ. It is regarded as a symmetric polynomial in n variables defined as

$$S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n) = \mathrm{Tr}(\rho_\lambda(\mathrm{D}Z_n))$$

where Tr is the usual trace symbol and $\mathrm{D}Z_n=\mathrm{diag}(Z_n)$ denotes the diagonal n by n matrix with z_i as its each (i, i) component. Note that since the polynomial representation ρ_λ is extended uniquely into a polynomial map on $M_{n,n}(\boldsymbol{C})$, S_λ is well defined even if some z_i are zero.

As is well known, we can write S_λ as a quotient of a Vandermonde-type determinant of rank n by the Vandermonde determinant of rank n. See Section 3, (3.2.1) and (3.2.1*).

For a partition λ of length n, we correspond a distinct partition $\lambda+\delta$ of length n defined by $(\lambda+\delta)_i=\lambda_i+n-i$.

NOTE. This δ corresponds to the "*half sum of the positive roots*" of the root system of type A_{n-1}.

Our theorem is the following:

THEOREM 2.1. *Let λ be a partition of length n and let $\alpha=\lambda+\delta$. Then the following formula* (2.1) *holds.*

$$(2.1)\qquad J(\alpha;t;z_1,z_2,\cdots,z_{n-1},z_n)=\left\{\prod_{j>i}(z_i+z_jt)\right\}S_\lambda(z_1,z_2,\cdots,z_{n-1},z_n)$$

where the product is taken over all pairs (i,j) satisfying $n\geqq j>i\geqq 1$.

To prove this theorem, we shall recall some branching rules of representations of general linear groups. We refer to [13] for the proof of the following lemma 2.2 and lemma 2.4.

$$L_n=\{g=(g_{i,j})_{i,j=1,2,\cdots,n}\in GL(n,\boldsymbol{C})\text{ satisfying } g_{n,j}=g_{j,n}=0 \text{ for any } j=1,2,\cdots,n-1\}$$

is a Levi subgroup of $GL(n,\boldsymbol{C})$, which is isomorphic to the direct product $GL(n-1,\boldsymbol{C})\times GL(1,\boldsymbol{C})$. First we shall recall the branching rule of the restricted representation of the irreducible representation ρ_λ of $GL(n,\boldsymbol{C})$ to L_n.

LEMMA 2.2 ([13], Section 66).

$$(2.2)\qquad \rho_\lambda|_{L_n}\cong\sum_{(a)}\rho_\mu\times e^{|\lambda|-|\mu|}$$

where ρ_μ is the irreducible representation of $GL(n-1,\boldsymbol{C})$ corresponding to the partition $\mu=(\mu_1,\mu_2,\cdots,\mu_{n-1})$, and e^q denotes the linear character of $\boldsymbol{C}^\times=GL(1,\boldsymbol{C})$ defined by $e^q(z)=z^q$ for any element z of $\boldsymbol{C}^\times$. The summation is taken over all partitions μ of length $n-1$ satisfying the condition

$$(\mathrm{a})\qquad \lambda_i\geqq\mu_i\geqq\lambda_{i+1}\quad \textit{for any } i=1,2,\cdots,n-1.$$

We shall write the identity 2.2 in terms of Schur functions. For convenience' sake, we shall write U_α for the Schur function S_λ where $\alpha=\lambda+\delta$. Then 2.2 becomes the equation 2.3 below.

$$(2.3)\qquad U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n)=\sum_{\alpha_i>\beta_i\geqq\alpha_{i+1}}U_\beta(z_1,z_2,\cdots,z_{n-1})\times z_n^{|\alpha|-|\beta|-n+1},$$

where the summation runs over all distinct partitions β of length $n-1$ satisfying the indicated condition $\alpha_i>\beta_i\geqq\alpha_{i+1}$.

Now we introduce the k-th exterior tensor product representation $\Lambda_{(k,m)}$ of $GL(m,\boldsymbol{C})$. For any element g of $GL(m,\boldsymbol{C})$, $\Lambda_{(k,m)}(g)$ is an automorphism of the k-th exterior tensor space $\overset{k}{\Lambda}\boldsymbol{C}^m$ of $\boldsymbol{C}^m$ defined by

$$\Lambda_{(k,m)}(g)(v_1\wedge v_2\wedge\cdots\wedge v_k)=g(v_1)\wedge g(v_2)\wedge\cdots\wedge g(v_k)$$

for any element $V_k=(v_1\wedge v_2\wedge\cdots\wedge v_k)$ of $\overset{k}{\Lambda}\boldsymbol{C}^m$.

In the notation using partitions, $\Lambda_{(k,m)}\cong\rho_{(1^k)}=\rho_{(\underbrace{1,1,\cdots,1}_{k})}$.

Let us recall the branching rule of the tensor product representation of $\Lambda_{(k,m)}$ and an arbitrary representation of $GL(m, \boldsymbol{C})$.

LEMMA 2.4 ([**13**], Section 79). *Let ρ_μ be an irreducible representation of $GL(m, \boldsymbol{C})$. Then,*

$$\rho_\mu \otimes \Lambda_{(k,m)} \cong \sum_{(b1),(b2)} \rho_\nu \tag{2.4}$$

where $\otimes$ is the inner tensor product symbol among the representations of $GL(m, \boldsymbol{C})$, and the summation runs over all partitions ν of length m satisfying both

$$|\nu| = |\mu| + k\,, \quad \text{and} \tag{b1}$$

$$\mu_i + 1 \geqq \nu_i \geqq \mu_i \quad \text{for any } i=1, 2, \cdots, m\,. \tag{b2}$$

Taking the character value at $\mathrm{D}Z_m$ of both sides of (2.4), we have

$$S_\mu(z_1, z_2, \cdots, z_m) \times \mathrm{Tr}\{\Lambda_{(k,m)}(\mathrm{D}Z_m)\} = \sum_{(b1),(b2)} S_\nu(z_1, z_2, \cdots, z_m)\,.$$

In another expression, using $\beta=\mu+\delta$ and $\gamma=\nu+\delta$ instead of μ and ν themselves,

$$U_\beta(z_1, z_2, \cdots, z_m) \times \mathrm{Tr}\{\Lambda_{(k,m)}(\mathrm{D}Z_m)\} = \sum_{\beta_i+1 \geqq \gamma_i \geqq \beta_i} U_\gamma(z_1, z_2, \cdots, z_m) \tag{2.4*}$$

where β and γ are distinct partitions of length m satisfying

$$|\gamma| = |\beta| + k\,. \tag{b1*}$$

Now we begin the proof of Theorem 2.1. Let us keep our eye upon the identity (2.1). Since both sides of (2.1) are polynomials, it suffices to show it under the condition (2.5) below.

(2.5) *t is a positive real number and z_i are real numbers which are larger than t for all $i=1, 2, \cdots, n$.*

From now on, we shall assume the condition (2.5).

Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\beta=(\beta_1, \beta_2, \cdots, \beta_{n-1})$ be distinct partitions. We say $\beta \leftarrow \alpha$ if $\alpha_i \geqq \beta_i \geqq \alpha_{i+1}$ for any $i=1, 2, \cdots, n-1$. When $\beta \leftarrow \alpha$, we set $s(\alpha, \beta)$ and $l(\alpha, \beta)$ to denote the numbers of indices i such that $\alpha_i > \beta_i > \alpha_{i+1}$ and $\beta_i = \alpha_i$ respectively. We shall use

$$\tilde{H}(\alpha\,;x\,;y\,;z_1, z_2, \cdots, z_{n-1}, z_n) \overset{\mathrm{def}}{=} H(\alpha\,;x\,;y\,;z_n, z_{n-1}, \cdots, z_2, z_1)$$

and

$$\tilde{J}(\alpha\,;t\,;z_1, z_2, \cdots, z_{n-1}, z_n) \overset{\mathrm{def}}{=} \tilde{H}(\alpha\,;t+1\,;t\,;z_1, z_2, \cdots, z_{n-1}, z_n)$$
$$= J(\alpha\,;t\,;z_n, z_{n-1}, \cdots, z_2, z_1)$$

instead of H and J themselves in order to shorten the expressions occurring in our proof.

The following formula is easily seen from the definition of the generating function of the strict Gelfand patterns.

$$\widetilde{H}(\alpha; x; y; z_1, z_2, \cdots, z_{n-1}, z_n)=\sum_{\beta\leftarrow\alpha}\widetilde{H}(\beta; x; y; z_1, z_2, \cdots, z_{n-1}) \times z_n^{|\alpha|-|\beta|}x^{s(\alpha,\beta)}y^{l(\alpha,\beta)}. \tag{2.6}$$

Fixing t and $(z_1, z_2, \cdots, z_{n-1}, z_n)$ satisfying (2.5). We can define a real number $p_m=\log(z_{m+1}{}^{-1}t)/\log(\prod_{j=1}^{m}z_j)$ and a representation Ψ_m of $GL(m, \boldsymbol{C})$ given as

$$\Psi_m(g)=\sum_{k=0}^{m}\{|\det(g)|^{kp_m}\times\Lambda_{(k,m)}(g)\} \qquad \text{where } g\in GL(m, \boldsymbol{C})$$

for each $m=1, 2, \cdots, n-1$.

Let us calculate the product of $\mathrm{Tr}\{\Psi_{n-1}(\mathrm{D}Z_{n-1})\}$, the character value of the representation Ψ_{n-1} at the diagonal element $\mathrm{D}Z_{n-1}$ of $GL(n-1, \boldsymbol{C})$, and the Schur function $U_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n)$ using the identities (2.3) and (2.4*).

We set

$$(L)=U_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n)\times\mathrm{Tr}\{\Psi_{n-1}(\mathrm{D}Z_{n-1})\}.$$

Applying (2.3),

$$(L)=\left\{\sum_{\alpha_i>\beta_i\geqq\alpha_{i+1}}U_\beta(z_1, z_2, \cdots, z_{n-1})\times z_n{}^{|\alpha|-|\beta|-n+1}\right\}\times\mathrm{Tr}\{\Psi_{n-1}(\mathrm{D}Z_{n-1})\}.$$

On the other hand, from (2.4*),

$$U_\beta(z_1, z_2, \cdots, z_{n-1})\times\mathrm{Tr}\{\Psi_{n-1}(\mathrm{D}Z_{n-1})\} =\sum_{\beta_i+1\geqq\gamma_i\geqq\beta_i}\left[\{U_\gamma(z_1, z_2, \cdots, z_{n-1})\}\times\left(\prod_{j=1}^{n-1}z_j\right)^{(|\gamma|-|\beta|)p_{n-1}}\right].$$

Then

$$(L)=\sum_{(c1)}\left[\sum_{(c2)}\left\{z_n{}^{|\alpha|-|\beta|-n+1}\times\left(\prod_{j=1}^{n-1}z_j\right)^{(|\gamma|-|\beta|)p_{n-1}}\times U_\gamma(z_1, z_2, \cdots, z_{n-1})\right\}\right]$$

$$=\sum_{(c1)}\left[\sum_{(c2)}\{z_n{}^{|\alpha|-|\beta|-n+1}\times(z_n{}^{-1}t)^{|\gamma|-|\beta|}\times U_\gamma(z_1, z_2, \cdots, z_{n-1})\}\right].$$

The summations run over the distinct partitions β and γ of length $n-1$ satisfying the indicated conditions

(c1) $\qquad \alpha_i>\beta_i\geqq\alpha_{i+1}$ and

(c2) $\qquad \beta_i+1\geqq\gamma_i\geqq\beta_i$ for any $i=1, 2, \cdots, n-1$.

It follows (c1) and (c2) that $\gamma\leftarrow\alpha$. Calculating the number of β satisfying both (c1) and (c2) for a fixed γ using the binomial theorem, we have

$$(L)=\sum_{\gamma\leftarrow\alpha}z_n{}^{|\alpha|-|\gamma|-n+1}(t+1)^{s(\alpha,\gamma)}t^{l(\alpha,\gamma)}\times U_\gamma(z_1, z_2, \cdots, z_{n-1}). \tag{2.7}$$

Let us consider the polynomial F defined below.

$$\begin{cases} F(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)\overset{\text{def}}{=}\left[\prod_{m=1}^{n-1}z_{m+1}{}^m\times\operatorname{Tr}\{\Psi_m(\mathrm{D}Z_m)\}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\times U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n) \quad \text{for} \quad n\geqq 2, \quad \text{and} \\ F((p)\,;t\,;z)\overset{\text{def}}{=}U_p(z)=z^p \quad \text{in case } n=1. \end{cases}$$

It follows (2.7) that

$$F(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)=\sum_{\gamma\leftarrow\alpha}F(\gamma\,;t\,;z_1,z_2,\cdots,z_{n-1})\times z_n{}^{|\alpha|-|\gamma|}(t+1)^{s(\alpha,\gamma)}t^{l(\alpha,\gamma)}. \tag{2.8}$$

Comparing (2.8) with (2.6), we deduce that both $\tilde{J}(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)$ and $F(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)$ satisfy a same inductive formula with respect to n. Moreover, since

$$\tilde{J}((p)\,;t\,;z)=z^p=F((p)\,;t\,;z),$$

the initial conditions coincide. So, by induction on n, we have

$$\begin{aligned}\tilde{J}(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)&=F(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)\\ &=\left[\prod_{m=1}^{n-1}z_{m+1}{}^m\times\operatorname{Tr}\{\Psi_m(\mathrm{D}Z_m)\}\right]\times U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n),\end{aligned}\tag{2.9}$$

while

$$\operatorname{Tr}\{\Psi_m(\mathrm{D}Z_m)\}=\sum_{k=0}^{m}(z_{m+1}{}^{-1})^k\operatorname{Tr}\{\Lambda_{(k,m)}(\mathrm{D}Z_m)\}=\prod_{k=1}^{m}(z_{m+1}{}^{-1}z_kt+1) \tag{2.10}$$

Substituting (2.10) into (2.9), we have

$$\tilde{J}(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)=\left\{\prod_{j\geqq i}(z_it+z_j)\right\}\times U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n).$$

Since U_α is a symmetric polynomial, exchanging the roles of suffixes i and j,

$$J(\alpha\,;t\,;z_1,z_2,\cdots,z_{n-1},z_n)=\left\{\prod_{j\geqq i}(z_jt+z_i)\right\}\times U_\alpha(z_1,z_2,\cdots,z_{n-1},z_n).$$

This is the identity we have been looking for.

3. Specializations.

Now we shall show how classical results come out of our formula. We may remark that we do not intend to give a 'new proof' of these, since we essentially used Weyl's character formula and Gelfand's parametrization in the proof of our theorem. What we would like to claim is that our formula is a deformation with respect to the parameter t which connects at least three important classical results. In this section, we fix a partition λ of length n and the distinct partition $\alpha=\lambda+\delta$.

3.1. Gelfand's parametrization. I. M. Gelfand and M. L. Zetlin used Gelfand patterns to parametrize the weight spaces of the representation spaces of general linear groups. In particular, in the representation space of ρ_λ, the dimension of the weight space corresponding to the weight $(m_1, m_2, \cdots, m_n)$ with respect to an arbitrary maximal torus coincides with the number of the Gelfand patterns T equipped with λ as their top rows and satisfying $m_i(T)=m_i$ for any i. That is,

$$\sum_{T\in G(\lambda)}\left\{\prod_{i=1}^{n} z_i^{m_i(T)}\right\} = S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n). \tag{3.1.1}$$

This formula comes from our formula as follows. Substituting 0 for t in (2.1), we have the following identity (3.1.2).

$$J(\alpha; 0; z_1, z_2, \cdots, z_{n-1}, z_n) = \left\{\prod_{i=1}^{n} z_i^{n-i}\right\}\times S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n). \tag{3.1.2}$$

From the definition of the generating function J, the left side of (3.1.2) equals

$$\sum_{\substack{T\in SG(\alpha)\\ l(T)=0}}\left\{\prod_{i=1}^{n} z_i^{m_i(T)}\right\}. \tag{3.1.3}$$

There is an injection Θ from $G(\lambda)$ into $SG(\alpha)$ defined as

$$\Theta: \begin{array}{lcl} G(\lambda) & \cdots\cdots\cdots\to & SG(\alpha) \\ T=(t_{i,j}) & \cdots\cdots\cdots\to & \Theta(T)=(t_{i,j}+n-j). \end{array}$$

It is easy to see that $m_i(\Theta(T))=m_i(T)+n-i$, and the image of Θ coinsides with the set of all strict patterns in $SG(\alpha)$ with no lefty element. By means of Θ, we exchange the summation occurring in (3.1.3) into the summation over $G(\lambda)$ to modify (3.1.3) as

$$\sum_{T\in G(\lambda)}\left\{\prod_{i=1}^{n} z_i^{m_i(T)+(n-i)}\right\}. \tag{3.1.4}$$

Substituting (3.1.4) for the left side of (3.1.2), we have (3.1.1) easily.

NOTE. The results of Gelfand and Zetlin are far deeper than (3.1.1) itself. These show a nice orthogonal basis corresponding to the symmetry breaking $GL(n, \boldsymbol{C})\supset GL(n-1, \boldsymbol{C})\supset\cdots\supset GL(1, \boldsymbol{C})$ by means of Gelfand patterns. That basis plays a significant role in quantum mechanics. See [**1**], [**4**] and [**11**] for their theory and its applications. Algebraically, that basis is often called "standard monomials".

3.2. Weyl's character formula. The standard form of Weyl's character formula for $GL(n, \boldsymbol{C})$ is the following. Cf. [**3**], [**6**] and [**12**].

$$\sum_{w\in S_n}\left\{(-1)^{i(w)}\prod_{i=1}^{n} z_i^{\alpha_{w(i)}}\right\}=\left\{\prod_{j>i}(z_i-z_j)\right\}\times S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n), \tag{3.2.1}$$

where S_n is the symmetric group of degree n, whose element w acts on the set $\{1, 2, \cdots, n\}$ in usual way, and $i(w)$ denotes the number of inversions of w. $i(w)$ is often called the length of w, cf. [5].

We remark that the left side of (3.2.1) can be written as a determinant, say $V_{\lambda+\delta}$, which resembles the Vandermonde determinant V_δ. Then we can rewrite (3.2.1) into the following (3.2.1*) refered in the introduction.

$$S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n)=\frac{V_{\lambda+\delta}}{\prod_{j>i}(z_i-z_j)}. \tag{3.2.1*}$$

The formula (3.2.1) comes from our formula as follows:

Substituting -1 for t in (2.1), we have

$$J(\alpha; -1; z_1, z_2, \cdots, z_{n-1}, z_n)=\left\{\prod_{j>i}(z_i-z_j)\right\}\times S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n). \tag{3.2.2}$$

From the definition of J, the left side of (3.2.2) equals

$$\sum_{\substack{t\in SG(\alpha)\\ s(T)=0}}\left\{(-1)^{l(T)}\prod_{i=1}^{n} z_i^{m_i(T)}\right\}. \tag{3.2.3}$$

For an element w of the symmetric group S_n, $T_{w,\alpha}$ denotes the strict Gelfand pattern belonging to $SG(\alpha)$ defined such as its any j-th row is the sequence consisting of all the elements of $\{\alpha_{w(j)}, \alpha_{w(j+1)}, \cdots, \alpha_{w(n)}\}$ arranged in the decreasing order.

EXAMPLE 3. Let $w=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$. Then $T_{w,(5,4,1)}=\begin{matrix}5\,4\,1\\5\,4\\4\end{matrix}$.

It is easy to see that $\{T\in SG(\alpha)\mid s(T)=0\}=\{T_{w,\alpha}\mid w\in S_n\}$. Moreover, $l(T_{w,\alpha})$ is independent of α, and is equal to the number of inversions of the permutation w. Since the i-th row $T_{w,\alpha}$ consists of both $\alpha_{w(i)}$ and the entries of its $(i+1)$-th row,

$$m_i(T_{w,i})=\alpha_{w(i)}.$$

Then (3.2.3) becomes

$$\sum_{w\in S_n}\left\{(-1)^{i(w)}\prod_{i=1}^{n} z_i^{\alpha_{w(i)}}\right\}. \tag{3.2.4}$$

Substituting (3.2.4) for the left side of (3.2.2), we get (3.2.1).

NOTE. Weyl's character formula was firstly given by H. Weyl for the classical groups in [12]. It has more general expressions applied for any reductive algebraic group and Kac-Moody algebra. Cf. [6].

3.3. Stanley's formula.

DEFINITION 3.3.1. The Hall-Littlewood polynomial of type α is

$$(3.3.1)\quad R_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n ; q)=\sum_{w\in S_n} w\left\{\left(\prod_{i=1}^n z_i^{\alpha_i}\right)\times\prod_{j>i}(z_i-qz_j)(z_i-z_j)^{-1}\right\},$$

where the permutation w acts naturally on the polynomials in $z_1, z_2, \cdots, z_{n-1}, z_n$.

NOTE. This definition is a convenient version. The Hall-Littlewood polynomial has a more general expression in which we can loosen the condition for α in (3.3.1) from a distinct partition to an arbitrary partition. Cf. [8].

The following formula was given by Stanley [14], and was also given by Mills et al. [9] for a special case using different methods.

$$(3.3.2)\qquad \sum_{T\in SG(\alpha)}\left\{2^{s(T)}\prod_{i=1}^n z_i^{m_i(T)}\right\}=R_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n ; -1).$$

This formula is an easy corollary of Theorem 2.1. Substituting 1 for t in (2.1), we get

$$(3.3.3)\qquad J(\alpha ; 1 ; z_1, z_2, \cdots, z_{n-1}, z_n)=\left\{\prod_{j>i}(z_i+z_j)\right\}\times S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n).$$

Substituting -1 for q in (3.3.1), we get

$$(3.3.4)\qquad R_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n ; -1)=\left\{\prod_{j>i}(z_i+z_j)\right\}\times S_\lambda(z_1, z_2, \cdots, z_{n-1}, z_n).$$

From (3.3.3) and (3.3.4),

$$(3.3.5)\qquad J(\alpha ; 1 ; z_1, z_2, \cdots, z_{n-1}, z_n)=R_\alpha(z_1, z_2, \cdots, z_{n-1}, z_n ; -1).$$

From the definition of the generating function J,

$$(3.3.6)\qquad J(\alpha ; 1 ; z_1, z_2, \cdots, z_{n-1}, z_n)=\sum_{T\in SG(\alpha)}\{2^{s(T)}\prod_{i=1}^n z_i^{m_i(T)}\}.$$

(3.3.5) and (3.3.6) mean (3.3.2).

The Hall-Littlewood polynomial is a basic tool to determine the characters of Chevalley groups over finite fields and local fields. The value of it at $q=-1$ is its most singular value, which shows very interesting behavior from both combinatorial and representation theoretic points of view. Cf. [8], [14]. In particular, Stanley showed (3.3.2) using an expansion formula of the Hall-Littlewood polynomials.

3.4. Weyl's denominator formula and alternating sign matrices. In case $\lambda=(0, 0, \cdots, 0)$, $\alpha=\delta=(n-1, n-2, \cdots, 1, 0)$ and Weyl's character formula (3.2.1) becomes

$$\sum_{w\in S_n}\left\{(-1)^{i(w)}\prod_{i=1}^{n} z_i^{w(n+1-i)-1}\right\} = \left\{\prod_{j>i}(z_i - z_j)\right\}. \tag{3.4.1}$$

This formula is called Weyl's denominator formula for $GL(n, \boldsymbol{C})$. Although it is known as a expansion formula of the Vandermonde matrix of rank n classically, it is generalized for the category of the reductive algebraic groups playing significant roles in the representation theory.

Now, what should we have from our formula (2.1) specializing it to the case $\lambda=(0, 0, \cdots, 0)$?

From the definition of the generating function J,

$$J(\delta\,; t\,; z_1, z_2, \cdots, z_{n-1}) = \sum_{T\in SG(\delta)}\left\{(t+1)^{s(T)}t^{l(T)}\prod_{i=1}^{n} z_i^{m_i(T)}\right\}.$$

Thus, it follows (2.1) that

$$\sum_{T\in SG(\delta)}\left\{(t+1)^{s(T)}t^{l(T)}\prod_{i=1}^{n} z_i^{m_i(T)}\right\} = \prod_{j>i}(z_i + z_j t). \tag{3.4.2}$$

To investigate the meaning of (3.4.2), we must study the set $SG(\delta)$.

DEFINITION 3.4.3. An n by n matrix X is called an alternating sign matrix if X satisfies the following four conditions:

1. Any entry of X is 1, 0, or -1.

2. In any row $(x_{i,1}, x_{i,2}, \cdots, x_{i,n})$ of X, its k-th partial row sum $R(i\,; k)$ satisfies

$$R(i\,; k) = \sum_{j=1}^{k} x_{i,j} = 0 \quad \text{or } 1 \qquad \text{for any } k=1, 2, \cdots, n.$$

3. In any column ${}^t(x_{1,j}, x_{2,j}, \cdots, x_{n,j})$ of X, its k-th partial column sum $C(j\,; k)$ satisfies

$$C(j\,; k) = \sum_{i=1}^{k} x_{i,j} = 0 \quad \text{or } 1 \qquad \text{for any } k=1, 2, \cdots, n.$$

4. For any i-th row and j-th column of X, the row sum and column sum $R(i\,; n)=C(j\,; n)=1$.

Any permutation matrix is an alternating sign matrix, although there exists some alternating sign matrices which are not permutation matrices.

EXAMPLE 4. The number of the 3 by 3 alternating sign matrices is 7, and that of 4 by 4 alternating matrices is 42. The following is the only 3 by 3 alternating sign matrix which is not a permutation matrix.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

DEFINITION 3.4.4. Let X be an n by n alternating sign matrix, then $i(X)$, the number of inversion of X, is defined by

$$i(X)=\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{x_{i,j}\times\sum_{k<i}\sum_{p>j}x_{k,p}\right\}.$$

REMARK. If X is a permutation matrix, $i(X)$ equals to the usual "number of inversion" of a permutation.

DEFINITION 3.4.5. The number of the entries -1 in X is denoted by $s(X)$, which is called the number of special elements in X.

Now we refer a proposition from [**9**]. For an element T of $SG(\delta)$, we define an n by n matrix $Q(T)$ as follows.

$Q(T)_{i,j}=1$ if and only if $n-j$ is an entry in the i-th row of T, and otherwise it is 0.

Let us consider the matrix $\Omega(T)=BQ(T)$, where B is the n by n matrix defined by

$$B_{i,j}=\begin{cases}1 & \text{if } j=i\\ -1 & \text{if } j=i+1\\ 0 & \text{otherwise.}\end{cases}$$

PROPOSITION 3.4.6. 1. *Ω is a canonical bijective mapping from $SG(\delta)$ onto Alt_n, the set of n by n alternating sign matrices.*

2. *For any element T of $SG(\delta)$, $s(\Omega(T))=s(T)$.*

3. *$i(\Omega(T))=l(T)+s(T)$.*

4. *$m_i(T)$ coincides with the i-th component of $\Omega(T)\delta$ where we consider δ as an n-dimensional vector.*

EXAMPLE 5. Suppose $T=\begin{matrix}3\,2\,1\,0\\3\,2\,0\\3\,1\\2\end{matrix}$. Then,

$$Q(T)=\begin{pmatrix}1&1&1&1\\1&1&0&1\\1&0&1&0\\0&1&0&0\end{pmatrix},\quad B=\begin{pmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{pmatrix}\quad\text{and}\quad \Omega(T)=\begin{pmatrix}0&0&1&0\\0&1&-1&1\\1&-1&1&0\\0&1&0&0\end{pmatrix}.$$

REMARK. $\Omega(T_{w,\delta})$ coincides with the permutation matrix corresponding to the permutation w.

Applying this proposition to (3.4.2), we get the following formula.

COROLLARY 3.4.7.

$$\prod_{j>i}(z_i+z_jt)=\sum_{X\in Alt_n}(t+1)^{s(X)}t^{(i(X)-s(X))}\prod_{i=1}^{n}z_i^{(X\delta)_i}, \tag{3.4.7}$$

where δ is the n-dimensional vector ${}^t(n-1, n-2, \cdots, 1, 0)$.

If we substitute -1 for t in (3.4.7), since an alternating matrix is a permutation matrix if and only if it has no -1 entry, we have

$$\prod_{j>i}(z_i-z_j)=\sum_{w\in S_n}(-1)^{i(w)}\prod_{i=1}^{n}z_i^{w(n+1-i)-1}.$$

This is nothing but Weyl's denominator formula.

For readers who know the notations of the representation theory of semisimple groups, we write (3.4.7) in "root language".

Let e^{γ} denote the linear character of the maximal torus of $SL(n, \boldsymbol{C})$ corresponding to the weight γ. We define the weight γ_i for $i=1, 2, \cdots, n$ such that $e^{\gamma_i}(Z_n)=z_i^{-1}$. Then $\{\gamma_i-\gamma_j | i\neq j\}$ is a root system Δ of type A_{n-1}, and $\Delta_+ := \{\gamma_i-\gamma_j \mid j>i\}$ is a positive root system of Δ. The action of an alternating matrix X on a weight γ of $SL(n, \boldsymbol{C})$ naturally comes from the action of X on the weight space of $GL(n, \boldsymbol{C})$, since X stabilizes the n-dimensional vector ${}^t(1, 1, \cdots, 1)$.

COROLLARY 3.4-(*). *Let δ be the half of the sum of the all positive roots. Then* (3.4.7) *is rewritten as follows:*

$$\prod_{\alpha\in\Delta_+}(1+te^{\alpha})=\sum_{X\in Alt_n}\left(1+\frac{1}{t}\right)^{s(X)}t^{i(X)}e^{\delta-X\delta}.$$

This corollary gives evidence that "alternating sign matrix" is a naturally extended notion of permutation, although Alt_n is not a group.

Some analogues of Gelfand pattern are given for other classical groups. But it is an open problem to define nice generating functions of "strict Gelfand patterns" for orthogonal groups (or symplectic groups) which describe some deformations of character formulas for those groups.

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