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Characterization of the class of upward first passage time distributions of birth and death processes and related results

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1. Introduction and main results.

We consider the minimal Markov process $\{X(t)\}_{t\geqq 0}$ on the nonnegative integers with a generator $A=(a_{ij})$ defined as follows. For nonnegative integers i and j,

$$(1.1)\qquad \begin{aligned} a_{ij} &= \beta_i && \text{if } i>0 \text{ and } j=i+1,\\ &= -(\beta_i+\delta_i) && \text{if } i>0 \text{ and } j=i,\\ &= \delta_i && \text{if } i>0 \text{ and } j=i-1,\\ &= 0 && \text{otherwise,} \end{aligned}$$

where $\delta_1\geqq 0$, $\delta_i>0$ for $i=2, 3, \cdots$, and $\beta_i>0$ for $i=1, 2, \cdots$. Such a process is called birth and death process. This process is strongly Markov by its minimality. Note that if $X(s)=0$ for some instant $s>0$, then $X(t)=0$ for all $t>s$, that is, the state 0 is a trap. Also note that the state 0 is attained from other states with positive probability whenever $\delta_1>0$. Let

$$\tau_n(\omega) = \inf\{t>0\,; X(t, \omega)=n\}$$

be the first passage time for $X(t)$ to n. Here we do not define $\tau_n(\omega)$ if $\{t\,; X(t, \omega)=n\}=\emptyset$. Let μ_{mn} be the distribution of τ_n when the process starts at m. We denote by $\sigma_{mn}(s)$ the Laplace transform of μ_{mn}, that is,

$$\sigma_{mn}(s) = E_m(e^{-s\tau_n}) = \int_0^\infty e^{-st}\mu_{mn}(dt).$$

Note that in the case $\delta_1>0$, the total mass of μ_{mn}, $1\leqq m<n$, is less than 1. We set $\bar{\mu}_{mn}=\mu_{mn}/\mu_{mn}([0, \infty))$ and $\bar{\sigma}_{mn}(s)=\sigma_{mn}(s)/\sigma_{mn}(0)$. Main purpose of this paper is to determine the class of μ_{mn}, $m<n$, for all birth and death processes.

Let $\boldsymbol{R}_+=[0, \infty)$. Let $\mathcal{P}(\boldsymbol{R}_+)$ be the totality of probability measures on $\boldsymbol{R}_+$. For $\mu\in\mathcal{P}(\boldsymbol{R}_+)$, we denote by $\mathcal{L}\mu(s)$ its Laplace transform. Let G be a pro-

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bability measure on $(0, \infty]$. We say that $\mu\in\mathscr{P}(\boldsymbol{R}_{+})$ is a mixture of exponential distributions with mixing distribution G if the distribution function $F_{\mu}(x)$ of μ is represented as

$$F_{\mu}(x)=\int_{(0,\infty]}(1-e^{-ux})G(du), \qquad x>0. \tag{1.2}$$

Here, we regard the delta measure at 0 as the degenerate exponential distribution. Denote by ME_{+} the class of mixtures of exponential distributions. Let $\mathrm{ME}_{+}(0)=\mathrm{ME}(0)$ be the class consisting of only one measure — the delta measure at 0. For $k\geqq 1$, denote by $\mathrm{ME}_{+}(k)$ the class of distributions μ in ME_{+} such that the support of the mixing distribution of μ consists of k points in $(0, \infty)$. For $k\geqq 1$, denote by $\mathrm{CE}_{+}(k)$ the subclass of $\mathscr{P}(\boldsymbol{R}_{+})$ consisting of convolutions of k distinct non-degenerate exponential distributions. For probability measures μ_1 and μ_2, we denote by $\mu_1*\mu_2$ the convolution of μ_1 and μ_2. Let $\mu_1\in\mathrm{CE}_{+}(m)$ $(m\geqq 1)$ with Laplace transform

$$\mathcal{L}\mu_1(s)=\prod_{k=1}^{m}a_k(s+a_k)^{-1}$$

where $0<a_1<a_2<\cdots<a_m<\infty$. For $0<b_1<b_2<\cdots<b_n<\infty$, let $\mu_2\in\mathrm{ME}_{+}(n)$ $(n\geqq 2)$ with $\{b_k\,;1\leqq k\leqq n\}$ as the support of its mixing distribution. Then there is a sequence $\{c_k\}_{1\leqq k\leqq n-1}$ such that

$$0<b_1<c_1<b_2<\cdots<c_{n-1}<b_n$$

and

$$\mathcal{L}\mu_2(s)=\prod_{k=1}^{n-1}c_k^{-1}(s+c_k)\prod_{k=1}^{n}b_k(s+b_k)^{-1}.$$

See Steutel [12]. We say that $\mu=\mu_1*\mu_2$ is a $\mathrm{CME}_{+}(m, n)$ distribution if

$$\{a_k\}\cap[\{b_k\}\cup\{c_k\}]=\emptyset.$$

We define classes $\mathrm{CME}_{+}(m, 1)$ for $m\geqq 0$ and $\mathrm{CME}_{+}(0, n)$ for $n\geqq 2$ by $\mathrm{CME}_{+}(m, 1)=\mathrm{CE}_{+}(m+1)$ and $\mathrm{CME}_{+}(0, n)=\mathrm{ME}_{+}(n)$, respectively. Set

$$\mathrm{CE}_{+}^{f}=\bigcup_{k=1}^{\infty}\mathrm{CE}_{+}(k) \quad\text{and}\quad \mathrm{CME}_{+}^{f}=\bigcup_{m=0}^{\infty}\bigcup_{n=1}^{\infty}\mathrm{CME}_{+}(m, n).$$

The superscript f stands for finite. We will discuss, in a forthcoming paper, extensions of the classes CE_{+}, ME_{+} and CME_{+} to distributions on the whole line. So we use the subscript '+' in order to denote the classes on the nonnegative real line. The following theorem is our main result.

THEOREM 1. *Let* $1\leqq m\leqq n$. *Then the following hold:*

(i) *There is* $k\,(\max\{1, 2m-n\}\leqq k\leqq m)$ *such that*

$$\bar{\mu}_{m,n+1}\in\mathrm{CME}_{+}(n-m, k).$$

(ii) *For any* $\mu \in \mathrm{CME}_+(n-m, k)$ *with* $\max\{1, 2m-n\} \leqq k \leqq m$, *there is a birth and death process for which* $\bar{\mu}_{m,n+1}$ *coincides with* μ.

COROLLARY 1. *The class of upward first passage time distributions of birth and death processes with reflecting boundary at* 1 *coincides with the class* CME_+^f.

It should be noted that 'upward' is not essential. The situation that we are dealing with is that there are finitely many states between the boundary and the hitting point, the starting point is located between them, and the paths can jump only to neighboring states.

Using Stone's result [**14**], Rösler [**11**] shows that first passage time distributions of generalized diffusion processes with local boundary conditions can be approximated by upward first passage time distributions of birth and death processes. Hence we have the following:

COROLLARY 2. *All first passage time distributions (normalized to be probability measures) of one-dimensional generalized diffusion processes with local boundary conditions are contained in the closure of the class* CME_+^f.

The definition of the generalized diffusion processes is found in Kotani and Watanabe [**9**]. These processes are also called gap diffusions in Knight [**8**].

There are many works about the first passage time distributions of birth and death processes. Among them, two works are very close to our result. Rösler shows that μ_{mn} is unimodal. Approximating diffusion processes by birth and death processes, he also shows the unimodality of first passage time distributions for diffusion processes with absorbing or reflecting boundary. Keilson [**5**] shows that an upward passage time distribution of a birth and death process is a convolution of a finite number of exponential distributions and a mixture of exponential distributions. Hence (i) of Theorem 1 is a refinement of his result. Our method of the proof is different from his. Our method has an advantage in refining his result as in (i) of Theorem 1 and in getting the converse result (ii) of Theorem 1. Our method also has an advantage in getting a condition under which the first passage time distribution is strongly unimodal. The definition of the strong unimodality is stated in Section 6. As Keilson points out, distributions in CE_+^f are strongly unimodal and distributions in ME_+ are unimodal and hence distributions in CME_+^f are unimodal, which gives an alternative proof of Rösler's result.

In order to prove Theorem 1, we restate Theorem 1 in Theorem 2 in Section 2. This restatement is also useful for the study of strong unimodality of the first passage time distributions. In Sections 3 and 4, we prove Theorem 2. In Theorem 2 (i), we describe a relation between zeros of polynomials $P_m(s)$ and $P_n(s)$, defined by (2.1). The relation is obtained by Stieltjes [**13**]. We prove this relation in a different way. We give in Section 5 the representation of the Laplace transforms of distributions in the closure of the class CME_+^f. We define, in Section 6, a class CME_+^d which contains, in addition to CME_+^f, many of first passage time distributions of one dimensional diffusion processes, and give a necessary and sufficient condition for a distribution in CME_+^d to be strongly unimodal. This condition partially extends an earlier result of the author [**17**]. We describe in Section 7 some examples and applications of our results from Sections 1 through 6.

2. Restatement of Theorem 1.

Define a sequence of polynomials $\{P_m(s)\}_{m\geqq 0}$ by

$$P_0(s)=1$$

and

$$P_m(s)=\beta_1\cdots\beta_m/\sigma_{1,m+1}(s) \tag{2.1}$$

for $m\geqq 1$. Since the process is strongly Markov and paths are 'continuous', the equality $\sigma_{1n}(s)=\sigma_{1m}(s)\sigma_{mn}(s)$ holds for $1<m<n$. Hence we have

$$\sigma_{mn}(s)=\beta_m\cdots\beta_{n-1}P_{m-1}(s)/P_{n-1}(s). \tag{2.2}$$

It is easy to show that $\{P_m(s)\}_{m\geqq 0}$ satisfies the recurrence relation

$$\begin{aligned} &P_0(s)=1, \\ &P_1(s)=s+\beta_1+\delta_1 \qquad \text{and} \\ &P_m(s)=(s+\beta_m+\delta_m)P_{m-1}(s)-\delta_m\beta_{m-1}P_{m-2}(s) \end{aligned} \tag{2.3}$$

for $m\geqq 2$. To prove this, again use the strong Markov property of the process and the 'continuity' of its paths. By this relation, we easily obtain the following: $P_m(s)$ is a polynomial of degree m. The leading coefficient is equal to one. The zeros of $P_m(s)$ are simple and negative. The zeros of $P_m(s)$ are interlaced with zeros of $P_{m+1}(s)$ for $m\geqq 1$. These properties of $\{P_m\}_{m\geqq 0}$ are proved in the same way as for a system of polynomials $\{Q_m(s)\}$ which is defined in Section 3. See Lemma 1 and Lemma 2.

For each positive integer m, denote by PSN(m) the class of polynomials of degree m whose zeros are all real, simple and negative and whose value at the origin is positive. For notational convenience, we denote by PSN(0) the class of positive constant functions and set PSN(-1)=PSN(0). Let $W_1(s)\in\mathrm{PSN}(m)$ and $W_2(s)\in\mathrm{PSN}(n)$ for $1\leqq m<n$. Let

$$0>a_1>a_2>\cdots>a_m$$

be zeros of $W_1(s)$ and put $a_0=0$ and $a_{m+1}=-\infty$. We say that W_2 interlaces W_1

in the weak sense and denote as $W_1 \leqq W_2$ if there is at least one zero of W_2 in each open interval (a_{i+1}, a_i), $0 \leqq i \leqq m$. We say that W_2 interlaces W_1 and denote as $W_1 \lhd W_2$ if $W_1 \leqq W_2$ and W_1 and W_2 have no common zero. If $n \geqq 1$, then we always say that $W_2 \in \mathrm{PSN}(n)$ interlaces $W_1 \in \mathrm{PSN}(0)$. Applying this definition to $\{P_m(s)\}_{m \geq 1}$, we have that $P_m(s) \in \mathrm{PSN}(m)$ and $P_m(s) \lhd P_{m+1}(s)$ for $m \geqq 0$. Let $\mu \in \mathrm{CME}_+(j, k)$ with $j \geqq 0$ and $k \geqq 1$. Then there are measures $\mu_1 \in \mathrm{CE}_+(j)$ and $\mu_2 \in \mathrm{ME}_+(k)$ such that $\mu = \mu_1 * \mu_2$. For μ_1 and μ_2, there are polynomials $W_1 \in \mathrm{PSN}(j)$, $W_2 \in \mathrm{PSN}(k-1)$ and $W_3 \in \mathrm{PSN}(k)$ such that

$$\mathcal{L}\mu_1(s) = 1/W_1(s), \quad \mathcal{L}\mu_2(s) = W_2(s)/W_3(s), \quad W_2 \lhd W_3 \quad \text{and} \quad W_2 \lhd W_3 W_1 .$$

Hence $\mu \in \mathrm{CME}_+(j, k)$ if and only if there are polynomials $W_1 \in \mathrm{PSN}(k-1)$ and $W_2 \in \mathrm{PSN}(j+k)$ such that

$$\mathcal{L}\mu(s) = W_1(s)/W_2(s) \quad \text{and} \quad W_1 \lhd W_2 .$$

If we establish the following theorem, then Theorem 1 is true by the above fact.

THEOREM 2. *Let* $1 \leqq m \leqq n$. *Then the following hold:*

(i) *The polynomial* P_n *interlaces* P_{m-1} *in the weak sense. Write* $\sigma_{m,n+1}(s)$ *as*

$$\sigma_{m,n+1}(s) = W_j(s)/W_k(s)$$

with $W_j(s) \in \mathrm{PSN}(j)$ *and* $W_k(s) \in \mathrm{PSN}(k)$ *so that* W_j *and* W_k *have no common zero point. Then*

$$k-j = n-m+1, \qquad \max\{0, 2m-n-1\} \leqq j \leqq m-1$$

and $W_j(s) \lhd W_k(s)$.

(ii) *Let* j *and* k *be nonnegative integers such that* $k-j=n-m+1$ *and* $\max\{0, 2m-n-1\} \leqq j \leqq m-1$. *For* $W_j \in \mathrm{PSN}(j)$ *and* $W_k \in \mathrm{PSN}(k)$ *with* $W_j \lhd W_k$, *we can construct a birth and death process for which*

$$W_j(s)/W_k(s) = \text{const.}\ \sigma_{m,n+1}(s) .$$

3. Proof of Theorem 2 (i).

Let $n \geqq 1$. Define a system of polynomials $\{Q_m(s)\}_{0 \leq m \leq n}$ by

$$
\begin{aligned}
&Q_0(s) = 1, \\
&Q_1(s) = s+\beta_n+\delta_n , \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&Q_m(s) = (s+\beta_{n-m+1}+\delta_{n-m+1})Q_{m-1}(s) - \delta_{n-m+2}\beta_{n-m+1}Q_{m-2}(s), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&Q_n(s) = (s+\beta_1+\delta_1)Q_{n-1}(s) - \delta_2\beta_1 Q_{n-2}(s) .
\end{aligned}
\tag{3.1}
$$

LEMMA 1. *For* $1 \leqq m \leqq n$,

$$Q_m(0) > \delta_{n-m+1} Q_{m-1}(0).$$

PROOF. We prove this lemma by induction in m. If $m=1$, then

$$Q_1(0) = \beta_n + \delta_n > \delta_n = \delta_n Q_0(0).$$

Suppose that the lemma is true for $m \leqq k$. Then we have

$$Q_{k+1}(0) = (\beta_{n-k} + \delta_{n-k}) Q_k(0) - \delta_{n-k+1} \beta_{n-k} Q_{k-1}(0) > \delta_{n-k} Q_k(0).$$

LEMMA 2. *For* $1 \leqq m \leqq n$, $Q_m(s) \in \mathrm{PSN}(m)$ *and* $Q_{m-1} \lhd Q_m$.

PROOF. We prove this lemma by induction in m. For $m=1$, it is obvious that $Q_1 \in \mathrm{PSN}(1)$ and $Q_0 \lhd Q_1$. Suppose that the lemma is true for $m \leqq k$. By the recurrence relation (3.1) $Q_{k+1}(s)$ is a polynomial of degree $k+1$. If s is a zero point of Q_k, then, by (3.1), $Q_{k+1}(s)$ and $Q_{k-1}(s)$ have alternative signs. Thus $Q_{k+1}(s)$ has at least $k-1$ zero points between zeros of Q_k. Since $Q_m(0) > 0$ for $1 \leqq m \leqq n$ by Lemma 1 and since $Q_{k-1} \lhd Q_k$, $Q_{k-1}(s_1) > 0$ for the largest zero s_1 of Q_k. Hence $Q_{k+1}(s_1) < 0$ and there is a zero of Q_{k+1} between s_1 and 0. Since Q_{k-1} and Q_{k+1} must have the same sign near $-\infty$ and have alternative signs at the smallest zero s_k of Q_k, Q_{k+1} must have a zero between $-\infty$ and s_k. This completes the proof.

LEMMA 3. *For* $1 \leqq m \leqq n$,

$$P_n(s) = P_{n-m}(s) Q_m(s) - \delta_{n-m+1} \beta_{n-m} P_{n-m-1}(s) Q_{m-1}(s). \tag{3.2}$$

PROOF. The proof is accomplished by induction in m. If $m=1$, then by (2.3) and (3.1) we have that

$$\begin{aligned} P_n(s) &= (s + \beta_n + \delta_n) P_{n-1}(s) - \delta_n \beta_{n-1} P_{n-2}(s) \\ &= P_{n-1}(s) Q_1(s) - \delta_n \beta_{n-1} P_{n-2}(s) Q_0(s). \end{aligned}$$

Suppose that (3.2) holds for $m \leqq k$. Then, for $m=k$

$$P_n(s) = P_{n-k}(s) Q_k(s) - \delta_{n-k+1} \beta_{n-k} P_{n-k-1}(s) Q_{k-1}(s)$$

holds. Applying (2.3) and (3.1) to the above equality, we have

$$\begin{aligned} P_n(s) &= \{(s + \beta_{n-k} + \delta_{n-k}) P_{n-k-1}(s) - \delta_{n-k} \beta_{n-k-1} P_{n-k-2}(s)\} Q_k(s) \\ &\quad - \delta_{n-k+1} \beta_{n-k} P_{n-k-1}(s) Q_{k-1}(s) \\ &= P_{n-k-1}(s) \{(s + \beta_{n-k} + \delta_{n-k}) Q_k(s) - \delta_{n-k+1} \beta_{n-k} Q_{k-1}(s)\} \\ &\quad - \delta_{n-k} \beta_{n-k-1} P_{n-k-2}(s) Q_k(s) \\ &= P_{n-k-1}(s) Q_{k+1}(s) - \delta_{n-k} \beta_{n-k-1} P_{n-k-2}(s) Q_k(s). \end{aligned}$$

The proof is complete.

Set

$$V_{m,n+1}(s) = \beta_m \cdots \beta_n / \sigma_{m,n+1}(s).$$

By (2.2) and (3.2), we have

$$V_{m,n+1}(s) = Q_{n-m}(s) R_{mn}(s) \tag{3.3}$$

where

$$R_{mn}(s) = P_m(s)/P_{m-1}(s) - \delta_{m+1}\beta_m Q_{n-m-1}(s)/Q_{n-m}(s).$$

PROOF OF THEOREM 2 (i). Note that for $1 \leqq m \leqq n$

$$P_m(0)/P_{m-1}(0) > \delta_{m+1}\beta_m Q_{n-m-1}(0)/Q_{n-m}(0) > 0$$

since $V_{m,n+1}(s) > 0$. Let $0 > a_1 > a_2 > \cdots > a_{m-1}$ be the zeros of $P_{m-1}(s)$ and let $a_0 = 0$ and $a_m = -\infty$. Since $P_{m-1} \lhd P_m$, $P_m(s)/P_{m-1}(s)$ is strictly increasing on (a_k, a_{k-1}) for $k = 1, 2, \cdots, m$, and the range on (a_k, a_{k-1}) coincides with $(-\infty, \infty)$ if $2 \leqq k \leqq m$ and coincides with $(-\infty, P_m(0)/P_{m-1}(0))$ if $k = 1$ (see Appendix). Let $0 > b_1 > b_2 > \cdots > b_{n-m}$ be the zeros of $Q_{n-m}(s)$ and let $b_0 = 0$ and $b_{n-m+1} = -\infty$. Since $Q_{n-m-1}(s) \lhd Q_{n-m}(s)$, $\delta_{m+1}\beta_m Q_{n-m-1}(s)/Q_{n-m}(s)$ is strictly decreasing on (b_k, b_{k-1}) for $k = 1, 2, \cdots, n-m+1$ and the range on (b_k, b_{k-1}) coincides with $(-\infty, \infty)$ if $2 \leqq k \leqq n-m+1$ and coincides with $(\delta_{m+1}\beta_m Q_{n-m-1}(0)/Q_{n-m}(0), \infty)$ if $k = 1$. Thus, for each k $(1 \leqq k \leqq m)$, there is a zero of $R_{mn}(s)$ in (a_k, a_{k-1}). That is, $P_{m-1} \trianglelefteq P_n$. From this fact, it is obvious that $W_j \lhd W_k$, $0 \leqq j \leqq m-1$, and $k - j = n - m + 1$, when we write

$$\sigma_{m,n+1}(s) = W_j(s)/W_k(s)$$

in the reduced form with $W_j \in \mathrm{PSN}(j)$ and $W_k \in \mathrm{PSN}(k)$. Note that P_m and P_{m-1} have no common zero. Zeros of P_{m-1} may cancel only with zeros of Q_{n-m}. Hence $2m - n - 1 \leqq j$. The proof is complete.

4. Proof of Theorem 2 (ii).

LEMMA 4. *Fix* $0 \leqq m \leqq n-1$. *Let* $f_{n-m-1}(s)$ *and* $f_{n-m}(s)$ *be polynomials in* $\mathrm{PSN}(n-m-1)$ *and* $\mathrm{PSN}(n-m)$, *respectively, with the leading coefficients equal* 1 *such that* $f_{n-m-1} \lhd f_{n-m}$. *Fix* δ_{m+1} *so that*

$$0 < \delta_{m+1} < f_{n-m}(0)/f_{n-m-1}(0).$$

Then there is a unique sequence of positive numbers $\{\beta_{m+1}, \cdots, \beta_n, \delta_{m+2}, \cdots, \delta_n\}$ *such that the sequence of polynomials* $\{Q_k(s)\}_{0 \leqq k \leqq n-m}$ *defined by the recurrence relation* (3.1) *satisfies*

$$Q_{n-m-1}(s) = f_{n-m-1}(s)$$

and

$$Q_{n-m}(s)=f_{n-m}(s).$$

LEMMA 5. *Fix* $m\geqq 1$. *Let* g_{m-1} *and* g_m *be polynomials in* PSN$(m-1)$ *and* PSN(m), *respectively, with the leading coefficients equal* 1 *such that* $g_{m-1}\lhd g_m$. *Fix* β_m *so that*

$$0<\beta_m\leqq g_m(0)/g_{m-1}(0).$$

Then there uniquely exist a nonnegative number δ_1 *and positive numbers* $\beta_1,\cdots,\beta_{m-1},\delta_2,\cdots,\delta_m$ *such that* $\{P_k(s)\}_{0\leq k\leq m}$ *defined by the recurrence relation* (2.3) *satisfies*

$$P_{m-1}(s)=g_{m-1}(s)$$

and

$$P_m(s)=g_m(s).$$

Moreover, $\delta_1=0$ *if and only if* $\beta_m=g_m(0)/g_{m-1}(0)$.

Proofs of Lemmas 4 and 5 are similar. So we prove only Lemma 5.

PROOF OF LEMMA 5. We prove this lemma by induction in m. Let $m=1$ and let $0>c_1$. It is obvious that if we fix β_1 so that

$$0<\beta_1\leqq -c_1,$$

then there is $\delta_1\geqq 0$ such that

$$P_1(s)=s+\beta_1+\delta_1=s-c_1.$$

Here $\delta_1=0$ if and only if $\beta_1=-c_1$. Suppose now that the lemma is true for $m\leqq k$. Let

$$g_{k+1}(s)=(s-c_1)\cdots(s-c_{k+1})$$

and

$$g_k(s)=(s-b_1)\cdots(s-b_k)$$

with

$$0>c_1>b_1>c_2>\cdots>b_k>c_{k+1}.$$

Put

$$p_i=g_{k+1}(b_i)\qquad\text{for}\quad i=1,\cdots,k.$$

Then we have

$$(-1)^i p_i>0\qquad\text{for}\quad i=1,\cdots,k. \tag{4.1}$$

There is one and only one polynomial

$$g_{k-1}(s)=\sum_{j=0}^{k-1}\alpha_j s^j$$

of degree not greater than $k-1$ such that

$$-g_{k-1}(b_i)=p_i\qquad\text{for}\quad i=1,2,\cdots,k.$$

Since $g_{k-1}(s)$ must have at least $k-2$ extremums by (4.1), we have $\alpha_{k-1}>0$ and

$\alpha_0>0$, that is, $g_{k-1}(s)\in\mathrm{PSN}(k-1)$ and $g_{k-1}\lhd g_k$. Set

$$G_{k+1}(s)=g_{k+1}(s)+g_{k-1}(s),$$

which is a polynomial of degree $k+1$. The points $b_1, \cdots, b_k$ are zero points of $G_{k+1}(s)$. Denote another zero point of $G_{k+1}(s)$ by b_{k+1}. We have

$$G_{k+1}(s)=(s-b_1)\cdots(s-b_{k+1})=(s-b_{k+1})g_k(s),$$

that is,

$$g_{k+1}(s)=(s-b_{k+1})g_k(s)-g_{k-1}(s).$$

Since $g_{k+1}(0)$, $g_k(0)$ and $g_{k-1}(0)$ are positive, the number $-b_{k+1}$ is positive. Fix β_{k+1} so that $0<\beta_{k+1}\leqq g_{k+1}(0)/g_k(0)$. Since

$$g_{k+1}(0)/g_k(0)=-b_{k+1}-g_{k-1}(0)/g_k(0)<-b_{k+1}, \tag{4.2}$$

we can find $\delta_{k+1}>0$ satisfying $\beta_{k+1}+\delta_{k+1}=-b_{k+1}$. Moreover, by (4.2), δ_{k+1} satisfies

$$g_{k-1}(0)/g_k(0)\leqq\delta_{k+1}.$$

Define β_k by $\beta_k=\alpha_{k-1}/\delta_{k+1}$. Then we have

$$0<\beta_k\leqq\alpha_{k-1}g_k(0)/g_{k-1}(0).$$

By the assumption of the induction, we can uniquely choose a sequence of a nonnegative number δ_1 and positive integers $\beta_1, \cdots, \beta_{k-1}, \delta_2, \cdots, \delta_k$ so that

$$P_{k-1}(s)=g_{k-1}(s)/\alpha_{k-1} \quad\text{and}\quad P_k(s)=g_k(s).$$

Hence

$$\begin{aligned} g_{k+1}(s)&=(s-b_{k+1})P_k(s)-\alpha_{k-1}P_{k-1}(s)\\ &=(s+\beta_{k+1}+\delta_{k+1})P_k(s)-\delta_{k+1}\beta_kP_{k-1}(s). \end{aligned}$$

Note that the uniqueness of β_k and δ_{k+1} is obvious. Therefore, we complete the proof letting $g_{k+1}(s)=P_{k+1}(s)$.

LEMMA 6. *Let j, k and m be nonnegative integers satisfying $j+1\leqq m\leqq k$. Let $U(s)$, $W_1(s)$ and $W_2(s)$ be polynomials in* PSN(k), PSN(j) *and* PSN($k-j-1$), *respectively, such that $W_1W_2\in\mathrm{PSN}(k-1)$ and $W_1W_2\lhd U$. Let c_{k-j-1} be the smallest zero of W_2 and set $R(s)=W_2(s)/(s-c_{k-j-1})$. Then we can find polynomials $U_1\in$* PSN($j+1$), *$T_1\in\mathrm{PSN}(m-1-j)$, $T_2\in\mathrm{PSN}(k-m)$ and $T_3\in\mathrm{PSN}(m-j-1)$ such that $W_1T_3\in\mathrm{PSN}(m-1)$, $U_1T_1T_2=U$, $T_2T_3\in\mathrm{PSN}(k-j-1)$, $W_1T_3\lhd U_1T_1$ and $R\lhd T_2T_3$.*

PROOF. Let

$$0>a_1>a_2>\cdots>a_j$$

be the zeros of W_1 and let $a_0=0$ and $a_{j+1}=-\infty$. For $i=1, 2, \cdots, j+1$, denote by b_i the smallest zero of U in (a_i, a_{i-1}). Set $U_1(s)=(s-b_1)\cdots(s-b_{j+1})$ and define $U_2(s)$ by

$$U_2(s) = U(s)/U_1(s) = (s-d_1)(s-d_2)\cdots(s-d_{k-j-1}).$$

Denote by

$$0 > c_1 > c_2 > \cdots > c_{k-j-1}$$

the zeros of $W_2(s)$. Then, by the choice of U_1, we have that $W_1 \lhd U_1$ and

$$0 > d_1 > c_1 > d_2 > c_2 > \cdots > d_{k-j-1} > c_{k-j-1}.$$

Obviously, $R \lhd U_2$. Note that neither U_2 nor W_2 has a zero in each interval (a_i, b_i), $i=1, 2, \cdots, k+1$. Also, neither U_1 nor W_1 has a zero in each interval (c_i, d_i), $i=1, 2, \cdots, k-j-1$. Choose e_i in (c_i, d_i) arbitrarily for $i=1, 2, \cdots, m-j-1$ and set

$$T_1(s) = (s-d_1)\cdots(s-d_{m-j-1}),$$

$$T_2(s) = (s-d_{m-j})\cdots(s-d_{k-j-1})$$

and

$$T_3(s) = (s-e_1)\cdots(s-e_{m-j-1}).$$

We have, by the above choice of polynomials, that

$$W_1T_3 \lhd U_1T_1 \quad \text{and} \quad R \lhd T_2T_3.$$

PROOF OF THEOREM 2 (ii). Let j and k be nonnegative integers satisfying $k-j=n-m+1$ and $j+1 \leqq m \leqq k$. Let $W_j \in \mathrm{PSN}(j)$ and $W_k \in \mathrm{PSN}(k)$ such that $W_j \lhd W_k$. Choose a polynomial $W \in \mathrm{PSN}(k-j-1)$ so that $W_jW \in \mathrm{PSN}(k-1)$ and $W_jW \lhd W_k$. Let d be the smallest zero of W and set $R(s)=W(s)/(s-d)$. Choose $p>0$ sufficiently small. Then, since zeros of a polynomial are continuous with respect to its coefficients,

$$V(s) = W_k(s)/W_j(s)R(s)+p$$

is represented by $U \in \mathrm{PSN}(k)$ as

$$V(s) = U(s)/W_j(s)R(s)$$

without changing the configuration of zeros. Hence, by Lemma 6, we can construct polynomials $U_1 \in \mathrm{PSN}(j+1)$, $T_1 \in \mathrm{PSN}(m-1-j)$, $T_2 \in \mathrm{PSN}(k-m)$ and $T_3 \in \mathrm{PSN}(m-j-1)$ such that $W_jT_3 \in \mathrm{PSN}(m-1)$, $U_1T_1T_2=U$, $T_2T_3 \in \mathrm{PSN}(k-j-1)$, $W_jT_3 \lhd U_1T_1$ and $R \lhd T_2T_3$. Since

$$\{U_1(0)T_1(0)/W_j(0)T_3(0)\}\{T_2(0)T_3(0)/R(0)\} = V(0) = W_k(0)/W_j(0)R(0)+p > p,$$

we can choose β_m and δ_{m+1} so that

$$\delta_{m+1}\beta_m = p,$$

$$\beta_m \leqq U_1(0)T_1(0)/W_j(0)T_3(0)$$

and

$$\delta_{m+1} < T_2(0)T_3(0)/R(0).$$

Hence by Lemmas 4 and 5 we can choose sequences of nonnegative numbers

$$\{\beta_1, \cdots, \beta_{m-1}, \delta_1, \cdots, \delta_m\} \tag{4.3}$$

and

$$\{\beta_{m+1}, \cdots, \beta_n, \delta_{m+2}, \cdots, \delta_n\} \tag{4.4}$$

such that the polynomials $\{P_k(s)\}_{0\leq k\leq m}$ and $\{Q_k(s)\}_{0\leq k\leq n-m}$ defined by the recurrence relations (2.3) and (3.1) satisfy

$$P_{m-1}(s) = W_j(s)T_3(s), \quad P_m(s) = U_1(s)T_1(s)$$

and

$$Q_{n-m-1}(s) = R(s), \qquad Q_{n-m}(s) = T_2(s)T_3(s).$$

We have

$$\begin{aligned} W_k/W_j &= (V-p)R \\ &= \{(U_1T_1/W_jT_3)(T_2T_3/R)-\delta_{m+1}\beta_m\}R \\ &= P_mQ_{n-m}/P_{m-1}-\delta_{m+1}\beta_mQ_{n-m-1}. \end{aligned}$$

Thus, we have by (3.3)

$$W_j(s)/W_k(s) = \text{const.}\ \sigma_{m,n+1}(s)$$

where $\sigma_{m,n+1}(s)$ is the Laplace transform of the first passage time distribution from m to $n+1$ of a birth and death process whose generator is defined by (1.1) with quantities (4.3) and (4.4). This completes the proof.

5. Closure of CME_+^f.

We denote by I_+ the class of infinitely divisible distributions on $\boldsymbol{R}_+$. Laplace transform $\mathcal{L}\mu(s)$ of $\mu\in I_+$ has the Lévy canonical representation:

$$\mathcal{L}\mu(s) = \exp\left[-\gamma s+\int_{(0,\infty)}(e^{-sx}-1)N(dx)\right]$$

where $\gamma\in\boldsymbol{R}_+$ and N is a measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} u(1+u)^{-1}N(dx) < \infty.$$

The measure N is called the Lévy measure of μ. Denote by BO the smallest subclass of $\mathcal{P}(\boldsymbol{R}_+)$ which contains ME_+ and is closed under convolutions and weak limits. We call this class BO the Bondesson class. A distribution μ in I_+ belongs to BO if and only if its Lévy measure $N(dx)$ is absolutely continuous and the density $n(x)$ of N is represented as

$$n(x)=\int_{(0,\infty)} e^{-xu} M(du)$$

where M is a measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} \min\{u^{-1}, u^{-2}\} M(du)<\infty .$$

It is easy to see that $\mu \in \mathrm{BO}$ if and only if its Laplace transform is represented by the above M as

$$\mathcal{L}\mu(s)=\exp\left[-\gamma s+\int_{(0,\infty)}\{(x+s)^{-1}-x^{-1}\} M(dx)\right].$$

See Bondesson [1]. A distribution $\mu \in \mathrm{BO}$ is determined by the pair $[\gamma, M]$. So, we identify the pair $[\gamma, M]$ with μ. Since the Laplace transform of $\mu \in \mathrm{CME}_{+}(m, n)$ is represented as

$$\mathcal{L}\mu(s)=W_{n-1}(s)/W_{m+n}(s)$$

by $W_{n-1} \in \mathrm{PSN}(n-1)$ and $W_{m+n} \in \mathrm{PSN}(m+n)$ satisfying $W_{n-1} \lhd W_{m+n}$, it is easy to show the following fact. A distribution μ belongs to $\mathrm{CME}_{+}(m, n)$ if and only if $\mu \in \mathrm{BO}$, the measure M is absolutely continuous and there are sequences $\{a_k\}_{0 \leq k \leq m+1}$, $\{b_k\}_{1 \leq k \leq n}$ and $\{c_k\}_{0 \leq k \leq n}$ satisfying

$$\{a_k ; 1 \leqq k \leqq m\} \cap[\{b_k ; 1 \leqq k \leqq n\} \cup\{c_k ; 1 \leqq k \leqq n-1\}]=\emptyset,$$

$$0=a_0<a_1<a_2<\cdots<a_m<a_{m+1}=\infty$$

and

$$0=c_0<b_1<c_1<b_2<c_2<\cdots<b_n<c_n=\infty$$

such that the density $m(x)$ of M is represented as

$$m(x)=m_1(x)+m_2(x)$$

where

$$m_1(x)=k \quad \text{if} \quad a_k \leqq x<a_{k+1}$$

for $k=0, 1, 2, \cdots, m$, and

$$\begin{aligned} m_2(x) &= 0 \quad \text{if} \quad c_k \leqq x<b_{k+1} \quad \text{and} \\ &= 1 \quad \text{if} \quad b_{k+1} \leqq x<c_{k+1} \end{aligned}$$

for $k=0, 1, 2, \cdots, n-1$. Thus the class CME_{+}^{f} is contained in BO.

We say that $\mu \in \mathcal{P}(\boldsymbol{R}_{+})$ is a CE_{+} distribution if $\mu \in \mathrm{BO}$, M is absolutely continuous and there is a nondecreasing sequence of positive numbers $\{a_k\}_{1 \leq k<p}$ $(p \leqq \infty)$ satisfying $\sum a_k^{-1}<\infty$ such that the density $m(x)$ of M is represented as follows:

In the case $p<\infty$,

$$\begin{aligned} m(x) &= 0 && \text{on } (0, a_1) \\ &= k && \text{on } (a_k, a_{k+1}) \text{ if } a_k<a_{k+1} \\ &= p-1 && \text{on } [a_{p-1}, \infty) \end{aligned}$$

for $1\leqq k<p-1$ and in the case $p=\infty$,

$$\begin{aligned} m(x) &= 0 && \text{on } (0, a_1) \\ &= k && \text{on } (a_k, a_{k+1}) \text{ if } a_k<a_{k+1} \end{aligned}$$

for $1\leqq k$. The class CE_+ coincides with the class of PF densities on $[0, \infty)$. See Karlin [4]. The classes CE_+ and ME_+ are closed in the weak convergence sense. These facts are found in Bondesson [1] (p. 50 Remark 5.2 and p. 46). Let CME_+ be the class of $\mu\in P(R_+)$ represented as $\mu=\mu_1*\mu_2$ with $\mu_1\in\mathrm{CE}_+$ and $\mu_2\in\mathrm{ME}_+$.

THEOREM 3. *The class* CME_+ *coincides with the closure of* CME_+^f *in the weak convergence sense.*

PROOF. It is easy to see that every CME_+ distribution is approximated by CME_+^f distributions. So, it is enough to show that the class CME_+ is closed. Let $\mu_n=\mu_n^1*\mu_n^2$, with $\mu_n^1\in\mathrm{CE}_+$ and $\mu_n^2\in\mathrm{ME}_+$, be a CME_+ distribution converging weakly as $n\to\infty$ to a distribution μ on $\boldsymbol{R}_+$. Regard these distributions as probability measures on $\bar{\boldsymbol{R}}_+=[0, \infty]$. Then $\{\mu_n^1\}_{n\geq 1}$ and $\{\mu_n^2\}_{n\geq 1}$ are relatively compact. Choose a subsequence $\{n'\}$ of natural numbers so that $\mu_{n'}^1$ and $\mu_{n'}^2$ converge weakly to probability measures μ^1 and μ^2 on $\bar{\boldsymbol{R}}_+$, respectively, as $n'\to\infty$. Since $\mu_n^1([0, K])\geqq\mu_n([0, K])\to 1$ uniformly in n as $K\to\infty$, we have $\mu^1(\{\infty\})=0$. Similarly $\mu^2(\{\infty\})=0$. Hence $\mu_{n'}^1$ and $\mu_{n'}^2$ converge weakly to μ^1 and μ^2 respectively as probability measures on $\boldsymbol{R}_+$. By the closedness of CE_+ and ME_+, we have $\mu^1\in\mathrm{CE}_+$ and $\mu^2\in\mathrm{ME}_+$. Hence $\mu=\mu^1*\mu^2\in\mathrm{CME}_+$. Thus CME_+ is closed in the weak convergence sense.

6. Strong unimodality of CME_+^d distributions.

A probability measure μ on $\boldsymbol{R}^1$ is said to be unimodal if there is $a\in\boldsymbol{R}^1$ such that the distribution function of μ is convex on $(-\infty, a)$ and concave on (a, ∞). We say that a probability measure μ on $\boldsymbol{R}^1$ is strongly unimodal if μ is unimodal and its convolution with any umimodal distribution is again unimodal. Ibragimov [3] shows that a probability measure μ on $\boldsymbol{R}^1$ is strongly unimodal if and only if μ is absolutely continuous, its support is an interval and the logarithm of its density is concave on the support. We say that $\mu\in\mathcal{P}(\boldsymbol{R}_+)$ is a CME_+^d distribution (the superscript d stands for discrete) if there are $\gamma\in\boldsymbol{R}_+$ and strictly increasing sequences of extended real numbers $\{a_k\}_{0\leq k\leq p}$, $\{b_k\}_{1\leq k\leq q}$ and $\{c_k\}_{0\leq k\leq r}$ $(p, q, r\leqq\infty)$, satisfying

$$r+1=q<\infty \quad \text{or} \quad r=q\leqq\infty,$$
$$a_0=c_0=0, \qquad a_p=b_q=c_r=\infty,$$
$$\{a_k\}_{1\leq k<p}\cap[\{b_k\}_{1\leq k<p}\cup\{c_k\}_{1\leq k<r}]=\emptyset,$$
$$0<b_k<c_k<b_{k+1} \qquad \text{for} \quad 1\leqq k<r$$

and $\sum_{1\leq k<p} a_k^{-1}<\infty$ such that the Laplace transform of μ is represented as

$$\mathcal{L}\mu(s)=\exp\left[-\gamma s+\int_{(0,\infty)}\{(x+s)^{-1}-x^{-1}\}\{m_1(x)+m_2(x)\}dx\right] \tag{6.1}$$

where

$$m_1(x)=k \qquad \text{on} \quad (a_k, a_{k+1})$$

for $0\leqq k<p$ and

$$\begin{aligned} m_2(x)&=1 \qquad \text{on} \quad (b_k, c_k) \quad \text{for } 1\leqq k<q \quad \text{and} \\ &=0 \qquad \text{on} \quad (c_k, b_{k+1}) \quad \text{for } 0\leqq k<r. \end{aligned}$$

It is easy to see that the Laplace transform of μ is represented by these γ, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ as

$$\mathcal{L}\mu(s)=e^{-\gamma s}\prod_{1\leq k<p}a_k(s+a_k)^{-1}\prod_{1\leq k<q}c_k^{-1}(s+c_k)b_k(s+b_k)^{-1}. \tag{6.2}$$

We regard $c_r^{-1}(s+c_r)=1$ in (6.2). Let $\sigma_1(s)=\prod_{1\leq k<q}c_k^{-1}(s+c_k)b_k(s+b_k)^{-1}$. Then $\sigma_1(s)$ is the Laplace transform of a mixture μ_1 of exponential distributions with a mixing distribution G on $\{b_k\}_{1\leq k<q}\cup\{\infty\}$. We remark that $G(\{\infty\})>0$ if and only if $\prod_{1\leq k<q}c_k/b_k$ converges. The distribution function $F_1(x)$ of μ_1 is represented with this G as

$$F_1(x)=\int_{[b_1,\infty]}(1-e^{-ux})G(du) \qquad \text{for} \quad x>0.$$

In Section 7-3, we will discuss first passage time distributions of Bessel diffusion processes. These distributions are not contained in CME_+^f but are contained in CME_+^d. So, we describe here a necessary and sufficient condition for strong unimodality of CME_+^d distributions for later application.

LEMMA 7. *Let* $\mu\in\mathrm{CME}_+^d$ *with Laplace transform* (6.2). *Assume that* $\gamma=0$ *and* $a_1<\infty$ *in* (6.2). *Then* μ *is absolutely continuous and the density* $f(x)$ *is a* C^∞ *function on* $(0,\infty)$. *If* $a_2=\infty$, *then with the above* G, *the density* $f(x)$ *of* μ *and its derivatives* $f'(x)$ *and* $f''(x)$ *are represented as*

$$f(x)=\int_{[b_1,\infty]}a_1u(u-a_1)^{-1}(e^{-a_1x}-e^{-ux})G(du), \tag{6.3}$$

(6.4) $$f'(x)=\int_{[b_1,\infty]}a_1u(u-a_1)^{-1}(-a_1e^{-a_1x}+ue^{-ux})G(du)$$

and

(6.5) $$f''(x)=\int_{[b_1,\infty]}a_1u(u-a_1)^{-1}(a_1^2e^{-a_1x}-u^2e^{-ux})G(du).$$

The proof is easy and omitted.

LEMMA 8. *Let $\mu\in \mathrm{CME}_+^d$ with Laplace transform* (6.2). *Suppose that $\gamma=0$ and $b_2<a_1<\infty$. Then* (i) *μ has a density $f(x)$ which is of C^∞ on* $(0,\infty)$ *and* (ii) *$f(x)$ and its derivatives $f'(x)$ and $f''(x)$ have the following asymptotics near infinity.*

$$f(x)=C_1e^{-b_1x}+C_2e^{-b_2x}+o(e^{-b_2x}),$$

$$f'(x)=-b_1C_1e^{-b_1x}-b_2C_2e^{-b_2x}+o(e^{-b_2x})$$

and

$$f''(x)=b_1^2C_1e^{-b_1x}+b_2^2C_2e^{-b_2x}+o(e^{-b_2x}).$$

Here C_1 and C_2 are positive constants.

PROOF. If $a_2=\infty$, then the conclusion is the direct consequence of Lemma 7. Suppose that $a_2<\infty$. In this case, by the inversion formula for Laplace transform ([**16**]), we have

$$f(x)=\lim_{T\to\infty}(2\pi i)^{-1}\int_{-iT}^{iT}e^{sx}\mathcal{L}\mu(s)ds.$$

We can apply the residue theorem to get

$$\left|\lim_{T\to\infty}(2\pi i)^{-1}\int_{-iT}^{iT}e^{sx}\mathcal{L}\mu(s)ds-\sum_{k=1}^{2}\operatorname*{residue}_{s=-b_k}[e^{sx}\mathcal{L}\mu(s)]\right|<Me^{-ux}$$

for $x>0$, where $b_2<u<\min\{b_3, a_1\}$ and $M>0$. Hence we have

(6.6) $$f(x)=C_1e^{-b_1x}+C_2e^{-b_2x}+o(e^{-b_2x})$$

where

$$C_1=(s+b_1)\mathcal{L}\mu(s)|_{s=-b_1}\quad\text{and}\quad C_2=(s+b_2)\mathcal{L}\mu(s)|_{s=-b_2}.$$

Note that C_1 and C_2 are positive. Let

$$\sigma_1(s)=\mathcal{L}\mu(s)a_1^{-1}(s+a_1).$$

Then $\sigma_1(s)$ is the Laplace transform of a CME_+^d distribution μ_1. Since $a_2<\infty$, μ_1 is absolutely continuous. The density $f_1(x)$ of μ_1 is of C^∞ class on $(0,\infty)$. We have

(6.7) $$f(x)=a_1\int_0^x e^{-a_1(x-y)}f_1(y)dy.$$

Differentiating (6.7), we have

(6.8) $$f'(x) = -a_1f(x)+a_1f_1(x).$$

In case $a_3=\infty$ apply Lemma 7 and in case $a_3<\infty$ apply to $f_1(x)$ the same argument which is previously used for $f(x)$. In both cases we have

(6.9) $$f_1(x) = C_1'e^{-b_1x}+C_2'e^{-b_2x}+o(e^{-b_2x})$$

with positive constants C_1' and C_2'. By (6.7), we have

(6.10) $$f(x) = a_1(a_1-b_1)^{-1}C_1'e^{-b_1x}+a_1(a_1-b_2)^{-1}C_2'e^{-b_2x}+o(e^{-b_2x})$$

as $x\to\infty$. Therefore, by (6.8)-(6.10), we get

$$C_1 = a_1(a_1-b_1)^{-1}C_1', \qquad C_2 = a_1(a_1-b_2)^{-1}C_2'$$

and

(6.11) $$f'(x) = -b_1C_1e^{-b_1x}-b_2C_2e^{-b_2x}+o(e^{-b_2x})$$

as $x\to\infty$. If $a_3=\infty$, then by Lemma 7

(6.12) $$f_1'(x) = -b_1C_1'e^{-b_1x}-b_2C_2'e^{-b_2x}+o(e^{-b_2x})$$

as $a\to\infty$. If $a_3<\infty$, then by the preceding argument, we have the same result. Differentiating the both sides of (6.8) and applying (6.11) and (6.12), we have

$$f''(x) = b_1^2C_1e^{-b_1x}+b_2^2C_2e^{-b_2x}+o(e^{-b_2x})$$

as $x\to\infty$. This completes the proof.

THEOREM 4. *Let μ be a CME_+^d distribution with the Laplace transform* (6.2). *Then μ is strongly unimodal if and only if $a_1<c_1$ or $a_1=c_1=\infty$.*

PROOF. Without loss of generality we may assume $\gamma=0$. In case $c_1=\infty$, the conclusion is obvious since exponential distributions are strongly unimodal. Hence we may assume that $c_1<\infty$. First, suppose that $a_1<c_1$. We can assume that $a_1<b_1$ interchanging a_1 and b_1 in case $b_1<a_1$. It is enough to show the strong unimodality in case $a_2=\infty$ since exponential distributions are strongly unimodal. In this case we already know that μ is strongly unimodal by [17]. But for completeness we state the proof here in this simpler case. By Lemma 7, μ has a density $f(x)$ which is of C^∞ class on $(0, \infty)$. By (6.3)-(6.5), we have

$$f'(x)^2-f(x)f''(x) = 2^{-1}\iint_{[b_1,\infty]\times[b_1,\infty]} a_1^2uv\{(u-a_1)(v-a_1)\}^{-1}A(u, v, x)G(du)G(dv),$$

where

$$A(u, v, x) = \{(u-a_1)^2e^{-(a_1+u)x}+(v-a_1)^2e^{-(a_1+v)x}-(u-v)^2e^{-(u+v)x}\}.$$

Since $a_1<b_1$, we have

$$\{(u-a_1)(v-a_1)\}^{-1}A(u, v, x)\geqq 2e^{-(u+v)x}.$$

Hence

$$f'(x)^2-f(x)f''(x) \geqq \iint_{[b_1,\infty]\times[b_1,\infty]} a_1^2 uve^{-(u+v)x}G(du)G(dv) \geqq 0$$

for all $x>0$. This shows that $f(x)$ is log concave on $(0, \infty)$ and hence μ is strongly unimodal by Ibragimov's theorem ([**3**]). Next, suppose that $a_1>c_1$. Without loss of generality, we may assume that $b_2<a_1$. By Lemma 8, we have

$$f'(x)^2-f(x)f''(x) = -C_1C_2(b_1-b_2)^2e^{-(b_1+b_2)x}+o(e^{-(b_1+b_2)x})$$

as $x\to\infty$. Hence we have

$$f'(x)^2-f(x)f''(x) < 0 \qquad \text{for large} \quad x>0.$$

The proof when $a_1=\infty$ goes on the same line since the density of discrete mixture of exponential distributions has a Dirichlet series with positive coefficients. The proof is complete.

7. Some examples and applications.

7-1. Strong unimodality of first passage time distributions of birth and death processes. Let $\mu\in\mathrm{CME}_+^{\ell}$ and $\mathcal{L}\mu(s)=W_j(s)/W_k(s)$ where $W_j\in\mathrm{PSN}(j)$ and $W_k\in\mathrm{PSN}(k)$ such that $W_j\lhd W_k$. In this case, Theorem 4 is restated as follows: μ is strongly unimodal if and only if the largest zero of W_j is smaller than the second largest zero of W_k. For $\mu_{m,n+1}$, by (3.3), this is equivalent to the following: $\mu_{m,n+1}$ is strongly unimodal if and only if the largest zero of $Q_{n-m}(s)$ is not less than the largest zero of $P_{m-1}(s)$. Of course in general, it is difficult to solve the equations

$$P_{m-1}(s)=0 \quad \text{and} \quad Q_{n-m}(s)=0.$$

We list the simplest cases as examples.

EXAMPLE 1. Let $m=2$ and $n=3$. Then

$$P_{m-1}(s)=P_1(s)=s+\beta_1+\delta_1=0$$

and

$$Q_{n-m}(s)=Q_1(s)=s+\beta_3+\delta_3=0.$$

Therefore, $\mu_{2,4}$ is strongly unimodal if and only if $\beta_3+\delta_3\leqq\beta_1+\delta_1$.

EXAMPLE 2. Let $m=3$ and $n=5$. Then

$$P_{m-1}(s)=P_2(s)=(s+\beta_2+\delta_2)(s+\beta_1+\delta_1)-\delta_2\beta_1=0 \tag{7.1}$$

and

$$Q_{n-m}(s)=Q_2(s)=(s+\beta_4+\delta_4)(s+\beta_5+\delta_5)-\delta_5\beta_4=0. \tag{7.2}$$

The roots of the equation (7.1) are

$$-(\beta_1+\beta_2+\delta_1+\delta_2)\pm[(\beta_1+\beta_2+\delta_1+\delta_2)^2-4\{(\beta_1+\delta_1)(\beta_2+\delta_2)-\delta_2\beta_1\}]^{1/2}$$

and the roots of the equation (7.2) are

$$-(\beta_4+\beta_5+\delta_4+\delta_5)\pm[(\beta_4+\beta_5+\delta_4+\delta_5)^2-4\{(\beta_4+\delta_4)(\beta_5+\delta_5)-\delta_5\beta_4\}]^{1/2}.$$

Therefore, $\mu_{3,6}$ is strongly unimodal if and only if

$$\begin{aligned}&-(\beta_1+\beta_2+\delta_1+\delta_2)+[(\beta_1+\beta_2+\delta_1+\delta_2)^2-4\{(\beta_1+\delta_1)(\beta_2+\delta_2)-\delta_2\beta_1\}]^{1/2}\\ &\leqq -(\beta_4+\beta_5+\delta_4+\delta_5)+[(\beta_4+\beta_5+\delta_4+\delta_5)^2-4\{(\beta_4+\delta_4)(\beta_5+\delta_5)-\delta_5\beta_4\}]^{1/2}.\end{aligned}$$

7-2. Relations between CME_+ **and other classes of infinitely divisible distributions.** We call the smallest subclass of $\mathcal{P}(\boldsymbol{R}_+)$ which contains gamma distributions and is closed under convolutions and weak convergence as Thorin class. We denote by T the Thorin class. It is known ([**1**]) that a probability measure μ $(=[\gamma, M])$ belongs to the Thorin class if and only if it belongs to BO, the measure M is absolutely continuous and the density $m(x)$ is non-decreasing. The names Thorin class and Bondesson class are due to Kent [**7**]. We see that

$$\mathrm{BO}\supset \mathrm{CME}_+\supset T\supset \mathrm{CE}_+.$$

A probability measure μ on $\boldsymbol{R}_+$ belongs to the class L if μ is infinitely divisible, the Lévy measure $N(dx)$ is absolutely continuous and the density $n(x)$ is represented as

$$n(x)=x^{-1}k(x)$$

with nondecreasing $k(x)$ (see [**2**]). Choosing non differentiable k, we get that $L\not\subset\mathrm{BO}$. Let

$$k(x)=e^{-ax}-e^{-bx}+e^{-cx}$$

with $0<a<b<c$. Since

$$x^{-1}k(x)=\int e^{-xu}m(u)du$$

where

$$\begin{aligned} m(u)&=0 && \text{for}\quad 0<u<a,\\ &=1 && \text{for}\quad a\leqq u<b,\\ &=0 && b\leqq u<c\quad\text{and}\\ &=1 && c\leqq u,\end{aligned}$$

the distribution μ with Lévy measure $x^{-1}k(x)dx$ belongs to CME_+. But if we choose a, b and c appropriately, then $k'(x)>0$ for some $x>0$. Let us show this fact. Differentiating $k(x)$, we have

$$k'(x)=-ae^{-ax}+be^{-bx}-ce^{-cx}.$$

Fix a and b so that $2a<b$. Put

$$x=(c-a)^{-1}\log(c/a)>0.$$

Then, for this x,

$$ce^{-cx}=ae^{-ax}$$

and

$$-k'(x)=\exp[-b(c-a)^{-1}\log(c/a)]\{2a\exp[(b-a)(c-a)^{-1}\log(c/a)]-b\}.$$

Since

$$(b-a)(c-a)^{-1}\log(c/a)\longrightarrow 0 \quad \text{as} \quad c\to\infty,$$

there is $x>0$ such that $k'(x)>0$ by the choice of a and b. Therefore $\mathrm{CME}_+\not\subset L$. It is obvious that $T\subset L$.

7-3. Location of zeros of Bessel functions with order $\alpha>-1$. Let α be a real number greater than -1. Let

$$J_\alpha(z)=(z/2)^\alpha\sum_{n=0}^{\infty}(-1)^n(z/2)^{2n}/(n\,!\,\Gamma(\alpha+n+1))$$

be the Bessel function with order α. Let

$$\begin{aligned}I_\alpha(z)&=e^{-i\alpha\pi/2}J_\alpha(e^{i\pi/2}z)\\&=\sum_{n=0}^{\infty}(z/2)^{\alpha+2n}/(n\,!\,\Gamma(\alpha+n+1))\end{aligned}$$

and

$$K_\alpha(z)=2^{-1}\pi\{I_{-\alpha}(z)-I_\alpha(z)\}/\sin\alpha\pi.$$

Let $\sigma^\alpha_{ab}(s)$ be the Laplace transform of the first passage time distribution μ^α_{ab} from a to b of the diffusion process, called Bessel process, determined by the differential operator

$$2^{-1}\left\{\frac{d^2}{dx^2}+(2\alpha+1)x^{-1}\frac{d}{dx}\right\}$$

with reflecting boundary condition at 0 in the case $-1<\alpha<0$ (in the case $\alpha\geqq 0$, the boundary 0 is an entrance boundary). In Kent [6], the following are found:

(i) If $0<a<b$, then

$$\sigma^\alpha_{ab}(s)=(b/a)^\alpha I_\alpha(a(2s)^{1/2})/I_\alpha(b(2s)^{1/2}). \tag{7.3}$$

(ii) If $0<b<a$, then

$$\sigma^\alpha_{ab}(s)=(b/a)^\alpha K_\alpha(a(2s)^{1/2})/K_\alpha(b(2s)^{1/2}).$$

Moreover in the case (ii), the first passage time distribution belongs to the Thorin class (see [1]). It is known that if $\alpha>-1$, then all zeros of $z^{-\alpha}J_\alpha(z)$ are real (see Watson [15] p. 483). Let $\{j_{\alpha,n}\}_{n\geq 1}$ be the set of positive zeros of

$z^{-\alpha}J_\alpha(z)$ arranged in ascending order in magnitude. We have

$$J_\alpha(z)=\Gamma(\alpha+1)^{-1}(z/2)^\alpha\prod_{n=1}^\infty(1-z^2/j_{\alpha,n}^2). \tag{7.4}$$

In the case (i), we rewrite (7.3) and get

$$\begin{aligned}\sigma_{ab}^\alpha(s)&=(b/a)^\alpha J_\alpha(e^{i\pi/2}a(2s)^{1/2})/J_\alpha(e^{i\pi/2}b(2s)^{1/2})\\ &=\prod_{n=1}^\infty(1+2sa^2/j_{\alpha,n}^2)\Big/\prod_{n=1}^\infty(1+2sb^2/j_{\alpha,n}^2).\end{aligned}$$

The poles of $\sigma_{ab}^\alpha(s)$ are

$$-2^{-1}(j_{\alpha,n}/b)^2 \qquad \text{for} \quad n=1, 2, \cdots.$$

The zeros of $\sigma_{ab}^\alpha(s)$ are

$$-2^{-1}(j_{\alpha,n}/a)^2 \qquad \text{for} \quad n=1, 2, \cdots.$$

Since, for any $0<a<b$, $\mu_{ab}^\alpha\in\mathrm{CME}_+$ by Corollary 2 of Theorem 2, there are Laplace transforms $\sigma_1(s)$ and $\sigma_2(s)$ of a CE_+ and an ME_+ distribution respectively such that

$$\sigma_{ab}^\alpha(s)=\sigma_1(s)\sigma_2(s).$$

Note that $\sigma_2(s)=\sigma_{ab}^\alpha(s)/\sigma_1(s)$ is a meromorphic function whose poles are located on the negative real line. We can apply the residue theorem to the inversion formula for Stieltjes transforms and we get that the mixing distribution of $\sigma_2(s)$ is discrete. Hence it can be shown that the meromorphic function $\sigma_1(s)\sigma_2(s)$ is the Laplace transform of a CME_+^d distribution. Detailed proof of this fact will be published elsewhere.

PROPOSITION. *For $k\geqq 1$, following inequality holds:*

$$j_{\alpha,k+1}/j_{\alpha,k}\geqq j_{\alpha,k+2}/j_{\alpha,k+1}. \tag{7.5}$$

PROOF. Suppose that for some integer k

$$j_{\alpha,k+1}/j_{\alpha,k}<j_{\alpha,k+2}/j_{\alpha,k+1}.$$

Then we can choose $0<a<b$ so that

$$j_{\alpha,k+1}/j_{\alpha,k}<b/a<j_{\alpha,k+2}/j_{\alpha,k+1}.$$

By the property of CME_+^d distributions $\sigma_{ab}^\alpha(s)$ has a pole between any adjoining zeros of $\sigma_{ab}^\alpha(s)$. Hence, if

$$j_{\alpha,k+1}/j_{\alpha,k}<b/a \quad \text{i.e.,} \quad (j_{\alpha,k+1}/b)^2<(j_{\alpha,k}/a)^2$$

then

$$(j_{\alpha,k+2}/b)^2\leqq(j_{\alpha,k+1}/a)^2 \quad \text{i.e.,} \quad b/a\geqq j_{\alpha,k+2}/j_{\alpha,k+1}.$$

This is a contradiction. Thus we have (7.5). The proof is complete.

Lorch [10] proved (7.5) without the equality sign. So, our result is not new but the proof is entirely different and it may be of some interest.

EXAMPLE 3. If $\alpha=-1/2$ then the process is a one-dimensional Brownian motion on $[0, \infty)$ with reflecting boundary at the origin. In this case,

$$I_{-1/2}(x) = 2^{-1}(e^{x}+e^{-x})(2/\pi x)^{1/2}$$

and

$$\sigma_{ab}^{-1/2}(s) = \cos(a(-2s)^{1/2})/\cos(b(-2s)^{1/2})$$

for $0<a<b$. The zeros of $\cos(a(-2s)^{1/2})$ are

$$-(n+1/2)^2\pi^2/2a^2 \qquad \text{for} \quad n=0, 1, 2, \cdots$$

and the zeros of $\cos(b(-2s)^{1/2})$ are

$$-(n+1/2)^2\pi^2/2b^2 \qquad \text{for} \quad n=0, 1, 2, \cdots.$$

For any $n\geqq 0$, there is a positive integer m such that

$$(b/a)(n+1/2)-1/2 < m < (b/a)(n+3/2)-1/2.$$

We have

$$(n+1/2)^2\pi^2/2a^2 < (m+1/2)^2\pi^2/2b^2 < (n+3/2)^2\pi^2/2a^2.$$

This shows that there is a zero of $\cos(b(-2s)^{1/2})$ between zeros of $\cos(a(-2s)^{1/2})$. By Theorem 4, in order that $\mu_{ab}^{-1/2}$ be strongly unimodal, it is necessary and sufficient

$$(3/2)^2\pi^2/2b^2 \leqq (1/2)^2\pi^2/2a^2,$$

equivalently, $b\geqq 3a$.

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Appendix.

Let

$$0 = b_0 > a_1 > \cdots > b_m > a_{m+1} > b_{m+1} = -\infty.$$

Let

$$g_m(s) = \prod_{k=1}^{m}(s-b_k), \qquad g_{m+1}(s) = \prod_{k=1}^{m+1}(s-a_k)$$

and set

$$U(s) = g_{m+1}(s)/g_m(s).$$

Then for $1 \leqq k \leqq m+1$

$$\begin{aligned} U(s) &> 0 \quad \text{for} \quad a_k < s < b_{k-1}, \\ &= 0 \quad \text{for} \quad s = a_k, \\ &< 0 \quad \text{for} \quad b_k < s < a_k, \end{aligned}$$

and $U'(s) > 0$ *for* $s \neq b_1, b_2, \cdots, b_m$.

PROOF. Differentiating $\log U(s)$ with respect to $s \neq b_1, b_2, \cdots, b_m$, we have

$$U'(s) = U(s)\left\{\sum_{k=1}^{m+1}(s-a_k)^{-1} - \sum_{k=1}^{m}(s-b_k)^{-1}\right\}.$$

Let $1 \leqq k \leqq m+1$. If $a_k < s < b_{k-1}$, then $(-1)^k g_m(s)$ and $(-1)^k g_{m+1}(s)$ are negative and

$$(s-a_1)^{-1} > (s-b_1)^{-1}, \cdots, (s-a_m)^{-1} > (s-b_m)^{-1}, (s-a_{m+1})^{-1} > 0.$$

Hence $U(s) > 0$ and $U'(s) > 0$. If $b_k < s < a_k$, then $(-1)^k g_m(s)$ is negative, $(-1)^k g_{m+1}(s)$ is positive and

$$(s-a_1)^{-1} < 0,$$

$$(s-a_2)^{-1} < (s-b_1)^{-1}, \cdots, (s-a_{m+1})^{-1} < (s-b_m)^{-1}.$$

Hence $U(s) < 0$ and $U'(s) > 0$. If $s = a_k$, then $(-1)^k g_m(s) < 0$, $g_{m+1}(s) = 0$ and $(-1)^k \prod_{j \neq k}(a_k - a_j) < 0$. Hence we have

$$U(a_k) = 0$$

and

$$U'(a_k) = \prod_{j \neq k}(a_k - a_j)/g_m(a_k) > 0.$$

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