J. Math. Soc. Japan
Vol. 31, No. 2, 1979

Weil's representations of the symplectic groups over finite fields*)

By Hiroyuki YOSHIDA

(Received May 15, 1978)

Introduction.

Let $\boldsymbol{F}(q)$ be the finite field with q elements where q is odd. Suppose that there is given a $2n\times 2n$ symmetric matrix S whose entries are in $\boldsymbol{F}(q)$ such that $\det S\neq 0$. Let $O_1(S)$ denote the special orthogonal group with respect to S and $Sp(2m)$ denote the symplectic group of genus m. We consider $O_1(S)$ and $Sp(2m)$ as connected semisimple algebraic groups defined over $\boldsymbol{F}(q)$ endowed with the Frobenius map F. Let $M_{2n,m}(\boldsymbol{F}(q))$ be the set of all $2n\times m$ matrices with entries in $\boldsymbol{F}(q)$ and $\mathcal{S}(M_{2n,m}(\boldsymbol{F}(q)))$ be the space of all complex valued functions on $M_{2n,m}(\boldsymbol{F}(q))$. Then we can construct, associated with S, so called Weil's representation $\pi_{S,m}$ of $Sp(2m)^F$ realized on $\mathcal{S}(M_{2n,m}(\boldsymbol{F}(q)))$. The representation $\pi_{S,m}$ can be decomposed naturally according to representations of $O_1(S)^F$. Thus we have a correspondence from the set of the equivalence classes of all representations of $O_1(S)^F$ to that of $Sp(2m)^F$. For a representation ρ of $O_1(S)^F$, let $\pi_{S,m}(\rho)$ denote the representation of $Sp(2m)^F$ which corresponds to ρ.

The purpose of this paper is to get some insight about the nature of this correspondence in the case $m=n$. A natural parametrization of most of the irreducible representations of $O_1(S)^F$ and $Sp(2n)^F$ is available from the work of Deligne-Lusztig [4]. In their paper, for an arbitrary connected reductive algebraic group G defined over $\boldsymbol{F}(q)$, a maximal F-stable torus T and a character θ of T^F, a virtual representation R_T^θ of G^F is constructed. Moreover it is shown that any irreducible representation of G^F occurs as a constituent of some R_T^θ and that $(-1)^{\sigma(G)-\sigma(T)}R_T^\theta$ is an irreducible representation if θ is in general position, where $\sigma(G)$ and $\sigma(T)$ denote the $\boldsymbol{F}(q)$-rank of G and T respectively. Now let T be a maximal F-stable torus of $O_1(S)$. Then there exists a maximal F-stable torus T' of $Sp(2n)$ such that T is isomorphic to T' over $\boldsymbol{F}(q)$ as algebraic tori. We fix the isomorphism between T^F and T'^F, which is similar to that between $T_0{}^F$ and $T_1{}^F$ given in § 2. Let θ be a character of T^F which

*) This work was partially supported by the Sakkokai Foundation.

is in general position in $Sp(2n)$ when we regard θ as a character of T'^F. We write $\pi_{S,n}$ as π_S. Then we naturally have

CONJECTURE. *$(-1)^{\sigma(O_1(S))-\sigma(T)}R_T^\theta(O_1(S)^F)$ corresponds to $(-1)^{n-\sigma(T')}R_{T'}^\theta(Sp(2n)^F)$ via Weil's representation π_S.*

In this paper, we shall settle this conjecture for some specific torus T. To be more precise, consider a map $f(x)=\mathrm{Tr}_{F(q^n)/F(q)}(N_{F(q^{2n})/F(q^n)}(x))$ from $\boldsymbol{F}(q^{2n})$ to $\boldsymbol{F}(q)$, where $\mathrm{Tr}_{F(q^n)/F(q)}$ (resp. $N_{F(q^{2n})/F(q^n)}$) denotes the trace (resp. the norm) map from $\boldsymbol{F}(q^n)$ to $\boldsymbol{F}(q)$ (resp. from $\boldsymbol{F}(q^{2n})$ to $\boldsymbol{F}(q^n)$). Then $f(x)$ defines a quadratic form on $\boldsymbol{F}(q)^{2n}$ when we fix an isomorphism $\boldsymbol{F}(q^{2n})\simeq\boldsymbol{F}(q)^{2n}$ as vector spaces over $\boldsymbol{F}(q)$. Let S be the $2n\times 2n$ symmetric matrix which corresponds to this quadratic form. The Witt index of S over $\boldsymbol{F}(q)$ is $n-1$, i.e. $O_1(S)^F$ is of non-split type. Put $H=\{x\in\boldsymbol{F}(q^{2n})\,|\,N_{F(q^{2n})/F(q^n)}(x)=1\}$. Then there exist maximal F-stable tori T_0 and T_1 of $O_1(S)$ and $Sp(2n)$ respectively such that T_0 and T_1 are isomorphic over $\boldsymbol{F}(q)$ as algebraic tori and that $T_0{}^F$ is isomorphic to H. We assume that $\theta\in H^\vee$ corresponds to a character in general position of T_1^F and $q>3$ if $n=2$ or 3 for some technical reasons. Then our Theorem 2 asserts that $(-1)^{n-1}R_{T_0}^\theta$ corresponds to $(-1)^nR_{T_1}^\theta$.

The method of our proof is as follows. In § 2, we compute the character of $\pi_S((-1)^{n-1}R_{T_0}^\theta))$ on regular elements of T_1^F, using Gaussian sums (Theorem 1). This character formula together with the formula (7.6.2) of Deligne-Lusztig [4] give us the information that $\pi_S((-1)^{n-1}R_{T_0}^\theta)$ must contain $(-1)^nR_{T_1}^{\theta'}$ as a constituent for some character θ' of T_1^F which is also in general position. From this fact, we can prove that if $\dim\pi_S((-1)^{n-1}R_{T_0}^\theta)=|Sp(2n)^F|/|T_1^F|q^{n^2}$, we may take $\theta'=\theta$ (Proposition 4). In § 3, we give an expression for $\dim\pi_S((-1)^{n-1}R_{T_0}^\theta)$ in terms of the Green function of T_0 and the number $A_r^{(n)}$ of unipotent elements u of $O_1(S)^F$ such that $\mathrm{rank}(1-u)=2r$ $(0\leqq r\leqq n-1)$. The Green function of T_0 and the number $A_r^{(n)}$ are calculated in Lusztig's paper [8] and [9] respectively. Thanks to these formulas, we can prove that $\dim\pi_S((-1)^{n-1}R_{T_0}^\theta)=|Sp(2n)^F|/|T_1^F|q^{n^2}$.

In § 1, we give a simple proof of the existence of Weil's representations. M. Saito [11] has given a proof of the same nature verifying many complicated relations among generators of the symplectic group. Our proof uses certain limit argument when the base field is local. Then it is sufficient for us to verify essentially one type of relations (i.e. relations of type (iii), cf. the proof of Proposition 1). From Weil's representations π in the case where the base field is local, we obtain Weil's representations of the symplectic groups over finite fields or finite rings obtained from a local field, by analyzing the restriction of the representations π to a maximal compact subgroup. (We should note that our method does not apply to the case where the number of variables of a quadratic form is odd and the base field is finite. This case is included in

[11]).

Assume that the base field is finite. If $n=m=1$, a complete decomposition of Weil's representations is contained in S. Tanaka [16]. When $n=m=2$, a complete decomposition of Weil's representations is obtained by J.S. Andrade [1], [2], [3]. Thus our Theorem 2 is a special case of their results if $n\leqq 2$. When the base field is $\boldsymbol{R}$ and the orthogonal group is compact and $n=m$, S. Gelbart [5] clarifies the correspondence between representations of the orthogonal group and the symplectic group. The result of this paper may be regarded as an $\boldsymbol{F}(q)$-analogue of [5], although our starting point was certain numerical data which seem to suggest the importance of the quadratic forms of the type $x\rightarrow \mathrm{Tr}_{F/\boldsymbol{Q}}(N_{K/F}(x))$ in the theory of Siegel's modular forms of genus 2, where K is a totally imaginary cyclic quartic extension of $\boldsymbol{Q}$ and F is the real quadratic subfield of K. The author would like to discuss this topic in his subsequent paper. We should also mention a paper of R. Howe [6], where quite general conjectures about correspondences between representations of "dual reductive pairs" are formulated. The conjecture stated above would be considered as a slightly sharpened special case of his conjecture.

The author would like to express his hearty thanks to Professor H. Hijikata and Dr. T. Asai for many stimulating discussions.

NOTATION. For a commutative ring R with a unit, $M_{n,m}(R)$ denotes the set consisting of all $n\times m$ matrices with entries in R and we abbreviate $M_{m,m}(R)$ to $M_m(R)$. For $A\in M_m(R)$, $\mathrm{Tr}(A)$ and $\det A$ denote the trace and the determinant of A respectively, and ${}^tA$ denotes the transpose of A. We denote by $GL(m, R)$ the group of all invertible elements of $M_m(R)$. We define an element $w(R)$ of $GL(2m, R)$ by $w(R)=\begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$, where 1_m and 0_m denote the identity and the zero matrix of $M_m(R)$ respectively. When no confusion is likely, we write $w(R)$ as $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The symplectic group over R is defined by $Sp(2m, R)=\{x\in GL(2m, R)\mid {}^txwx=w\}$. We usually write $g\in Sp(2m, R)$ as $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are elements of $M_m(R)$. We denote a subgroup $\left\{\begin{pmatrix} a & * \\ 0 & {}^ta^{-1} \end{pmatrix}=g\mid g\in Sp(2m, R),\ a\in GL(m, R)\right\}$ by $B(R)$. For an algebraic group G defined over $\boldsymbol{F}(q)$, F denotes the Frobenius map of G and G^F denotes the group formed by all fixed points of G under F. If G is connected reductive and T is a maximal F-stable torus of G, R_T^θ denotes the virtual representation constructed in Deligne-Lusztig [4], where θ is a character of T^F. Since we deal two groups simultaneously, we sometimes denote R_T^θ by $R_T^\theta(G^F)$ to avoid the confusion. For a finite group H, $H^\vee$ denotes the set of the equivalence classes of all irreducible representations of H over $\boldsymbol{C}$ and $R(H)$ denotes the Grothendieck group of the finite dimensional representations of H over $\boldsymbol{C}$. For $\rho, \tau \in R(H)$, we define the intertwining number $\langle \rho, \tau \rangle \in \boldsymbol{Z}$ by $\langle \rho, \tau \rangle = \sum_{h \in H} \operatorname{Tr} \rho(h) \operatorname{Tr} \bar{\tau}(h)/|H|$, where $\bar{\tau}$ denotes the complex conjugate of τ. For a commutative field F and a finite separable extension K of F, $\operatorname{Tr}_{K/F}$ and $N_{K/F}$ denote the trace and the norm map from K to F respectively.

§ 1. Weil's representations.

In this section, we shall give an elementary proof of the existence of Weil's representations of the symplectic groups over local fields or finite fields or finite rings obtained from local fields. Our proof is based on the "transformation formula" of characters of the second degree, which is given in Weil [17]. First we consider the case of local fields.

LEMMA 1. *Let k be a local field and put $B=B(k)$. Then we have $Sp(2m, k) = BwBwB$.*

PROOF. We can give a similar proof to that of Lemma 2 of Shimura [13]. Namely, put $Y=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2m, k) \mid \det c \neq 0\right\}$. Then Y is the set of k-rational points of a Zariski open subset defined over k of the symplectic group and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Y$ can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -{}^t c^{-1} & -a \\ 0 & -c \end{pmatrix} w \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} \in BwB .$$

By Lemma 6.1 of Shimura [12], any $g \in Sp(2m, k)$ can be written as $g=y_1 y_2$ with $y_1, y_2 \in Y$, hence our assertion.

Let ψ be a non-trivial additive character of k and $S \in M_{2n}(k)$ be a symmetric matrix such that $\det S \neq 0$. We put $V=M_{2n,m}(k)$ and consider V as a locally compact abelian group. For $x \in V$, define an $m \times m$-matrix $Q(x)$ by $Q(x)={}^t x S x$. We define a character of the second degree f of V by $f(x)=\psi(\operatorname{Tr}(Q(x)))$ for $x \in V$ and a self duality $\langle\, ,\rangle$ of V by $\langle x, y \rangle = f(x+y) f(x)^{-1} f(y)^{-1}$. Let dx denote the selfdual measure of V and $\mathcal{S}(V)$ the space of all Schwartz-Bruhat functions on V. For $\Phi \in \mathcal{S}(V)$, the Fourier transform $\Phi^* \in \mathcal{S}(V)$ is defined by $\Phi^*(x)=\int_V \Phi(y) \langle x, y \rangle dy$. For $a \in GL(m, k)$, let $|a|_V$ denote the module of the automorphism $x \to xa$ of V. The "transformation formula" of Weil (cf. Weil [17], p. 162, Corollary 1) can be read as follows.

$$(1) \qquad \int (\Phi * f)(y) \langle x, y \rangle dy = \gamma(f) \Phi^*(x) f(x)^{-1} \qquad \text{for any } \Phi \in \mathcal{S}(V).$$

Here $\gamma(f)$[1] is a complex number which depends only on f (i. e. on S and ψ) such that $|\gamma(f)|=1$ and the convolution $\Phi*f$ is defined by

$$(2)\qquad (\Phi*f)(y)=\int\Phi(u)f(y-u)\,du\,.$$

We assume, for simplicity, that the characteristic of k is not 2. We shall associate a character $\omega(S)$ of $k^{\times}$ to S. If $(-1)^{n}\det S$ is a square in $k^{\times}$, we put $\omega(S)=1$. If $(-1)^{n}\det S$ is not a square in $k^{\times}$, define a separable quadratic extension K of k by $K=k(\sqrt{(-1)^{n}\det S})$. We take $\omega(S)$ as the nontrivial character of $k^{\times}$ which is trivial on $N_{K/k}(K^{\times})$.

PROPOSITION 1. *Let the notation and the assumption be as above. Put* $\gamma=\gamma(f)$ *and* $\omega=\omega(S)$. *Then there exists a unique representation* π *of* $Sp(2m, k)$ *on* $\mathcal{S}(V)$ *which satisfies the following* 1)$\sim$3) *for any* $\Phi\in\mathcal{S}(V)$.

1) $\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\Phi(x)=\psi(\mathrm{Tr}(bQ(x)))\Phi(x)$,

2) $\pi\begin{pmatrix}a & 0\\ 0 & {}^{t}a^{-1}\end{pmatrix}\Phi(x)=\omega(\det a)|a|_{V}^{1/2}\Phi(xa)$,

3) $\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi(x)=\gamma\Phi^{*}(x)$.

PROOF. By Lemma 1, $Sp(2m, k)$ is generated by the elements of the form $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}a & 0\\ 0 & {}^{t}a^{-1}\end{pmatrix}$ and w. Hence the uniqueness of π is obvious. Put $B=B(k)$. It is easy to verify that a representation of B on $\mathcal{S}(V)$ is defined by 1) and 2). Therefore it is sufficient to show that the relations between the elements of B and w are preserved by π. We first find the following relations, as in the case of $SL(2, k)$.

(i) $w\begin{pmatrix}a & 0\\ 0 & {}^{t}a^{-1}\end{pmatrix}=\begin{pmatrix}{}^{t}a^{-1} & 0\\ 0 & a\end{pmatrix}w$, $\quad a\in GL(m, k)$.

(ii) $w^{2}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.

(iii) $w\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}w=\begin{pmatrix}-b^{-1} & 0\\ 0 & -b\end{pmatrix}\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & -b^{-1}\\ 0 & 1\end{pmatrix}$, $\quad b\in GL(m, k)$, ${}^{t}b=b$.

1) From the definition of $\gamma(f)$, we can verify easily that $\gamma(f)$ remains invariant when we replace f by $f'=\psi(\mathrm{Tr}({}^{t}x{}^{t}aSax))$ for $a\in GL(2n, k)$, if we make the obvious modifications of the self duality and the selfdual measure according to this replacement.

We can see easily that (i) is preserved by π if and only if

$$\omega^2=1 \tag{3}$$

and (ii) is preserved by π if and only if

$$\gamma^2=(\omega(-1))^m. \tag{4}$$

By the definition of ω, (3) is obvious. A verification of (iii) can be done along the similar line to that in Jacquet-Langlands [7], § 1. By the transformation formula (1), we can obtain (iii) for $b=1_m$ with a little computation. To get (iii) for any $b\in GL(m, k)$ such that ${}^tb=b$, we take a character of the second degree $f'(x)=\psi(\mathrm{Tr}(bQ(x)))$ of V and replace the self-duality of V and the selfdual measure of V according to the change $f\rightarrow f'$. The formula (1) holds under this modification with another constant $\gamma(f')$. Using this formula for f', we can see that the relation (iii) for b is equivalent to

$$\gamma(f')/\gamma(f)=\omega(\det b), \tag{5}$$

if we assume (4). Let f_0 be a character of the second degree of $V_0=k^{2n}$ defined by $f_0(x)=\psi({}^txSx)$, $x\in V_0$. By Proposition 3 of [17], we have $\gamma(f)=\gamma(f_0)^m$. We can take $c\in GL(m, k)$ so that $cb{}^tc=\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{pmatrix}$, where $d_i\in k^\times$ $(1\leqq i\leqq m)$. Then we see, by changing the basis of V, the character of the second degree f' can be written as $f'(x_1, x_2, \cdots, x_m)=\prod_{i=1}^m f_0(d_ix)$, where $x_i\in V_0$ is identified with the i-th column vector of x[2]. Therefore we get $\gamma(f')=\prod_i \gamma(f_i)$, where f_i is a character of the second degree of V_0 defined by $f_i(x)=f_0(d_ix)$. We shall show that

$$\gamma(f_i)=\omega_0(d_i)\gamma(f_0),\ \gamma(f_0)^2=\omega_0(-1) \quad \text{and} \quad \omega=\omega_0, \tag{6}$$

with a character ω_0 of $k^\times$. Assuming (6), we obtain (4) and (5). To prove (6), we decompose V_0 into a direct sum of 2-dimensional subspaces U_j; $V_0=\bigoplus_{j=1}^n U_j$ so that we can write $f_0(y)=g_1(y_1)g_2(y_2)\cdots g_n(y_n)$ if $y=(y_1, y_2, \cdots, y_n)$, $y_j\in U_j$, where g_j is a character of the second degree of U_j. We may assume that g_j is written as $g_j(y_j)=\psi({}^ty_jS_jy_j)$, $S_j\in M_2(k)$, ${}^tS_j=S_j$, $\det S_j\neq 0$. Put $g_j'(y_j)=\psi(d_i{}^ty_jS_jy_i)$. We have $\gamma(f_0)=\prod_j \gamma(g_j)$ and $\gamma(f_i)=\prod_j \gamma(g_j')$. In Jacquet-Langlands [7], § 1, it is shown that

2) We replace $f'(x)=\psi(\mathrm{Tr}(b{}^txSx))$ by $f''(x)=\psi(\mathrm{Tr}(b{}^t(xc)S(xc)))=\psi(\mathrm{Tr}(cb{}^tc{}^txSx))$. Then we can verify that $\gamma(f')=\gamma(f'')$ if we modify the self-duality and the selfdual measure according to the change $f'\rightarrow f''$.

(7) $$\gamma(g_j')=\omega_j(d_j)\gamma(g_j),\quad \gamma(g_j)^2=\omega_j(-1),$$

where ω_j is a character of $k^\times$ such that $\omega_j^2=1$. Put $\omega_0=\prod_{j=1}^{n}\omega_j$. Then by (7), the first two equalities of (6) are satisfied. We shall show that $\omega=\omega_0$ i.e. $\omega_0=\omega(S)$. For this, we remind that ω_j is determined in the following way (cf. p. 4~6 of [7]). We may write S_j as $S_j=\begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}$, $a_j, b_j\in k^\times$. If $-a_j/b_j$ is a square in $k^\times$, then $\omega_j=1$. If $-a_j/b_j$ is not a square, then ω_j is the character which corresponds to the separable quadratic extension $k(\sqrt{-a_j/b_j})$ of k by local class field theory. We can verify easily that if two characters α and β correspond to separable quadratic extensions $k(\sqrt{x})$ and $k(\sqrt{y})$ respectively, then $\alpha\beta$ corresponds to the extension $k(\sqrt{xy})$, where $x, y\in k^\times$. Hence we have $\omega_0=\omega(S)$. Therefore the relation (iii) is verified.

By Lemma 1, it is clear that any relation among the elements of B and w can be reduced to the followings;

(iv) $b_1wb_2wb_3=wb_4w$, $b_i\in B$ $(1\leqq i\leqq 4)$,

(v) $wb_1wb_2w=b_3wb_4wb_5$, $b_i\in B$ $(1\leqq i\leqq 5)$.

First let us consider the relation of type (iv). This is equivalent to $wb_4wb_3^{-1}w^{-1}=b_1wb_2$. Using the relation (i), we may assume that $b_4=\begin{pmatrix} 1 & c_4 \\ 0 & 1 \end{pmatrix}$, $b_3=\begin{pmatrix} 1 & c_3 \\ 0 & 1 \end{pmatrix}$, $c_i\in M_m(k)$, ${}^tc_i=c_i$, $i=3, 4$. Since $w\begin{pmatrix} 1 & c_4 \\ 0 & 1 \end{pmatrix}w\begin{pmatrix} 1 & -c_3 \\ 0 & 1 \end{pmatrix}w^{-1}=\begin{pmatrix} c_3 & 1 \\ -1-c_4c_3 & -c_4 \end{pmatrix}\in BwB$, we have $\det(-1-c_4c_3)\neq 0$. Hence we get

(8) $$w\begin{pmatrix} 1 & c_4 \\ 0 & 1 \end{pmatrix}w\begin{pmatrix} 1 & -c_3 \\ 0 & 1 \end{pmatrix}w^{-1}=\begin{pmatrix} {}^t(1+c_4c_3)^{-1} & -c_3 \\ 0 & 1+c_4c_3 \end{pmatrix}w\begin{pmatrix} 1 & (1+c_4c_3)^{-1}c_4 \\ 0 & 1 \end{pmatrix}$$

and (iv) is reduced to (8) and the relation of the form (*) $b_1wb_2=b_5wb_6$, $b_5, b_6\in B$. The relation (*) can easily be reduced to the relations (i)~(iii). Suppose that $\det c_3\neq 0$. Then we have $w\begin{pmatrix} 1 & -c_3 \\ 0 & 1 \end{pmatrix}w^{-1}\in BwB$ and (8) is reduced to the relations of the form $wb_1'w=b_2'wb_3'$ with $b_1', b_2', b_3'\in B$.

LEMMA 2. *A relation of the form $wb_1w=b_2wb_3$, $b_1, b_2, b_3\in B$ can be derived from the relations* (i), (ii) *and* (iii).

PROOF. Using (i), we may assume that $b_1=\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}$, $c_1\in M_m(k)$, ${}^tc_1=c_1$. We have $\det c_1\neq 0$. Using (iii), we get $w\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}w=b_4wb_5$, $b_4, b_5\in B$. Hence our relation is reduced to $b_4wb_5=b_2wb_3$ i.e., $wb_5b_3^{-1}w^{-1}=b_4^{-1}b_2$. Then we must have $b_5b_3^{-1}=\begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}$, $b_4^{-1}b_2=\begin{pmatrix} {}^ta^{-1} & 0 \\ 0 & a \end{pmatrix}$ with some $a\in GL(m, k)$, hence our assertion.

By Lemma 2, we have seen that the relation (8) is preserved by π if $\det c_3 \neq 0$. Assume that k is non-archimedean. We can easily see that for any $\Phi \in \mathcal{S}(V)$, there exists a positive integer N such that

$$\pi(w)\pi\begin{pmatrix}1 & c_4\\ 0 & 1\end{pmatrix}\pi(w)\pi\begin{pmatrix}1 & -c_3\\ 0 & 1\end{pmatrix}\pi(w^{-1})\Phi=\pi(w)\pi\begin{pmatrix}1 & c_4\\ 0 & 1\end{pmatrix}\pi(w)\pi\begin{pmatrix}1 & -c_3'\\ 0 & 1\end{pmatrix}\pi(w^{-1})\Phi$$

and

$$\pi\begin{pmatrix}{}^t(1+c_4c_3)^{-1} & -c_3\\ 0 & 1+c_4c_3\end{pmatrix}\pi(w)\pi\begin{pmatrix}1 & (1+c_4c_3)^{-1}c_4\\ 0 & 1\end{pmatrix}\Phi$$

$$=\pi\begin{pmatrix}{}^t(1+c_4c_3')^{-1} & -c_3'\\ 0 & 1+c_4c_3'\end{pmatrix}\pi(w)\pi\begin{pmatrix}1 & (1+c_4c_3')^{-1}c_4\\ 0 & 1\end{pmatrix}\Phi$$

if $c_3' \in M_m(k)$, ${}^tc_3'=c_3'$ and $c_3' \equiv c_3 \bmod \varpi^N$, where ϖ denotes a prime element of k. It is easy to see that we can take c_3' so that $c_3' \equiv c_3 \bmod \varpi^N$, ${}^tc_3'=c_3'$ and $\det c_3' \neq 0$. Therefore (8) is proved when k is non-archimedean. If k is archimedean, we define, as usual, the L^2-norm $\|\Phi\|$ of $\Phi \in \mathcal{S}(V)$ by $\|\Phi\|^2=\int_V |\Phi(x)|^2 dx$. Then we can see that the effects of both sides of (8) to Φ under π can be made arbitrarily small with respect to this norm, when we replace c_3 to a sufficiently close c_3', $c_3' \in M_m(k)$, ${}^tc_3'=c_3'$, $\det c_3' \neq 0$ (use Plancherel's formula to show that $\|\pi(w)\Phi\|$ is small if $\|\Phi\|$ is). Hence the relation (8) follows.

Now let us consider the relations of type (v). Although (v) can be derived algebraically from the relations (i)$\sim$(iv), we prefer to use the following simpler argument. Using (i), we may assume that $b_1=\begin{pmatrix}1 & c_1\\ 0 & 1\end{pmatrix}$, $b_2=\begin{pmatrix}1 & c_2\\ 0 & 1\end{pmatrix}$, $b_5=\begin{pmatrix}1 & c_5\\ 0 & 1\end{pmatrix}$. Since we have $w\begin{pmatrix}1 & c_1\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & c_2\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & -c_5\\ 0 & 1\end{pmatrix}w^{-1}=\begin{pmatrix}-1-c_2c_5 & -c_2\\ c_1-c_5+c_1c_2c_5 & -1+c_1c_2\end{pmatrix}\in BwB$, we have $\det(c_1-c_5+c_1c_2c_5) \neq 0$ and we see that the relation (v) can be reduced to

$$(9)\qquad w\begin{pmatrix}1 & c_1\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & c_2\\ 0 & 1\end{pmatrix}w\begin{pmatrix}1 & -c_5\\ 0 & 1\end{pmatrix}w^{-1}$$

$$=\begin{pmatrix}-{}^t(c_1-c_5+c_1c_2c_5)^{-1} & 1+c_2c_5\\ 0 & -(c_1-c_5+c_1c_2c_5)\end{pmatrix}w\begin{pmatrix}1 & (c_1-c_5+c_1c_2c_5)^{-1}(-1+c_1c_2)\\ 0 & 1\end{pmatrix}.$$

Using (iii), we can see that (9) is reduced to (iv) if $\det c_2 \neq 0$. If $\det c_2=0$, we can take $c_2' \in M_m(k)$, ${}^tc_2'=c_2'$, $\det c_2' \neq 0$ so that c_2' is sufficiently close to c_2. Applying the same argument as above, we get (9). This completes the proof of Proposition 1.

REMARK 1. By Proposition 3 of [17] and the results of [7], § 1, we see immediately that $\gamma=\gamma(f)$ is determined in the following way. In the Witt group, the quadratic form $x \to {}^t xSx$ on k^{2n} is equivalent to one of the followings; (i) the norm form of the division quaternion algebra over k, (ii) the norm form of a quadratic extension of k, (iii) $a\times$the norm form of a quadratic extension K of k, where $a\in k^{\times}$, $a\notin N_{K/k}(K^{\times})$, (iv) the trivial form. In the case (i), $\gamma=(-1)^m$. In the case (ii), $\gamma=\lambda(K/k, \psi)^m$ (about λ, see [7], p. 6). In the case (iii), $\gamma=(-1)^m\lambda(K/k, \psi)^m$. In the case (iv), $\gamma=1$.

LEMMA 3. *$Sp(2m, \boldsymbol{F}(q))$ is generated by $B(\boldsymbol{F}(q))$ and w.*

PROOF. We put $G=Sp(2m, \boldsymbol{F}(q))$ and $B=B(\boldsymbol{F}(q))$. Define subgroups A and N of G by $A=\left\{\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix}; a\in GL(m, \boldsymbol{F}(q))\right\}$, $N=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}; b\in M_m(\boldsymbol{F}(q)), {}^t b=b\right\}$. The map defined by the multiplication $A\times N\times N\to ANwN\subset G$ is an injection. Therefore we have $|B\cup BwB|=|A\|N|+|A\|N|^2$. We have $|A|=(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})$, $N=q^{m(m+1)/2}$, $G=q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$. Hence we get $|G|=q^{m^2+m(m+1)}(1-q^{-2})(1-q^{-4})\cdots(1-q^{-2m})$ and $|A\|N|^2=q^{m^2+m(m+1)}(1-q^{-1})(1-q^{-2})\cdots(1-q^{-m})$. If $q\geqq 3$, we have $(1-q^{-1})(1-q^{-2})\cdots(1-q^{-m})>(q-2)/(q-1)\geqq 1/2$. Hence we get $|G|/|BwB|<2$ and G must be generated by B and w. Assume that $q=2$. Then we have $|G|/|BwB|=(1+2^{-1})(1+2^{-2})\cdots(1+2^{-m})<3$. Hence if B and w does not generate G, they must generate a subgroup of G of index 2. Since G is known to be simple if $m>2$, we may assume that $m\leqq 2$. Then we get $|G|/|BwB|\leqq 15/8<2$, hence our assertion.

Let k, $\mathfrak{O}$ and ϖ be a non-archimedean local field, the maximal compact subring of k and a prime element of k respectively. For a positive integer N, let us define a congruence subgroup $\Gamma(N)$ of $Sp(2m, \mathfrak{O})$ by

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in Sp(2m, \mathfrak{O})\,\middle|\, a\equiv d\equiv 1_m \bmod \varpi^N,\ b\equiv c\equiv 0_m \bmod \varpi^N\right\}.$$

LEMMA 4. *$\Gamma(N)$ is generated by the elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}$, $b\equiv 0 \bmod \varpi^N$, ${}^t b=b$, $\begin{pmatrix} 1 & 0 \\ c & 1\end{pmatrix}$, $c\equiv 0 \bmod \varpi^N$, ${}^t c=c$ and $\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix}$, $a\equiv 1 \bmod \varpi^N$.*

PROOF. Let γ be an element of $\Gamma(N)$. We can write $\gamma=\begin{pmatrix} 1+\varpi^N a & \varpi^N b \\ \varpi^N c & 1+\varpi^N d\end{pmatrix}$ with $a, b, c, d\in M_m(\mathfrak{O})$. Put

$$\gamma_1=\begin{pmatrix} {}^t(1+\varpi^N d)^{-1} & \varpi^N b \\ 0 & 1+\varpi^N d\end{pmatrix}=\begin{pmatrix} 1 & \varpi^N b(1+\varpi^N d)^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} {}^t(1+\varpi^N d)^{-1} & 0 \\ 0 & 1+\varpi^N d\end{pmatrix},$$

$$\gamma_2=\begin{pmatrix} 1 & 0 \\ \varpi^N(1+\varpi^N d)^{-1}c & 1\end{pmatrix} \text{ and } \gamma_3=\gamma_1\gamma_2=\begin{pmatrix} {}^t(1+\varpi^N d)^{-1}+\varpi^{2N}b(1+\varpi^N d)^{-1}c & \varpi^N b \\ \varpi^N c & 1+\varpi^N d\end{pmatrix}.$$

Since $\gamma\in Sp(2m, k)$, we have $b(1+\varpi^N d)^{-1}={}^t(1+\varpi^N d)^{-1}\,{}^t b$ and ${}^t(1+\varpi^N a)(1+\varpi^N d)=1+\varpi^{2N}\,{}^t cb$. Hence we have $\gamma_1\in\Gamma(N)$ and $1+\varpi^N a={}^t(1+\varpi^N d)^{-1}(1+\varpi^{2N}\,{}^t bc)={}^t(1+\varpi^N d)^{-1}+\varpi^{2N}\,{}^t(1+\varpi^N d)^{-1}\,{}^t bc={}^t(1+\varpi^N d)^{-1}+\varpi^{2N}b(1+\varpi^N d)^{-1}c$. Therefore we get $\gamma=\gamma_3=\gamma_1\gamma_2$. Hence $\gamma_2=\gamma_1^{-1}\gamma\in\Gamma(N)$ and $(1+\varpi^N d)^{-1}c$ must be symmetric. This completes the proof.

LEMMA 5. *$Sp(2m, \mathfrak{O})$ is generated by $B(\mathfrak{O})$ and $w(\mathfrak{O})$. $Sp(2m, \mathfrak{O}/\varpi^l)$ is generated by $B(\mathfrak{O}/\varpi^l)$ and $w(\mathfrak{O}/\varpi^l)$, where l is a positive integer.*

PROOF. The first assertion follows immediately from Lemma 3 and Lemma 4. The second assertion is an obvious consequence from the first one.

Hereafter in this section, we will assume that the residual characteristic of k is not 2. To get Weil's representation of $Sp(2m, \mathfrak{D}/\varpi^l)$, $l\geqq 1$, we consider the module $V_0=\mathfrak{O}/\varpi^{r_1}\oplus\mathfrak{O}/\varpi^{r_2}\oplus\cdots\oplus\mathfrak{O}/\varpi^{r_{2n}}$, where n is a positive integer and r_i $(1\leqq i\leqq 2n)$ is a positive integer such that $r_i\leqq l$. We take elements $u_i\in(\mathfrak{O}/\varpi^l)^\times$, $1\leqq i\leqq 2n$. For $y=(y_1, y_2, \cdots, y_{2n})$, $y'=(y_1', y_2', \cdots, y_{2n}')$, $y_i\in\mathfrak{O}/\varpi^{r_i}$, $y_i'\in\mathfrak{O}/\varpi^{r_i}$, $y, y'\in V_0$, we define an element $B(y, y')$ of $\mathfrak{O}/\varpi^l$ by $B(y, y')=\sum_{i=1}^{2n} u_i\varpi^{l-r_i}y_iy_i'$. We put $V=V_0^m$. For $a\in GL(m, \mathfrak{O}/\varpi^l)$, we obtain a well defined element $xa\in V$ in an obvious manner. For $x=(x_1, x_2, \cdots, x_m)\in V$, $x_i\in V_0$, we define $Q(x)\in M_m(\mathfrak{O}/\varpi^l)$ by $(Q(x))_{ij}=B(x_i, x_j)$, where $(Q(x))_{ij}$ denotes the ij-entry of $Q(x)$. Let ψ be a character of $\mathfrak{O}/\varpi^l$ such that the restriction of ψ to $\varpi^{l-1}\mathfrak{O}/\varpi^l\mathfrak{O}$ is non-trivial. We denote by $\mathcal{S}(V)$ the vector space of all complex valued functions on V. For $\Phi\in\mathcal{S}(V)$, the Fourier transform Φ^* of Φ is defined by

$$\Phi^*(x)=q^{-(m\sum_i r_i)/2}\sum_{y\in V}\Phi(y)\langle x, y\rangle, \tag{10}$$

where we put $\langle x, y\rangle=\psi(\mathrm{Tr}(Q(x+y)-Q(x)-Q(y)))$ and q denotes the module of k.

For $1\leqq i\leqq 2n$, we take an element $\tilde{u}_i\in\mathfrak{O}^\times$ so that $\tilde{u}_i \bmod \varpi^l=u_i$ and define a matrix $\tilde{S}\in M_{2n}(k)$ by $\tilde{S}=\begin{pmatrix} \tilde{u}_1\varpi^{l-r_1} & & & 0 \\ & \tilde{u}_2\varpi^{l-r_2} & & \\ & & \ddots & \\ 0 & & & \tilde{u}_{2n}\varpi^{l-r_{2n}} \end{pmatrix}$. Then the character $\omega(\tilde{S})=\tilde{\omega}^*$ of $k^\times$ defined before does not depend on the choice of $\tilde{u}_i$'s, since every element $u\in\mathfrak{O}^\times$ such that $u\equiv 1 \bmod \varpi^l$ is a square in $\mathfrak{O}^\times$. Moreover the conductor of $\tilde{\omega}^*$ is not greater than 1. Define a character ω of $(\mathfrak{O}/\varpi^l)^\times$ of order 2 by

$$\omega(u \bmod \varpi^l)=\tilde{\omega}^*(u) \qquad \text{for} \quad u\in\mathfrak{O}^\times. \tag{11}$$

Let $\tilde{\psi}$ be an additive character of k such that $\tilde{\psi}(u)=\psi(u \bmod \varpi^l)$ for any $u\in\mathfrak{O}$. Define a character of the second degree $\tilde{f}$ of $M_{2n,m}(k)$ by $\tilde{f}(x)=\tilde{\psi}(\mathrm{Tr}({}^t x\tilde{S}x))$, $x\in M_{2n,m}(k)$. Then we see easily that $\gamma(\tilde{f})$ does not depend on the choice of $\tilde{u}_i$'s and $\tilde{\psi}$ (cf. Remark 1 and [7], p. 4~6). We put

$$\gamma=\gamma(\tilde{f}). \tag{12}$$

PROPOSITION 2. *Let the notation and the assumption be as above. There exists a unique representation π of $Sp(2m, \mathfrak{O}/\varpi^l)$ on $\mathcal{S}(V)$ which satisfies the following* 1)$\sim$3) *for any* $\Phi\in\mathcal{S}(V)$.

1) $\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\Phi(x)=\psi(\mathrm{Tr}(bQ(x))\Phi(x)$,

2) $\pi\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}\Phi(x)=\omega(\det a)\Phi(xa)$,

3) $\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi(x)=\gamma\Phi^*(x)$,

where ω is a character of $(\mathfrak{O}/\varpi^l)^\times$ *defined by* (11) *and γ is a complex number defined by* (12).

PROOF. Let $\tilde{\pi}$ be the Weil representation of $Sp(2m, k)$ associated with $\tilde{S}$ which is realized on $\mathcal{S}(M_{2n,m}(k))$, whose existence is shown in Proposition 1. Put $\tilde{V}=M_{2n,m}(k)$. Let us define a subspace W of $\mathcal{S}(\tilde{V})$ consisting of all functions $\Phi\in\mathcal{S}(\tilde{V})$ which satisfy the following conditions (i) and (ii).

(i) The support of Φ is contained in $M_{2n,m}(\mathfrak{O})$.

(ii) If (x_{ij}) and $(x'_{ij})\in M_{2n,m}(\mathfrak{O})$ satisfy $x_{ij}\equiv x'_{ij} \bmod \varpi^{r_i}$ for any ij-entries, then $\Phi((x_{ij}))=\Phi((x'_{ij}))$.

First we will verify that W is invariant by $Sp(2m, \mathfrak{O})$. To see this, it is enough to show that W is invariant by the actions of $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, $b\in M_m(\mathfrak{O}), {}^tb=b$, $\begin{pmatrix}a & 0\\ 0 & {}^ta^{-1}\end{pmatrix}$, $a\in GL(m, \mathfrak{O})$ and w by Lemma 5. We take an element $\Phi\in W$. Put $\Psi(x)=\tilde{\psi}(\mathrm{Tr}(b{}^tx\tilde{S}x))\Phi(x)$. It is clear that the support of Ψ is contained in $M_{2n,m}(\mathfrak{O})$. If $x=(x_{ij})$ and $x'=(x'_{ij})\in M_{2n,m}(\mathfrak{O})$ satisfy $x_{ij}\equiv x'_{ij} \bmod \varpi^{r_i}$ (for $\forall i, \forall j$), then we have ${}^tx\tilde{S}x\equiv{}^tx'\tilde{S}x' \bmod \varpi^l$. Hence we get $\mathrm{Tr}(b{}^tx\tilde{S}x)\equiv\mathrm{Tr}(b{}^tx'\tilde{S}x') \bmod \varpi^l$. Therefore we have $\Psi\in W$. Next put $\Psi(x)=\tilde{\omega}^*(\det a)\Phi(xa)$ for $a\in GL(m, \mathfrak{O})$. It is obvious that Ψ satisfies (i) and (ii). We consider the case $\Psi(x)=\int_{\tilde{V}}\Phi(y)\widetilde{\langle x, y\rangle}dy$, where $\widetilde{\langle\,,\,\rangle}$ is the self-duality of $\tilde{V}$ defined by $\widetilde{\langle x, y\rangle}=\tilde{\psi}(\mathrm{Tr}(2{}^tx\tilde{S}y))$ and dy denotes the selfdual measure on $\tilde{V}$. We see that if $x_{ij}\equiv x'_{ij} \bmod \varpi^{r_i}$ for $x=(x_{ij})$ and $x'=(x'_{ij})\in M_{2n,m}(\mathfrak{O})$ (for $\forall i, \forall j$), we have $\widetilde{\langle x, y\rangle}=\widetilde{\langle x', y\rangle}$ for any $y\in M_{2n,m}(\mathfrak{O})$. Therefore Ψ satisfies (ii). Define a submodule M' of $M_{2n,m}(\mathfrak{O})$ by $M'=\{x=(x_{ij})\in M_{2n,m}(\mathfrak{O})\mid x_{ij}\equiv 0 \bmod \varpi^{r_i} \text{ for } \forall i, \forall j\}$. Let $\{y_k\}$ be a complete set of representatives of $M_{2n,m}(\mathfrak{O})/M'$. We have $\int_{M_{2n,m}(\mathfrak{O})}\Phi(y)\widetilde{\langle x, y\rangle}dy=\sum_k\int_{M'}\Phi(y_k)\widetilde{\langle x, y_k+v\rangle}dv=\sum_k\Phi(y_k)\widetilde{\langle x, y_k\rangle}\int_{M'}\widetilde{\langle x, v\rangle}dv$. We can easily see that the map $v\rightarrow\widetilde{\langle x, v\rangle}$ defines a non-trivial character of M' if $x\notin M_{2n,m}(\mathfrak{O})$.

Therefore $\Phi^*(x)=0$ if $x\notin M_{2n,m}(\mathfrak{O})$ and (i) is verified.

It is clear that $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, $b\equiv 0 \bmod \varpi^l$ acts trivially on W. Since the conductor of $\tilde{\omega}^*$ is not greater than 1, $\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}$, $a\equiv 1 \bmod \varpi^l$ acts trivially on W. Hence $\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}=w\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}w^{-1}$ also acts trivially on W if $c\equiv 0 \bmod \varpi^l$. By Lemma 4, $\Gamma(l)$ acts trivially on W. Therefore the restriction of $\tilde{\pi}$ to W defines a representation π' of $Sp(2m, \mathfrak{O})/\Gamma(l)\simeq Sp(2m, \mathfrak{O}/\varpi^l)$. Let ι be the natural isomorphism from W to $\mathcal{S}(V)$, i.e. $\iota(\Phi)((x_{ij}) \bmod \varpi^{r_i}))=\Phi((x_{ij}))$, $(x_{ij})\in M_{2n,m}(\mathfrak{O})$, $\Phi\in W$. By shifting the structure, we obtain a representation π of $Sp(2m, \mathfrak{O}/\varpi^l)$ on $\mathcal{S}(V)$. We can see easily that the volume of $M_{2n,m}(\mathfrak{O})$ measured by dy is equal to $q^{(m\sum_i r_i)/2}$. Then it is clear that π satisfies 1), 2) and 3) and the "Fourier transform" Φ^* of $\Phi\in\mathcal{S}(V)$ is given by (10). The uniqueness of π is obvious by Lemma 5. This completes the proof.

§ 2. A computation of characters on regular semisimple elements.

For a $2n\times 2n$ non-singular symmetric matrix S with entries in $\boldsymbol{F}(q)$, let $O(S)$ denote the orthogonal group with respect to S. Hereafter we assume that q is odd. We put $O_1(S)=\{g\in O(S)\mid \det g=1\}$. We consider $O_1(S)$ and $Sp(2m)$ as connected semisimple algebraic groups defined over $\boldsymbol{F}(q)$ with the Frobenius map F, where m is a positive integer. Let $\mathcal{S}(M_{2n,m}(\boldsymbol{F}(q))$ be the vector space of all complex valued functions on $M_{2n,m}(\boldsymbol{F}(q))$. We put $\varepsilon(S)=1$ if the Witt index of S is n (i.e. $(-1)^n\det S\in(\boldsymbol{F}(q)^\times)^2$) and $\varepsilon(S)=-1$ if the Witt index of S is $n-1$ (i.e. $(-1)^n\det S\notin(\boldsymbol{F}(q)^\times)^2$)[3]. Let ψ be a non-trivial additive character of $\boldsymbol{F}(q)$. We put $V=M_{2n,m}(\boldsymbol{F}(q))$, $\langle x, y\rangle=\psi(\mathrm{Tr}(2{}^t xSy))$ for $x, y\in V$ and $\Phi^*(x)=q^{-mn}\sum_{y\in V}\Phi(y)\langle x, y\rangle$. Then, by Proposition 2, there exists a unique representation π of $Sp(2m)^F\simeq Sp(2m, \boldsymbol{F}(q))$ realized on $\mathcal{S}(M_{2n,m}(\boldsymbol{F}(q))$ which satisfies the following 1)～3) for any $\Phi\in\mathcal{S}(M_{2n,m}(\boldsymbol{F}(q))$.

$$
(13)\qquad
\begin{aligned}
&1)\quad \pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\Phi(x)=\psi(\mathrm{Tr}(b\,{}^t xSx))\Phi(x)\,,\\
&2)\quad \pi\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}\Phi(x)=\Phi(xa)\,,\\
&3)\quad \pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Phi(x)=(\varepsilon(S))^m\Phi^*(x)\,.
\end{aligned}
$$

3) This choice of $\varepsilon(s)$ follows from the fact that $\lambda(K/k,\psi)=-1$ if K is the unramified quadratic extension of a non-archimedean local field k and ψ is an additive character of k such that $\varpi\mathfrak{O}$ is the largest ideal on which ψ is trivial.

We denote π as $\pi=\pi_{S,m}$[4]. Let ρ be a representation of $O_1(S)^F$ whose representation space is W. It is obvious that $\pi_{S,m}$ can be extended to the representation $\pi_{S,m,W}$ of $Sp(2m)^F$ realized on the space of all W-valued functions on V in such a way that $\pi_{S,m,W}$ satisfies (13) for any W-valued function Φ on V. Let $\widetilde{W}(\rho)$ denote the space of all W-valued functions Φ on V which satisfy that $\Phi(gx)=\rho(g)\Phi(x)$ for any $g\in O_1(S)^F$ and any $x\in V$. We have

PROPOSITION 3. *The space $\widetilde{W}(\rho)$ is $\pi_{S,m,W}$-invariant.*

PROOF. We can verify immediately that $\pi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\Phi$, $\pi\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}\Phi$ and $\pi(w)\Phi$ belongs to $\widetilde{W}(\rho)$ if $\Phi\in\widetilde{W}(\rho)$, where $\pi=\pi_{S,m,W}$. Hence our assertion follows by Lemma 3.

Let $\pi_{S,m}(\rho)$ denote the representation of $Sp(2m)^F$ realized on $\widetilde{W}(\rho)$. We are interested in the correspondence $\rho\to\pi_{S,m}(\rho)$ from the set of equivalence classes of all representations of $O_1(S)^F$ to that of $Sp(2m)^F$. This correspondence can be extended to a $\boldsymbol{Z}$-linear homomorphism $\pi_{S,m}$ from $R(O_1(S)^F)$ to $R(Sp(2m)^F)$ by $\pi_{S,m}(\sum_i a_i\rho_i)=\sum_i a_i\pi_{S,m}(\rho_i)$, $\rho_i\in(O_1(S)^F)^\vee$, $a_i\in\boldsymbol{Z}$.

Let ι be an isomorphism $\boldsymbol{F}(q^{2n})\cong\boldsymbol{F}(q)^{2n}$ as vector spaces over $\boldsymbol{F}(q)$ and S denotes the $2n\times 2n$ symmetric non-singular matrix with entries in $\boldsymbol{F}(q)$ defined by

$$\mathrm{Tr}_{\boldsymbol{F}(q^n)/\boldsymbol{F}(q)}(N_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n)}(x))={}^t\iota(x)S\iota(x)\,. \tag{14}$$

It is known that the Witt index of S is $n-1$ i.e. $O(S)^F$ is of non-split type (cf. Lemma 2.2 of Milnor [10]). We put $H=\{x\in\boldsymbol{F}(q^{2n})\,|\,N_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n)}(x)=1\}$. We will give explicit embeddings of H into $O_1(S)^F$ and $Sp(2n)^F$. First H can be embedded into $O(S)^F$ in an obvious manner, since H makes the quadratic form $x\to\mathrm{Tr}_{\boldsymbol{F}(q^n)/\boldsymbol{F}(q)}(N_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n)}(x))$ on $\boldsymbol{F}(q^{2n})\cong\boldsymbol{F}(q)^{2n}$ invariant. By an inspection, we can see that the image of H is contained in $O_1(S)^F$. For an embedding into $Sp(2n)^F$, let us write $\boldsymbol{F}(q^{2n})=\boldsymbol{F}(q^n)(\sqrt{d})$ with $d\in\boldsymbol{F}(q^n)$. Let σ be the non-trivial element of $\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n))$. Define an alternating form $f(x, y)$ on $\boldsymbol{F}(q^{2n})\times\boldsymbol{F}(q^{2n})$ by $f(x, y)=\mathrm{Tr}_{\boldsymbol{F}(q^n)/\boldsymbol{F}(q)}((xy^\sigma-x^\sigma y)/2\sqrt{d})$. It is clear that H makes f invariant. We write $x=x_1+x_2\sqrt{d}$, $y=y_1+y_2\sqrt{d}$ with $x_1, x_2, y_1, y_2\in\boldsymbol{F}(q^n)$. Then we have $f(x, y)=\mathrm{Tr}_{\boldsymbol{F}(q^n)/\boldsymbol{F}(q)}(x_1y_2-x_2y_1)$. Define a symmetric matrix $S_0\in M_n(\boldsymbol{F}(q))$ by $\mathrm{Tr}_{\boldsymbol{F}(q^n)\boldsymbol{F}(q)}(xy)={}^t\iota_0(x)S_0\iota_0(y)$ for $x, y\in\boldsymbol{F}(q^n)$, where ι_0 denotes an isomorphism $\boldsymbol{F}(q^n)\cong\boldsymbol{F}(q)^n$ as vector spaces over $\boldsymbol{F}(q)$. Let τ be the regular representation of $\boldsymbol{F}(q^n)$ into $M_n(\boldsymbol{F}(q))$ with respect to the basis of $\boldsymbol{F}(q^n)$ which determines the isomorphism ι_0. For $\alpha+\beta\sqrt{d}\in H$, $(\alpha, \beta\in\boldsymbol{F}(q^n))$, put $g(\alpha, \beta)=\begin{pmatrix}\tau(\alpha) & \tau(\beta d)\\ \tau(\beta) & \tau(\alpha)\end{pmatrix}$. Then we get ${}^tg(\alpha, \beta)\begin{pmatrix}0 & S_0\\ -S_0 & 0\end{pmatrix}g(\alpha, \beta)=\begin{pmatrix}0 & S_0\\ -S_0 & 0\end{pmatrix}$. Since

---

4) Since S and aS are equivalent for $a\in\boldsymbol{F}(q)^\times$, the equivalence class of $\pi_{S,m}$ does not depend on the choice of ψ.

${}^t\begin{pmatrix} S_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & S_0 \\ -S_0 & 0 \end{pmatrix}\begin{pmatrix} S_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}=w$, we have $\begin{pmatrix} S_0 & 0 \\ 0 & 1 \end{pmatrix}g(\alpha, \beta)\begin{pmatrix} S_0^{-1} & 0 \\ 0 & 1 \end{pmatrix}=$ $\begin{pmatrix} S_0\tau(\alpha)S_0^{-1} & S_0\tau(\beta d) \\ \tau(\beta)S_0^{-1} & \tau(\alpha) \end{pmatrix}\in Sp(2n)^F$. We embed H into $Sp(2n)^F$ by this map: $\alpha+\beta\sqrt{d}$ $\rightarrow \begin{pmatrix} S_0\tau(\alpha)S_0^{-1} & S_0\tau(\beta d) \\ \tau(\beta)S_0^{-1} & \tau(\alpha) \end{pmatrix}$. We can see that there exists an F-stable maximal torus T_1 of $Sp(2n)$ which splits over $\boldsymbol{F}(q^{2n})$ such that T_1^F is equal to the image of H. Let T_0 be an F-stable maximal torus of $O_1(S)$ such that T_0^F is equal to the image of H. Thus we get a natural identification of T_0^F with T_1^F. Hence we may identify $(T_0^F)^\vee$ with $(T_1^F)^\vee$, using this isomorphism.

LEMMA 6. *If η is an element of T_1^F, then the set of all elements of T_1^F which are conjugate to η in $Sp(2n)^F$ is given by $\{\eta, \eta^q, \eta^{q^2}, \cdots, \eta^{q^{2n-1}}\}$. If η is an element of T_0^F, then the set of all elements of T_0^F which are conjugate to η in $O_1(S)^F$ is given by $\{\eta, \eta^{q^2}, \eta^{q^4}, \cdots \eta^{q^{2n-2}}\}$.*

PROOF. First we consider the case of the symplectic group. Assume that $\eta\in T_1^F$ corresponds to $\kappa\in H$. Then all eigenvalues of η are given by κ^{q^i}, $0\leqq i\leqq 2n-1$. If $\eta'\in T_1^F$ corresponds to $\kappa'\in H$ and η' is conjugate to η in $Sp(2n)^F$, we must have $\kappa'=\kappa^{q^i}$ for some i, $0\leqq i\leqq 2n-1$. This shows that $\eta'=\eta^{q^i}$. Hence it is sufficient to show that η and η^q are conjugate in $Sp(2n)^F$. Fix an isomorphism $\iota: \boldsymbol{F}(q^{2n})\simeq \boldsymbol{F}(q)^{2n}$ and an alternating form f on $\boldsymbol{F}(q)^{2n}\times\boldsymbol{F}(q)^{2n}$ defined by $f(\iota(x), \iota(y))=\mathrm{Tr}_{\boldsymbol{F}(q^n)/\boldsymbol{F}(q)}((xy^\sigma-x^\sigma y)/\sqrt{d})$ with $\sigma\in\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n))$, $\sigma\neq 1$. Define $A\in GL(2n, \boldsymbol{F}(q))$ by $A\iota(x)=\iota(x^q)$, $x\in\boldsymbol{F}(q^{2n})$. We can take $\mu\in\boldsymbol{F}(q^{2n})^\times$ so that $(\mu\mu^\sigma)^q=\sqrt{d}/(\sqrt{d})^q$ since $\sqrt{d}/(\sqrt{d})^q\in\boldsymbol{F}(q^n)^\times$. Define $\tilde{\mu}\in GL(2n, \boldsymbol{F}(q))$ by $\tilde{\mu}\iota(x)=\iota(\mu x)$. Then we have $((A\tilde{\mu})^{-1}\eta^q A\tilde{\mu})\iota(x)=(\tilde{\mu}^{-1}A^{-1}\eta^q A)\iota(\mu x)=(\tilde{\mu}^{-1}A^{-1}\eta^q)\iota(\mu^q x^q)=\tilde{\mu}^{-1}A^{-1}(\iota(\kappa^q\mu^q x^q))=\mu^{-1}(\iota(\kappa\mu x))=\iota(\kappa x)=\eta(\iota(x))$ for any $x\in\boldsymbol{F}(q^{2n})$. Hence we get $(A\tilde{\mu})^{-1}\eta^q A\tilde{\mu}=\eta$. On the other hand, we can verify easily that $f(A\tilde{\mu}\iota(x), A\tilde{\mu}\iota(y))=f(\iota(x), \iota(y))$ for any x, $y\in\boldsymbol{F}(q^{2n})$. Therefore we see that $A\tilde{\mu}$ belongs to the symplectic group with respect to f. From this, we see easily that η and η^q are conjugate in $Sp(2n)^F$.

Now let us consider the case of the orthogonal group. We define a non-degenerate symmetric bilinear form g on $\boldsymbol{F}(q)^{2n}\times\boldsymbol{F}(q)^{2n}$ by $g(\iota(x), \iota(y))=\mathrm{Tr}_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q)}(xy^\sigma)$. Then we see that $O(S)$ is isomorphic over $\boldsymbol{F}(q)$ to the orthogonal group with respect to g. Let $A\in GL(2n, \boldsymbol{F}(q))$ be the matrix defined as above. Then we have $g(A\iota(x), A\iota(y))=g(\iota(x^q), \iota(y^q))=\mathrm{Tr}_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q)}((xy^\sigma)^q)=g(\iota(x), \iota(y))$ for any x, $y\in\boldsymbol{F}(q^{2n})$. Hence $A\in O(S)^F$. We get $A^{-1}\eta^q A=\eta$ for any $\eta\in T_0^F$ in the same manner as above. Therefore η and η^q are conjugate in $O(S)^F$ for any $\eta\in T_0^F$. Since $A^{-2}\eta^{q^2}A^2=\eta$, η and η^{q^2} are conjugate in $O_1(S)^F$. Then, to prove our assertion, it is sufficient to show that η and η^q are not conjugate in $O_1(S)^F$ if $\eta\neq\eta^q$. If $A'^{-1}\eta^q A'=\eta$ with $A'\in O_1(S)^F$, then $A'^{-1}A$ centralizes η. From $\eta\neq\eta^q$, we see that η is a regular element of a maximal torus even in $GL(2n, \boldsymbol{F}(q))$. Using this fact, we see easily that $A'^{-1}A$ must belong to T_0^F. Hence it is sufficient to show that $A \notin O_1(S)^F$ i.e. $\det A=-1$. Choosing a suitable basis of $\boldsymbol{F}(q^{2n})$, we can see that A is represented by a cyclic permutation matrix of length $2n$. Hence we get $\det A=(-1)^{2n-1}=-1$ and complete the proof.

Let χ be a character of T_0^F which is in general position. By Deligne-Lusztig [4], there exists an irreducible representation $(-1)^{n-1}R_{T_0}^{\chi}$ of $O_1(S)^F$. Let $t \in T_0^F$ be a regular element. Then by Theorem 4.2 of [4] and Lemma 6, we have

$$\mathrm{Trace}\,(-1)^{n-1}R_{T_0}^{\chi}(t)=(-1)^{n-1}\sum_{i=0}^{n-1}\chi(t^{q^{2i}})\,.$$

THEOREM 1. *Let S be the $2n\times 2n$ symmetric matrix defined by* (14). *Put $\pi=\pi_{S,n}$. If $t \in T_1^F$ is a regular element, then*

$$\mathrm{Trace}\,\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))(t)=(-1)^n\sum_{i=0}^{2n-1}\chi(t^{q^i})\,.$$

PROOF. Let $g_0=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ be an element of $Sp(2n)^F$ which satisfies that $\det\gamma \neq 0$. Since $g_0=\begin{pmatrix}-{}^t\gamma^{-1} & -\alpha\\ 0 & -\gamma\end{pmatrix}w\begin{pmatrix}1 & \gamma^{-1}\delta\\ 0 & 0\end{pmatrix}$, g_0 is conjugate in $Sp(2n)^F$ to $g_1=\begin{pmatrix}-{}^t\gamma^{-1} & 0\\ 0 & -\gamma\end{pmatrix}\begin{pmatrix}1 & {}^t\gamma(\alpha+\gamma^{-1}\delta\gamma)\\ 0 & 1\end{pmatrix}w$. Let ρ be a finite dimensional representation of $O_1(S)^F$ with a representation space W. We put $V=M_{2n,n}(\boldsymbol{F}(q))$. By a direct computation, we have

$$\pi(\rho)(g_1)\Phi(x)=(-1)^n q^{-n^2}\psi(\mathrm{Tr}\,(\gamma^{-1}\delta+\alpha\gamma^{-1})Q(x))\sum_{y\in V}\Phi(y)\langle -x{}^t\gamma^{-1}, y\rangle\,, \tag{15}$$

for $\Phi\in\widetilde{W}(\rho)$, where $Q(x)={}^t xSx \in M_n(\boldsymbol{F}(q))$.

A standard basis of $\widetilde{W}(\rho)$ can be given as follows. Let $\{x_i\}$ be a complete set of representatives of the orbits of $O_1(S)^F$ in V. For $x\in V$, put $Z(x)=\{g\in O_1(S)^F \mid gx=x\}$. For x_i, let $\{v_{ij}\}$ be a basis of the vector subspace of W which consists of all vectors which are fixed by $Z(x_i)$. Define an element Φ_i^j of $\widetilde{W}(\rho)$ by

$$\Phi_i^j(x)=\begin{cases}\rho(g)v_{ij} & \text{if} \quad x=gx_i,\ g\in O_1(S)^F\,,\\ 0 & \text{if} \quad x\notin O_1(S)^F x_i\,.\end{cases}$$

Then we can verify easily that the Φ_i^j's make a basis of $\widetilde{W}(\rho)$. From (15), we obtain

$$\begin{aligned}&\pi(\rho)(g_1)\Phi_i^j(x_i)\\ &=(-1)^n q^{-n^2}\psi(\mathrm{Tr}\,(\gamma^{-1}\delta+\alpha\gamma^{-1})Q(x_i))\sum_{g\in O_1(S)^F}\rho(g)\langle -x_i{}^t\gamma^{-1}, gx_i\rangle |Z(x_i)|^{-1}v_{ij}\,.\end{aligned} \tag{16}$$

Therefore we get

$$\begin{aligned}&\operatorname{Trace}\pi(\rho)(g_0)\\ (17)\qquad &=(-1)^n q^{-n^2}\sum_i\sum_{g\in O_1(S)^F}\psi(\operatorname{Tr}(\gamma^{-1}\delta+\alpha\gamma^{-1})Q(x_i))\\ &\cdot\operatorname{Trace}\rho(g)\langle -x_i{}^t\gamma^{-1}, gx_i\rangle|Z(x_i)|^{-1}.\end{aligned}$$

For $\xi\in O_1(S)^F$, let $\{\xi\}$ denote the conjugacy classes of $O_1(S)^F$ which contains ξ and $Z(\xi)$ denotes the centralizer of ξ in $O_1(S)$. We can divide the right hand side of (17) into the contributions of the conjugacy classes of $O_1(S)^F$. Thus we get

$$(18)\qquad \operatorname{Trace}\pi(\rho)(g_0)=(-1)^n q^{-n^2}\sum_{\{\xi\}}\operatorname{Trace}\rho(\xi)I(\xi)/|Z(\xi)^F|,$$

$\{\xi\}$ runs over the conjugacy classes of $O_1(S)^F$, where

$$(19)\qquad I(\xi)=\sum_{x\in V}\psi(\operatorname{Tr}(\gamma^{-1}\delta+\alpha\gamma^{-1})Q(x))\langle -x^t\gamma^{-1}, \xi x\rangle.$$

Since $\langle -x^t\gamma^{-1}, \xi x\rangle=\psi(-\operatorname{Tr}(2\gamma^{-1t}xS\xi x))$, we have

$$(20)\qquad I(\xi)=\sum_{x\in V}\psi[\operatorname{Tr}\{(\gamma^{-1}\delta+\alpha\gamma^{-1})^t xSx-2\gamma^{-1t}xS\xi x\}].$$

The map $x\to\operatorname{Tr}\{(\gamma^{-1}\delta+\alpha\gamma^{-1})^t xSx-2\gamma^{-1t}xS\xi x\}$ defines a quadratic form on $V\simeq \boldsymbol{F}(q)^{2n^2}$. We will describe the symmetric matrix in $M_{2n^2}(\boldsymbol{F}(q))$ which corresponds to this quadratic form. Let us identify each column vector of an element of $M_{2n,n}(\boldsymbol{F}(q))$ with an element of $\boldsymbol{F}(q)^{2n}$. Then $M_{2n,n}(\boldsymbol{F}(q))$ can be identified with $\boldsymbol{F}(q)^{2n}\otimes\boldsymbol{F}(q)^n$ by $(aE_{ij})\to(0,\cdots,\underbrace{a}_{i},\cdots,0)\otimes(0,\cdots,\underbrace{1}_{j},\cdots,0)\in\boldsymbol{F}(q^{2n})\otimes\boldsymbol{F}(q)^n$, where $a\in\boldsymbol{F}(q)$ and E_{ij} denotes the matrix in which only the ij-entry is non zero and equals 1. Let $\varepsilon\in M_n(\boldsymbol{F}(q))$ and $\mu\in M_{2n}(\boldsymbol{F}(q))$ be symmetric matrices and consider the quadratic form $x\to\operatorname{Tr}(\varepsilon^t x\mu x)$ on $M_{2n,n}(\boldsymbol{F}(q))$. Then, by an easy calculation, we can see that the symmetric matrix $\mu\otimes\varepsilon$ corresponds to this quadratic form. Therefore we have

$$(21)\qquad I(\xi)=\sum_{x\in V'}\psi({}^t xTx),$$

where $V'=\boldsymbol{F}(q)^{2n^2}$, and $T=S\otimes\{(\alpha\gamma^{-1}+\gamma^{-1}\delta)+{}^t(\alpha\gamma^{-1}+\gamma^{-1}\delta)\}/2-\{S\xi+{}^t(S\xi)\}/2\otimes(\gamma^{-1}+{}^t\gamma^{-1})\in M_{2n^2}(\boldsymbol{F}(q))$.

Let χ_0 denote the quadratic residue character of $\boldsymbol{F}(q)^\times$ and define the Gauss sum $G(\chi_0,\psi)$ by $G(\chi_0,\psi)=\sum_{x\in\boldsymbol{F}(q)^\times}\chi_0(x)\psi(x)$. It is well known that $G(\chi_0,\psi)^2=\chi_0(-1)q$ and that

$$(22)\qquad \sum_{x\in\boldsymbol{F}(q)}\psi(ax^2)=\chi_0(a)G(\chi_0,\psi)\qquad\text{if}\quad a\in\boldsymbol{F}(q)^\times.$$

Now let us specialize to the case $\rho=(-1)^{n-1}R^{\chi}_{T_0}(O_1(S)^F)$. Assume that α'

$+\beta'\sqrt{d}\in H$ corresponds to a regular element t of T_1^F, where α', $\beta'\in \boldsymbol{F}(q^n)$. Put $t=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. Then we have $\alpha=S_0\tau(\alpha')S_0^{-1}$, $\delta=\tau(\alpha')$, $\beta=S_0\tau(\beta'd)$, $\gamma=\tau(\beta')S_0^{-1}$. We can verify easily that $S_0\tau(x)$ is symmetric for any $x\in \boldsymbol{F}(q^n)$. Therefore we get $\{\alpha\gamma^{-1}+\gamma^{-1}\delta)+{}^t(\alpha\gamma^{-1}+\gamma^{-1}\delta)\}/2=S_0\tau(\alpha'\beta'^{-1})+{}^t(S_0\tau(\alpha'\beta'^{-1}))=2S_0\tau(\alpha'\beta'^{-1})$ and $\gamma^{-1}+{}^t\gamma^{-1}=S_0\tau(\beta')^{-1}+{}^t(S_0\tau(\beta')^{-1})=2S_0\tau(\beta')^{-1}$. Hence we have $T=S\otimes 2S_0\tau(\alpha'\beta'^{-1})-S(\xi+\xi^{-1})/2\otimes 2S_0\tau(\beta')^{-1}$. Let $\xi=su$ be the Jordan decomposition of ξ with s semisimple and u unipotent. We know that $\mathrm{Tr}\rho(\xi)=0$ if s is not conjugate to an element of T_0^F (cf. [4]). Therefore, replacing ξ by its conjugate if necessary, we may assume that $s\in T_0^F$. Let us assume that s corresponds to an element $s_1+s_2\sqrt{d}\,(s_1,\ s_2\in \boldsymbol{F}(q^n))$ of H. We will show that if $s_1\neq\alpha^\eta$ for any $\eta\in\mathrm{Gal}(\boldsymbol{F}(q^n)/\boldsymbol{F}(q))$, then $\det T\neq 0$. It is clear that $\operatorname{rank}T=\operatorname{rank}(1_{2n}\otimes\tau(\alpha')-(\xi+\xi^{-1})/2\otimes 1_n)$. We note that $\tau(\alpha')$ can be diagonalized over the algebraic closure $\overline{\boldsymbol{F}(q)}$ of $\boldsymbol{F}(q)$ to the form

$$\begin{bmatrix}\alpha' & & & 0\\ & \alpha'^q & & \\ & & \ddots & \\ 0 & & & \alpha'^{q^{n-1}}\end{bmatrix}.$$

When we triangulize ξ over $\overline{\boldsymbol{F}(q)}$, it takes the form

$$\begin{bmatrix}s_1+s_2\sqrt{d} & & & *\\ & (s_1+s_2\sqrt{d})^q & & \\ & & \ddots & \\ 0 & & & (s_1+s_2\sqrt{d})^{q^{2n-1}}\end{bmatrix}.$$

Therefore $\operatorname{rank}T<2n^2$ is equivalent to that $2\alpha'=2s_1^{\eta_0}$ for some $\eta_0\in\mathrm{Gal}(\boldsymbol{F}(q^n)/\boldsymbol{F}(q))$. If $\alpha'=s_1^{\eta_0}$ for some $\eta_0\in\mathrm{Gal}(\boldsymbol{F}(q^n)/\boldsymbol{F}(q))$, we have $\beta'=\pm s_2^{\eta_0}$ since $\alpha'^2-\beta'^2d=s_1^2-s_2^2d=1$. This implies that $\alpha'+\beta'\sqrt{d}=(s_1+s_2\sqrt{d})^\eta$ for some $\eta\in\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q))$. From (22), we see immediately that $I(\xi)=\chi_0(\det T)G(\chi_0,\psi)^{2n^2}$ if $\det T\neq 0$.

We will show that if $\alpha'=s_1^{\eta_0}$ with some $\eta_0\in\mathrm{Gal}(\boldsymbol{F}(q^n)/\boldsymbol{F}(q))$, then $u=1$. If $\alpha'=s_1^{\eta_0}$, we have $\alpha'+\beta'\sqrt{d}=(s_1+s_2\sqrt{d})^\eta$ with some $\eta\in\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q))$. Since $\alpha'+\beta'\sqrt{d}$ corresponds to a regular element t of T_1^F, we see easily that $(\alpha'+\beta'\sqrt{d})\neq(\alpha'+\beta'\sqrt{d})^{q^i}$ for any i, $0\leqq i\leqq 2n-1$ by Lemma 6. Hence all eigenvalues of s must be different each other and we get $u=1$.

Therefore we may concentrate in the computation of $I(\xi)$, when $\xi\in T_0^F$. For the convenience of the computation, let us identify $M_{2n,n}(\boldsymbol{F}(q))$ with $(\boldsymbol{F}(q^{2n}))^n$, identifying each column vector with an element of $\boldsymbol{F}(q^{2n})$, and write $x\in(\boldsymbol{F}(q^{2n}))^n$ as $x=(x_1, x_2, \cdots, x_n)$, $x_i\in \boldsymbol{F}(q^{2n})$. Let σ be the generator of $\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q^n))$. Then the $n\times n$-matrix $Q(x)={}^txSx$ is equal to $(\mathrm{Tr}_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q)}(x_ix_j^\sigma)/2)$. Assume that $\xi\in T_0^F$ corresponds to $\kappa\in H$. Then we have $2{}^txS\xi x=(\mathrm{Tr}_{\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q)}(\kappa^\sigma x_ix_j^\sigma+\kappa x_ix_j^\sigma)/2)$. Hence we obtain

(23) $I(\xi)=\sum_{x\in V'}\psi[\mathrm{Tr}((\gamma^{-1}\delta+\alpha\gamma^{-1})(\mathrm{Tr}_{F(q^{2n})/F(q)}(x_i x_j^{\sigma})/2))$

$-\mathrm{Tr}(\gamma^{-1}(\mathrm{Tr}_{F(q^{2n})/F(q)}(\kappa^{\sigma}x_i x_j^{\sigma}+\kappa x_i x_j^{\sigma})/2))]$, where $V'=F(q^{2n})^n$.

Let us write $\kappa=\kappa_1+\kappa_2\sqrt{d}$ with κ_1, $\kappa_2\in F(q^n)$ and define a non-trivial character ψ' of $F(q^n)$ by $\psi'=\psi_0\circ\mathrm{Tr}_{F(q^n)/F(q)}$. Then (23) can be written as

(24)
$$\begin{aligned}I(\xi)=&\sum_{x\in V'}\psi'[\mathrm{Tr}\{(\gamma^{-1}(\delta+\gamma\alpha\gamma^{-1})/2)(\mathrm{Tr}_{F(q^{2n})/F(q^n)}(x_i x_j^{\sigma}))\\&-\kappa_1(\mathrm{Tr}_{F(q^{2n})/F(q^n)}(x_i x_j^{\sigma}))\}]\\=&\sum_{x\in V'}\psi'[\mathrm{Tr}\,\gamma^{-1}((\delta+\gamma\alpha\gamma^{-1})/2-\kappa_1\cdot 1_n)(\mathrm{Tr}_{F(q^{2n})/F(q^n)}(x_i x_j^{\sigma}))].\end{aligned}$$

Using $\alpha=S_0\tau(\alpha')S_0^{-1}$, $\gamma=\tau(\beta')S_0^{-1}$ and $\delta=\tau(\alpha')$, we get $(\delta+\gamma\alpha\gamma^{-1})/2=\tau(\alpha')$. We put $x_i=y_i+\sqrt{d}z_j$ with y_i, $z_i\in F(q^n)$. Let us define an $n\times n$-matrix A by $A=2(S_0\tau(\alpha'\beta'^{-1})-\kappa_1 S_0\tau(\beta'^{-1}))$ and a $2n\times 2n$-matrix $\tilde{T}$ by $\tilde{T}=\begin{pmatrix}A & 0\\ 0 & -dA\end{pmatrix}$. Put $A=(a_{ij})$, where a_{ij} denotes the ij-entry of A. Then we have

$$\mathrm{Tr}(A\,\mathrm{Tr}_{F(q^{2n})/F(q^n)}(x_i x_j^{\sigma})/2)=\mathrm{Tr}(A(y_i y_j-dz_i z_j))=\sum_i\sum_k a_{ik}(y_k y_i-dz_k z_i).$$

Therefore we get

(25)
$$I(\xi)=\sum_{x\in V'}\psi'[{}^t x\tilde{T}x].$$

Replacing $\tilde{T}$ by ${}^t g\tilde{T}g$ with some $g\in GL(2n, F(q))$, we can bring $\tilde{T}$ to the form

$$\tilde{T}=\begin{pmatrix}t_1 & & & & & 0\\ & \ddots & & & & \\ & & t_n & & & \\ & & & -dt_1 & & \\ & & & & \ddots & \\ 0 & & & & & -dt_n\end{pmatrix}.$$

First we consider the case where $\kappa\neq(\alpha'+\beta'\sqrt{d})^{\eta}$ for any $\eta\in\mathrm{Gal}(F(q^n)/F(q))$. Then we have $t_i\neq 0$ $(1\leqq i\leqq n)$ and using (22), we get

$$\begin{aligned}I(\xi)&=\prod_{i=1}^{n}\sum_{x\in F(q^n)}\psi'(t_i x^2)\prod_{i=1}^{n}\sum_{x\in F(q^n)}\psi'(-dt_i x^2)\\&=\prod_{i=1}^{n}(\tilde{\chi}(t_i)G(\tilde{\chi},\psi'))\prod_{i=1}^{n}(\tilde{\chi}(-dt_i)G(\tilde{\chi},\psi'))\\&=\tilde{\chi}(-1)^n(-1)^n G(\tilde{\chi},\psi')^{2n}=\tilde{\chi}(-1)^n(-1)^n\tilde{\chi}(-1)^n(q^n)^n=(-1)^n q^{n^2},\end{aligned}$$

where $\tilde{\chi}$ denotes the quadratic residue character of $F(q^n)^{\times}$.

Next assume that $\kappa=(\alpha'+\beta'\sqrt{d})^{\eta}$ with some $\eta\in\mathrm{Gal}(F(q^{2n})/F(q))$. Put rank $T=2l$. Then by a similar computation to the above one, we get $I(\xi)=(-1)^l q^{2n^2-nl}$. We have $l=\mathrm{rank}\,A=\mathrm{rank}\,(\tau(\alpha')-\kappa_1\cdot 1_n)$. Since $F(q)(\alpha')=F(q^n)$, all eigenvalues of $\tau(\alpha')$ are different each other. Hence we have $l=n-1$. Therefore we obtain

$$\mathrm{Trace}\ \pi(\rho)(t)=(-1)^n q^{-n^2}\Big[\sum_{\{\xi\}}{}' (-1)^n q^{n^2}\ \mathrm{Trace}\ \rho(\xi)/|Z(\xi)^F| + \sum_{\{\xi\}}{}'' (-1)^{n-1} q^{n^2+n}\ \mathrm{Trace}\ \rho(\xi)/|Z(\xi)^F|\Big],$$

where $\sum''$ extends over conjugacy classes of $O_1(S)^F$ whose semisimple parts correspond to $(\alpha'+\beta'\sqrt{d})^\eta$ with some $\eta\in\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q))$ and $\sum'$ extends over conjugacy classes of $O_1(S)^F$ which do not appear in $\sum''$. Since ρ does not contain the trivial representation, we have $\sum_{\{\xi\}}\mathrm{Trace}\ \rho(\xi)/|Z(\xi)^F|=0$, where $\{\xi\}$ extends over all conjugacy classes of $O_1(S)^F$. Hence we obtain $\mathrm{Trace}\ \pi(\rho)(t)=-\sum_{\{\xi\}}{}'' (q^n+1)\ \mathrm{Trace}\ \rho(\xi)/|Z(\xi)^F|$. Suppose that ξ corresponds to $(\alpha'+\beta'\sqrt{d})^\eta$ with some $\eta\in\mathrm{Gal}(\boldsymbol{F}(q^{2n})/\boldsymbol{F}(q))$. Then ξ must be a regular element of T_0^F. By Lemma 6, such ξ fall into two $O_1(S)^F$-conjugacy classes; one corresponds to $(\alpha'+\beta'\sqrt{d})^{q^{2i}}$, $0\leqq i\leqq n-1$ and the other corresponds to $(\alpha'+\beta'\sqrt{d})^{q^{2i+1}}$. Since ξ is regular, we have $Z(\xi)=T_0$ and $|Z(\xi)^F|=|T_0^F|=|H|=q^n+1$. Hence we obtain $\mathrm{Trace}\ \pi(\rho)(t)=-(\mathrm{Trace}\ \rho(\xi_1)+\mathrm{Trace}\ \rho(\xi_2))$, where ξ_1 corresponds to $\alpha'+\beta'\sqrt{d}$ and ξ_2 corresponds to $(\alpha'+\beta'\sqrt{d})^q$. Using (15), we obtain finally

$$\mathrm{Trace}\ \pi(\rho)(t)=-\Big[(-1)^{n-1}\Big(\sum_{i=0}^{n-1}\chi(t^{q^{2i}})+\sum_{i=0}^{n-1}\chi(t^{q^{2i+1}})\Big)\Big]=(-1)^n\sum_{i=0}^{2n-1}\chi(t^{q^i})$$

and complete the proof of Theorem 1.

We call an element $\kappa\in H$ regular if $\boldsymbol{F}(q)(\kappa)=\boldsymbol{F}(q^{2n})$. Then $\kappa\in H$ is regular if and only if $\kappa\neq\kappa^{q^i}$ for $1\leqq i\leqq 2n-1$. Assume that $t\in T_1^F$ corresponds to $\kappa\in H$. By Lemma 6, t is a regular element of $Sp(2n)$ if and only if κ is regular. We define a subset H' of H as follows. Let r_0 be a positive integer such that r_0 divides n and n/r_0 is an odd integer greater than 1. Let $H(r_0)$ be the set $\{x\in\boldsymbol{F}(q^{2r_0})\,|\,N_{\boldsymbol{F}(q^{2r_0})/\boldsymbol{F}(q^{r_0})}(x)=1\}$. From $x^{q^{r_0}+1}=1$, we get $x^{q^n+1}=1$, for $x\in\overline{\boldsymbol{F}(q)}$. Therefore we have $H(r_0)\subseteqq H$. We put $H'=\{\pm1\}\cup(\bigcup_{r_0}H(r_0))$, where r_0 extends over all positive integers such that n/r_0 is odd and $n/r_0>1$. Then we have

LEMMA 7. *An element $\kappa\in H$ is not regular if and only if $\kappa\in H'$.*

PROOF. It is obvious that any element of H' is not regular. Assume that $\kappa\in H$ is not regular and $\kappa\neq\pm1$. Then we have $\boldsymbol{F}(q)(\kappa)=\boldsymbol{F}(q^m)$, $m\,|\,2n$, $m<2n$. Hence we get $\kappa^{q^m-1}=1$ and $\kappa^{q^n+1}=1$. Let U be the greatest common divisor of q^n+1 and q^m-1. Then we have $\kappa^U=1$. We may assume that $U>2$. Let r be the least positive integer such that $q^r\equiv1 \bmod U$. We have $q^m\equiv1 \bmod U$ and $q^n\equiv-1 \bmod U$. Hence we must have $r\,|\,m$. Put $n=ur+r_0$ with $0\leqq r_0<r$, $u\geqq0$, r_0, $u\in\boldsymbol{Z}$. If $r_0=0$, we get $2\equiv0 \bmod U$ and this contradicts the assumption $U>2$. Therefore we have $q^{r_0}\equiv-1 \bmod U$, $0<r_0<r$. This implies $q^{2r_0}\equiv1 \bmod U$, $0<2r_0$

$<2r$, hence $r=2r_0$. It is obvious that r_0 is the least positive integer such that $q^{r_0}\equiv -1 \bmod U$. Therefore we get $r_0 | n$ and n/r_0 is odd. If $r_0=n$, we get $r=2r_0=2n$, hence κ is regular. Therefore we have $n/r_0>1$ and complete the proof.

Now let us investigate the relation between $\pi(\rho)$, $\rho=(-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F)$ and $(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$, where χ is a character of T_1^F which is in general position. By [4], (7.6.2), we have

$$\mathrm{Tr}\,\pi(\rho)(t)=\sum_{\theta\in(T_1^F)^\vee}\theta(t)\langle\pi(\rho),\ R_{T_1}^{\theta}\rangle\,, \tag{26}$$

if $t\in T_1^F$ is a regular element. Let H_r denote the set of all regular element of H i.e. $H_r=H-H'$. From (26) and Theorem 1, we have

$$\sum_{i=0}^{2n-1}\chi(\kappa^{q^i})=\sum_{\theta\in H^\vee}\theta(\kappa)\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta}\rangle \tag{27}$$

if $\kappa\in H_r$, where we identified $(T_1^F)^\vee$ with $H^\vee$.

LEMMA 8. *We exclude the cases $n=2$, $q=3$ and $n=3$, $q=3$. Let χ be a character of T_0^F. Assume that χ is in general position in $Sp(2n)^F$ when we regard χ as a character of T_1^F. Then the representation $\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))$ contains $(-1)^n R_{T_1}^{\chi'}(Sp(2n)^F)$ as a subrepresentation, where χ' is a character of T_1^F which is in general position in $Sp(2n)^F$.*

PROOF. The case $n=1$ is well known (cf. [**16**]). We assume that $n>1$. Put $\rho=(-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F)$ and assume that $\langle\pi(\rho),\ R_{T_1}^{\theta}\rangle=0$ if $\theta\in(T_1^F)^\vee$ is in general position. For $1\leqq i\leqq 2n-1$, let χ^{q^i} denote the character of H such that $\chi^{q^i}(u)=\chi(u^{q^i})$, $u\in H$ (we identify $(T_0^F)^\vee$ with $(T_1^F)^\vee$ and $H^\vee$). Suppose that $\sum_{i=0}^{2n-1}\chi^{q^i}(\kappa)=0$ for any $\kappa\in H_r$. Then we have $|H_r|=\sum_{\kappa\in H'}\sum_{i=1}^{2n-1}\chi^{q^i}(\kappa)\chi^{-1}(\kappa)\leqq(2n-1)|H'|$. We will show below the inequality

$$(2n-1)|H'|<|H_r|\,, \tag{28}$$

and let us assume (28) for a moment. Then we must have $\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta}\rangle\neq 0$ for some $\theta\in(T_1^F)^\vee$ from (27). Let θ_1 be a character of T_1^F for which $|\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta}\rangle|$ takes the maximum value. Then, from (27), we get

$$\begin{aligned}|H_r|\cdot\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta_1}\rangle&=\sum_{\kappa\in H_r}\Bigl(\sum_{i=0}^{2n-1}\chi^{q^i}(\kappa)\theta_1^{-1}(\kappa)\\&\qquad-\sum_{\substack{\theta\in H^\vee\\ \theta\neq\theta_1}}\theta(\kappa)\theta_1^{-1}(\kappa)\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta}\rangle\Bigr)\\&=\sum_{\kappa\in H'}\Bigl(-\sum_{i=0}^{2n-1}\chi^{q^i}(\kappa)\theta_1^{-1}(\kappa)+\sum_{\substack{\theta\in H^\vee\\ \theta\neq\theta_1}}\theta(\kappa)\theta_1^{-1}(\kappa)\langle\pi(\rho),\ (-1)^n R_{T_1}^{\theta}\rangle\Bigr).\end{aligned}$$

Taking absolute values of the both sides, we get

$$|H_r||\langle\pi(\rho),(-1)^n R_{T_1}^{\theta_1}\rangle|\leqq 2n|H'|+\sum_{\substack{\theta\in H^\vee\\ \theta\neq\theta_1}}|H'||\langle\pi(\rho),(-1)^n R_{T_1}^{\theta}\rangle| .$$

It is easy to see that the number of characters of T_1^F which are not in general position is equal to $|H'|$. Hence we must have $|H_r|\leqq 2n|H'|+|H'|(|H'|-1)$. We will show that

$$|H_r|>2n|H'|+|H'|^2 . \tag{29}$$

We note that (29) implies (28). If n is a power of 2, Lemma 7 shows that $|H'|=2$. Then, except for the case $n=2$, $q=3$, (29) follows immediately since we have $3^n>4n+5$ if $n\geqq 3$. Let $n=2^\beta p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ be a factorization of n into primes with odd primes p_i $(1\leqq i\leqq m)$. Then, by Lemma 7, we have $|H'|\leqq\sum_{i=1}^m(q^{n/p_i}+1)$. If $n=3$, then we have $|H'|=q+1$, hence (29) reduces to the inequality $q^3>q^2+9q+7$ which holds if $q\geqq 5$. Therefore it is sufficient to show

$$q^n+1>(2n+1)\sum_{i=1}^m(q^{n/p_i}+1)+\Big(\sum_{i=1}^m(q^{n/p_i}+1)\Big)^2 , \tag{30}$$

when $n\geqq 5$. If $2n+1\geqq\sum_{i=1}^m(q^{n/p_i}+1)$, the (30) reduces to

$$3^n+1>2(2n+1)^2 \tag{31}$$

and we can immediately verify (31) for $n\geqq 5$. Hence we may assume that $2n+1<\sum_{i=1}^m(q^{n/p_i}+1)$ and (30) reduces to

$$q^n+1\geqq 2\Big(\sum_{i=1}^m(q^{n/p_i}+1)\Big)^2 . \tag{32}$$

Since $p_i\geqq 3$, we have $m\leqq[\log_3 n]$ and (32) reduces to

$$q^n+1\geqq(32/9)\cdot[\log_3 n]^2 q^{2n/3} \tag{33}$$

and (33) obviously reduces to the inequality

$$3^{n/3}\geqq 4[\log_3 n]^2 \qquad \text{for} \quad n\geqq 5 . \tag{34}$$

We can easily verify (34). Hence we get (29) and complete the proof.

PROPOSITION 4. *Let the assumption on n, q, and χ be as in Lemma 8. Assume that* $\dim\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))=\dim(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$. *Then* $\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))$ *and* $(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$ *are equivalent.*

PROOF. By Lemma 8, $\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))$ contains $(-1)^n R_{T_1}^{\chi'}(Sp(2n)^F)$ as a subrepresentation, where χ' is a character of T_1^F which is in general position in $Sp(2n)^F$. Since $\dim\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))=\dim(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$, $\pi((-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F))$ must be equivalent to $(-1)^n R_{T_1}^{\chi'}(Sp(2n)^F)$. Therefore we have

(35) $$\sum_{i=1}^{2n-1}\chi(\kappa^{q^i})=\sum_{i=1}^{2n-1}\chi'(\kappa^{q^i}) \qquad \text{if} \quad \kappa\in H_r\,.$$

If $\chi'\neq\chi^{q^i}$ for any $1\leqq i\leqq 2n-1$, we get

(36) $$|H_r|\leqq(4n-1)|H'|$$

in a similar way as in the proof of Lemma 8. We may assume that $n\geqq 2$, since the case $n=1$ was settled in [**16**]. Then we can verify the inequality $|H_r|>(4n-1)|H'|$ except for the cases $n=2, q=3$ and $n=3, q=3$ by a similar method to the proof of Lemma 8. Therefore we must have $\chi'=\chi^{q^i}$ for some i, $1\leqq i\leqq 2n-1$. Then $R_{T_1}^{\chi'}(Sp(2n)^F)$ is equivalent to $R_{T_1}^{\chi}(Sp(2n)^F)$, hence our assertion.

REMARK 2. Let the assumptions be as in Proposition 4. The equivalence of $\pi(\rho)$, $\rho=(-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F)$ and $(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$ follows immediately if we could show that $\pi(\rho)$ is irreducible. However, for general n, a direct proof of the irreducibility seems rather difficult to obtain. We shall compensate this defect by showing $\dim\pi(\rho)=\dim(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$ in the next section.

§ 3. The dimension of $\pi(\rho)$.

Let S be a $2n\times 2n$ symmetric matrix with entries in $\boldsymbol{F}(q)$ such that $\det S\neq 0$. For a positive integer m and a finite dimensional representation ρ of $O_1(S)^F$, $\pi_{S,m}(\rho)$ denotes the representation of $Sp(2m)^F$ which is defined in § 2 (cf. Proposition 3). For $g\in O_1(S)^F$, let $S_m(g)$ denote the set of all elements x of $M_{2n,m}(\boldsymbol{F}(q))$ such that $gx=x$. Then $S_m(g)$ makes a vector space over $\boldsymbol{F}(q)$. We have

(37) $$\dim S_m(g)=m(2n-\operatorname{rank}(1-g))\,.$$

PROPOSITION 5. *Let the notation be as above. Then we have*

$$\dim\pi_{S,m}(\rho)=\sum_{g\in O_1(S)^F}\operatorname{Trace}\rho(g)q^{m(2n-\operatorname{rank}(1-g))}/|O_1(S)^F|\,.$$

PROOF. Put $V=M_{2n,m}(\boldsymbol{F}(q))$. For $x\in V$, put $Z(x)=\{g\in O_1(S)^F\,|\,gx=x\}$. Let $m(x)$ denote the multiplicity with which the trivial representation occurs in the restriction of ρ to $Z(x)$. We have

(38) $$m(x)=\sum_{g\in Z(x)}\operatorname{Trace}\rho(g)/|Z(x)|\,.$$

A standard basis of the representation space $\widetilde{W}(\rho)$ of $\pi_{S,m}(\rho)$ can be given in a similar way as in the proof of Theorem 1. Namely let $\{x_i\}$ be a complete set of representatives of the orbits of $O_1(S)^F$ in V. For x_i, let $\{v_{ij}\}$ be a basis of the vector subspace of W (i.e. a representation space of ρ) which transforms according to the trivial representation of $Z(x_i)$. Define Φ_i^j of $\widetilde{W}(\rho)$ by

$$\Phi_i^j(x)=\begin{cases}\rho(g)v_{ij} & \text{if} \quad x=gx_i,\ g\in O_1(S)^F,\\ 0 & \text{if} \quad x\notin O_1(S)^F x_i.\end{cases}$$

Then we can verify easily that the Φ_i^j's make a basis of $\widetilde{W}(\rho)$. Therefore we get

$$\begin{aligned}\dim \pi_{S,m}(\rho)&=\sum_i m(x_i)=\sum_{x\in V} m(x)|Z(x)|/|O_1(S)^F|\\ &=\sum_{x\in V}\frac{|Z(x)|}{|O_1(S)^F|}\Big(\sum_{g\in Z(x)}\frac{\operatorname{Trace}\rho(g)}{|Z(x)|}\Big)=\sum_{x\in V}\sum_{g\in Z(x)}\operatorname{Trace}\rho(g)/|O_1(S)^F|\\ &=\Big(\sum_{g\in O_1(S)^F}\sum_{x\in S_m(g)}\operatorname{Trace}\rho(g)\Big)/|O_1(S)^F|\\ &=\sum_{g\in O_1(S)^F}\operatorname{Trace}\rho(g)|S_m(g)|/|O_1(S)^F|.\end{aligned}$$

Using (37), we obtain the desired expression for $\dim \pi_{S,m}(\rho)$.

We denote by η_m the representation of $O_1(S)^F$ on $\mathcal{S}(V)$ given by $(\eta_m(g)\Phi)(x)=\Phi(gx)$ for $\Phi\in\mathcal{S}(V)$. We obviously have $\eta_m=\underbrace{\eta_1\otimes\eta_1\otimes\cdots\otimes\eta_1}_{m\text{-times}}$.

PROPOSITION 6. *For any irreducible representation ρ of $O_1(S)^F$, the dimension of $\pi_{S,m}(\rho)$ is equal to the multiplicity with which ρ occurs in η_m.*

PROOF. We can see easily that

$$\operatorname{Trace}\eta_m(g)=|S_m(g)|,\ g\in O_1(S)^F. \tag{39}$$

By (37) and Proposition 5, we have

$$\dim \pi_{S,m}(\rho)=\sum_{g\in O_1(S)^F}\operatorname{Trace}(\rho\otimes\eta_m)(g)/|O_1(S)^F|. \tag{40}$$

Therefore $\dim \pi_{S,m}(\rho)$ is equal to the multiplicity with which the trivial representation of $O_1(S)^F$ occurs in $\rho\otimes\eta_m$. By the orthogonality relation of irreducible characters, we can see that this multiplicity is equal to the multiplicity with which $\bar\rho$ occurs in η_m, where $\bar\rho$ denotes the complex conjugate of ρ. Since ρ and $\bar\rho$ occur in η_m with the same multiplicity, we get our conclusion.

PROPOSITION 7. *Let S be the matrix given by* (14). *Let χ be a character of T_0^F which is in general position and put $\rho=(-1)^{n-1}R_{T_0}^{\chi}(O_1(S)^F)$. Then we have $\dim \pi_{S,m}(\rho)=\sum'\operatorname{Trace}\rho(u)(q^{m(2n-\operatorname{rank}(1-u))}-1)/|O_1(S)^F|$, where $\sum'$ extends over $u\in O_1(S)^F$ which is unipotent.*

PROOF. For $g\in O_1(S)^F$, let $g=su$ be the Jordan decomposition of g with s semisimple and u unipotent. We have $\operatorname{Trace}\rho(g)=0$ if s is not conjugate to an element of T_0^F. If s is conjugate to an element of T_0^F which is not 1, we have $S_m(g)=\{0\}$, since all eigenvalues of s are different from 1. Therefore, by Proposition 5, we have

$$\dim \pi_{S,m}(\rho)=\sum{}'\operatorname{Trace}\rho(u)q^{m(2n-\operatorname{rank}(1-u))}/|O_1(S)^F|+\sum{}''\operatorname{Trace}\rho(g)/|O_1(S)^F|,$$

where $\sum''$ extends over $g\in O_1(S)^F$ which is not unipotent. By $\sum_{g\in O_1(S)^F}$ Trace $\rho(g)=0$, we obtain the desired formula.

Hereafter we take the matrix defined by (14) as S. The function $u\rightarrow$ Trace $R^{\chi}_{T_0}(u)$ from unipotent elements of $O_1(S)^F$ to $\boldsymbol{C}$ is so called the Green function of the F-stable maximal torus T_0, which is independent of the choice of χ. For our specific torus T_0 (the Coxeter torus), the Green function is computed explicitly by Lusztig [8]. Since the Witt index of S is $n-1$, $\mathrm{rank}(1-u)$ is even if $u\in O_1(S)^F$ is unipotent. Put $2r=\mathrm{rank}(1-u)$. Then the formula (38.1) of [8] can be read as follows.

(41) $\quad$ Trace $\rho(u)=(-1)^r(q^2-1)(q^4-1)\cdots(q^{2n-2r-2}-1),\ 0\leqq r\leqq n-1,\ \rho=(-1)^{n-1}R^{\chi}_{T_0}.$

(This formula should be read as Trace $\rho(u)=(-1)^{n-1}$ if $r=n-1$).

Let us denote by $A^{(n)}_r$ the number of unipotent elements of $O_1(S)^F$ such that $\mathrm{rank}(1-u)=2r$. By Theorem 3.1 of Lusztig [9][5], $A^{(n)}_r$ is given by the following formula.

$$A^{(n)}_r=\frac{q^{r^2-r}(q^{2n}-1)(q^{2n-2}-1)^2(q^{2n-4}-1)^2\cdots(q^{2n-2r+2}-1)^2(q^{2n-2r}-1)}{(q^2-1)(q^4-1)\cdots(q^{2r}-1)}. \tag{42}$$

(This formula should be read as $A^{(n)}_0=1$, $A^{(n)}_1=(q^{2n}-1)(q^{2n-2}-1)/(q^2-1)$ if $r=0$ or 1).

By (41) and Proposition 7, the desired relation $\dim\pi_{S,n}(\rho)=\dim(-1)^nR^{\chi}_{T_1}(Sp(2n)^F)$ reduces to the following. (Note that $\dim(-1)^nR^{\chi}_{T_1}(Sp(2n)^F)=|Sp(2n)^F|/|T^F_1|q^{n^2}$, cf. Theorem 7.1 of [4]).

$$\sum_{r=0}^{n-1}(q^2-1)(q^4-1)\cdots(q^{2n-2r-2}-1)(-1)^rA^{(n)}_r(q^{n(2n-2r)}-1) \tag{43}$$
$$=(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n-1)|O_1(S)^F|.$$

($(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)$ should be read as 1 if $n=1$).

We have

$$|O_1(S)^F|=q^{n^2-n}(q^n+1)(q^2-1)(q^4-1)\cdots(q^{2n-2}-1). \tag{44}$$

We shall prove (43) and the formula

$$\sum_{r=0}^{n-1}(q^2-1)(q^4-1)\cdots(q^{2n-2r-2}-1)(-1)^rA^{(n)}_r(q^{m(2n-2r)}-1)=0\quad\text{if}\quad 0\leqq m\leqq n-1, \tag{45}$$

using Lusztig's formula (42).

PROPOSITION 8. *The formulas* (43) *and* (45) *hold.*

---

5) This literature was kindly communicated to the author by Professor N. Kawanaka.

PROOF. First we shall prove (45). For this purpose, we will solve (45) and verify that the "solution" $A_r^{(n)}$ are given by (42). Put $B_r^{(n)}=(q^2-1)(q^4-1)\cdots(q^{2n-2r-2}-1)(-1)^r A_r^{(n)}$. Then (45) is equivalent to

$$\sum_{r=0}^{n-1} B_r^{(n)} q^{m(2n-2r)}=\sum_{r=0}^{n-1} B_r^{(n)} \qquad (0\leqq m\leqq n-1). \tag{46}$$

By (7.11.4) of [4], we get

$$\sum_{r=0}^{n-1} B_r^{(n)}=(-1)^{n-1}|O_1(S)^F|/|T_0^F|=(-1)^{n-1}q^{n^2-n}\prod_{j=1}^{n-1}(q^{2j}-1).$$

The non-vanishing of the Vandermonde determinant implies that $B_r^{(n)}$, $0\leqq r\leqq n-1$ can be uniquely determined by (46). For an indeterminant x, let us consider a simultaneous equation

$$C_1^{(n)}x^m+C_2^{(n)}x^{2m}+\cdots+C_n^{(n)}x^{nm}=1 \qquad \text{for} \quad 0\leqq m\leqq n-1. \tag{47}$$

Write $C_i^{(n)}=C_i^{(n)}(x)$. Then we have

$$B_r^{(n)}=\Big[(-1)^{n-1}q^{n^2-n}\prod_{j=1}^{n-1}(q^{2j}-1)\Big]C_{n-r}^{(n)}(q^2). \tag{48}$$

Multiply the formula (47) for $m-1$ by x^n and subtract (47) for m from it $(1\leqq m\leqq n-1)$. Then we get

$$\sum_{i=1}^{n-1} C_i^{(n)}x^{im}(x^{n-i}-1)=x^n-1 \qquad \text{for} \quad 1\leqq m\leqq n-1. \tag{49}$$

Therefore we have

$$C_i^{(n)}=C_i^{(n-1)}(x^n-1)/(x^{n-i}-1)x^i \qquad \text{if} \quad 1\leqq i\leqq n-1. \tag{50}$$

By (48), we get

$$B_r^{(n)}=-B_{r-1}^{(n-1)}q^{2r-2}(q^{2(n-1)}-1)(q^{2n}-1)/(q^{2r}-1) \qquad \text{if} \quad 1\leqq r\leqq n-1. \tag{51}$$

Hence we obtain a recurrence formula

$$A_r^{(n)}=A_{r-1}^{(n-1)}q^{2r-2}(q^{2(n-1)}-1)(q^{2n}-1)/(q^{2r}-1) \qquad \text{if} \quad 1\leqq r\leqq n-1, \tag{52}$$

for the "solution" $A_r^{(n)}$ of (45). We shall show that

$$C_n^{(n)}(x)=(-1)^{n-1}x^{-(n^2-n)/2}. \tag{53}$$

Let D_i $(0\leqq i\leqq n-1)$ be the solution of

$$D_0+D_1x^m+D_2x^{2m}+\cdots+D_{n-1}x^{m(n-1)}=x^{mn} \qquad (0\leqq m\leqq n-1). \tag{54}$$

(53) is obviously equivalent to $D_0=(-1)^{n-1}x^{(n^2-n)/2}$. Multiply (54) for $m-1$ by x^{n-1} and subtract (54) for m $(1\leqq m\leqq n-1)$. Then we obtain $D_0=D_0'x^{-1}(1-x)/(x^{n-1}-1)$, where D_i' $(0\leqq i\leqq n-2)$ are given as the solution of

(55) $$\sum_{i=0}^{n-2} D_i' x^{mi} = x^{mn} \qquad (1 \leqq m \leqq n-1).$$

Multiply (55) for $m-1$ by x^{n-2} and subtract (55) for m $(2 \leqq m \leqq n-1)$. Then we obtain $D_0' = D_0'' x^{-2}(1-x^2)/(x^{n-2}-1)$, where D_i'' $(0 \leqq i \leqq n-3)$ are given as the solution of

(56) $$\sum_{i=0}^{n-3} D_i'' x^{mi} = x^{mn} \qquad (2 \leqq m \leqq n-1).$$

Repeating this process, we get $D_0 = \{x^{-1}(1-x)/(x^{n-1}-1)\}\{x^{-2}(1-x^2)/(x^{n-2}-1)\}\{x^{-3}(1-x^3)/(x^{n-3}-1)\} \cdots \{x^{-n+1}(1-x^{n-1})/(x-1)\} x^{n(n-1)} = (-1)^{n-1} x^{(n-1)n/2}$. Therefore we obtain (53). By (48), we see immediately that $A_0^{(n)}=1$ if $A_r^{(n)}$ $(1 \leqq r \leqq n-1)$ satisfy (45) for $0 \leqq m \leqq n-1$. From the recurrence formula (52), we see easily that the "solution" $A_r^{(n)}$ $(1 \leqq r \leqq n-1)$ is given by (42). Therefore (45) holds.

Now we shall show (43). We see that (43) is equivalent to

(57) $$\sum_{r=0}^{n-1} B_r^{(n)} q^{n(2n-2r)} = q^{n^2-n}(q^{2n}-1)(q^{2n-2}-1)^2(q^{2n-4}-1)^2 \cdots (q^2-1)^2 + (-1)^{n-1} q^{n^2-n}(q^{2n-2}-1)(q^{2n-4}-1) \cdots (q^2-1).$$

We put $I_n = \sum_{r=0}^{n-1} B_r^{(n)} q^{n(2n-2r)}$ and for the solution $C_i^{(n)}$ of (47) we put $J_n(x) = \sum_{i=1}^{n} C_i^{(n)} x^{ni}$. We multiply (47) for $n-1$ by x^n and subtract $J_n(x)$. Then we get

(58) $$x^n - J_n(x) = (x^n-1) J_{n-1}(x).$$

From (48), we have

(59) $$I_n = \left[(-1)^{n-1} q^{n^2-n} \prod_{j=1}^{n-1} (q^{2j}-1)\right] J_n(q^2).$$

Therefore we obtain

$$I_{n+1}/(-1)^n q^{n^2+n}(q^2-1)(q^4-1) \cdots (q^{2n}-1)$$
$$= q^{2(n+1)} - (q^{2(n+1)}-1) I_n / (-1)^{n-1} q^{n^2-n}(q^2-1)(q^4-1) \cdots (q^{2n-2}-1).$$

We use an induction on n i.e. assume that I_n is given by the right hand side of (57). Then we get $I_{n+1} = (-1)^n q^{n^2+3n+2}(q^2-1)(q^4-1) \cdots (q^{2n}-1) + q^{2n}(q^{2n}-1)(q^{2n+2}-1) I_n = q^{n(n+1)}(q^2-1)^2(q^4-1)^2 \cdots (q^{2n}-1)^2(q^{2n+2}-1) + (-1)^n q^{n(n+1)}(q^2-1)(q^4-1) \cdots (q^{2n}-1)$. Since the case $n \leqq 2$ can be verified immediately, (43) follows. This completes the proof.

THEOREM 2. *We exclude the cases $n=2$, $q=3$ and $n=3$, $q=3$. Let χ be a character of T_0^F which is in general position in $Sp(2n)^F$, when regarded as a character of T_1^F. Then $\pi_{S,n}(\rho)$ is equivalent to $(-1)^n R_{T_1}^{\chi}(Sp(2n)^F)$ for $\rho = (-1)^{n-1} R_{T_0}^{\chi}(O_1(S)^F)$.*

PROOF. By Proposition 7 and 8, we have $\dim \pi_{S,n}(\rho)=\dim(-1)^n R_{T_1}^{\chi}$. Hence our theorem follows from Proposition 4.

PROPOSITION 9. *Let χ be a character of T_0^F which is in general position in $O_1(S)^F$. Then $\pi_{S,m}(\rho)$, $\rho=(-1)^{n-1}R_{T_0}^{\chi}$ is the zero representation of $Sp(2m)^F$ for $1\leqq m\leqq n-1$.*

PROOF. By Proposition 7 and (45), we have $\dim \pi_{S,m}(\rho)=0$ for $1\leqq m\leqq n-1$, hence our assertion.

COROLLARY. *Let the notation be as in Proposition 9. Take the natural action of $O_1(S)^F$ on $\boldsymbol{F}(q)^{2n}$ as linear transformations. Let $v_1, v_2, \cdots, v_m$ be any vectors of $\boldsymbol{F}(q)^{2n}$ and put $Z(v_1, v_2, \cdots, v_m)=\{g\in O_1(S)^F \mid gv_i=v_i,\ 1\leqq i\leqq m\}$. If $m\leqq n-1$, the restriction of ρ to $Z(v_1, v_2, \cdots, v_m)$ does not contain the trivial representation.*

REMARK 3. If $m<n/2$, Proposition 9 follows only from Proposition 6. Since $\dim \eta_m=q^{2nm}$ and $\dim(-1)^{n-1}R_{T_0}^{\chi}\sim q^{n^2-n}$ $(q\to\infty)$, we get (45) if $m<n/2$ and q is sufficiently large (use the fact that there exists $q^n+O(q^{n/3})$ characters in general position of T_0^F). Since $A_r^{(n)}$ is a polynomials in q, which can be verified without using (42), we must have (45) for any odd q. This argument also applies to an arbitrary torus. Namely let T be a maximal F-stable torus of $O_1(S)$ and χ be a character in general position of T^F. Then we see that $\pi_{S,m}(R_T^{\chi}(O_1(S)^F)$ is the zero representation of $Sp(2m)^F$ if $m<(n-1)/2$ and q is sufficiently large. It might not be too reckless to conjecture that $\pi_{S,m}(R_T^{\chi}(O_1(S)^F)$ is the zero representation if $m<n$ and χ is in general position. We leave this question for future investigations.

References

[1] J.S. Andrade, Représentations de Weil de $GSp(4, F_q)$ associée à un espace quadratique de dimension 4 sur F_q, C.R. Acad. Sci. Paris, **278** (1974), 227-230.

[2] J.S. Andrade, La représentations de Weil de $GSp(4, F_q)$ et les représentations de $Sp(4, F_q)$, C.R. Acad. Sci. Paris, **278** (1974), 321-324.

[3] J.S. Andrade, Constructions géometrique de certaines series discretes, Lecture note in mathematics 597, 32-55, Springer, 1977.

[4] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math., **103** (1976), 103-161.

[5] S. Gelbart, Holomorphic discrete series for the real symplectic group, Invent. Math., **19** (1973), 49-58.

[6] R. Howe, Dual reductive pairs, to appear.

[7] H. Jacquet and R.P. Langlands, Automorphic forms on $GL(2)$, Lecture note in mathematics 114, Springer, 1970.

[8] G. Lusztig, On the Green polynomials of classical groups, Proc. London Math. Soc., (3) **33** (1976), 443-475.

[9] G. Lusztig, A note on counting nilpotent matrices of fixed rank, Bull. London Math. Soc., **8** (1976), 77-80.

[10] J. Milnor, On isometries of inner product spaces, Invent, Math., **8** (1969), 83-97.
[11] M. Saito, Représentations unitaires des groupes symplectique, J. Math. Soc. Japan, **24** (1972), 232-251.
[12] G. Shimura, Arithmetic of unitary groups, Ann. of Math., **79** (1964), 369-409.
[13] G. Shimura, Theta functions with complex multiplication, Duke Math. J., **43** (1976), 673-696.
[14] T. A. Springer and R. Steinberg, Conjugacy classes, Lecture note in mathematics 131, 167-266, Springer, 1970.
[15] S. Tanaka, On irreducible unitary representations of some special linear groups of the second order, I, II, Osaka J. Math., **3** (1966), 217-227, 229-242.
[16] S. Tanaka, Construction and classification of irreducible representations of special linear group of the second order over a finite field, Osaka J. Math., **4** (1967), 65-84.
[17] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math., **111** (1964), 143-211.

Hiroyuki YOSHIDA
Department of Mathematics
Faculty of Science
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto, Japan