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The orbits of affine symmetric spaces under the action of minimal parabolic subgroups

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Introduction

An affine symmetric space is a triple (G, H, σ) consisting of a connected Lie group G, a closed subgroup H of G and an involutive automorphism σ of G such that H lies between G_σ and the identity component of G_σ, where G_σ denotes the closed subgroup of G consisting of all the elements fixed by σ. Suppose that G is real semi-simple. We are interested in the double coset decomposition $H\backslash G/P$, where P is a minimal parabolic subgroup of G. These double cosets are considered as H-orbits on G/P or as P-orbits on $H\backslash G$.

If H is a maximal compact subgroup of G (when G is of finite center) and σ is the corresponding Cartan involution, this orbit structure is trivial in view of the Iwasawa decomposition $G=KA_{\mathfrak{p}}N^+$, where $P=MA_{\mathfrak{p}}N^+$ and $H=K$. If the affine symmetric space is $(G\times G, \Delta G, \sigma)$ where G is real semi-simple, ΔG denotes the diagonal of $G\times G$ and σ is the mapping $(x, y)\rightarrow(y, x)$, then the orbit structure can be easily reduced to the Bruhat decomposition $G=\bigcup_{w\in W} PwP$. In the case of (G_C, G, σ), where G_C is a complex semi-simple Lie group, G is a real form of G_C and σ is the conjugation of G_C with respect to G, then the orbit structure is studied in Aomoto [1] and Wolf [8].

In this paper the orbit structure is determined for an arbitrary affine symmetric space such that G is real semi-simple.

Let (G, H, σ) be an affine symmetric space such that G is real semi-simple, and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let θ be a Cartan involution commutative with σ (cf. Berger [2]), and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Since the factor space G/P is identified with the set of all the minimal parabolic subalgebras of $\mathfrak{g}$, the following theorem and corollary which are the extension of [1] Theorem 3 and of [8] 2.6 Theorem give a complete characterization of H-orbits on G/P.

THEOREM 1. (i) *Let $\mathfrak{P}$ be a minimal parabolic subalgebra of $\mathfrak{g}$. Then there exists a σ-stable maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ and a positive system Σ^+ of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ such that $\mathfrak{P}$ is H_0-conjugate to $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ (where H_0 is the identity component of H, $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}^+$, $\mathfrak{m}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, $\mathfrak{n}^+=\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$, and*

$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}\,|\,[Y, X]=\alpha(Y)X$ *for all* $Y\in\mathfrak{a}_\mathfrak{p}\}$).

(ii) *Let* $\mathfrak{a}_\mathfrak{p}$ *and* $\mathfrak{a}'_\mathfrak{p}$ *be* σ*-stable maximal abelian subspaces of* $\mathfrak{p}$, *and* Σ^+ *and* Σ'^+ *be positive systems of root systems* $\Sigma(\mathfrak{a}_\mathfrak{p})$ *and* $\Sigma(\mathfrak{a}'_\mathfrak{p})$ *respectively. If* $\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ *and* $\mathfrak{P}(\mathfrak{a}'_\mathfrak{p}, \Sigma'^+)$ *are H-conjugate, then* $\mathfrak{a}_\mathfrak{p}$ *and* $\mathfrak{a}'_\mathfrak{p}$ *are* K_+*-conjugate* ($K_+=H\cap K$).

If $\mathfrak{a}_\mathfrak{p}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, we can define a subgroup $W(\mathfrak{a}_\mathfrak{p}, K_+)$ of the Weyl group $W(\mathfrak{a}_\mathfrak{p})$ by $W(\mathfrak{a}_\mathfrak{p}, K_+)=(M^*(\mathfrak{a}_\mathfrak{p})\cap K_+)/(M(\mathfrak{a}_\mathfrak{p})\cap K_+)$, where $M^*(\mathfrak{a}_\mathfrak{p})=N_K(\mathfrak{a}_\mathfrak{p})$ and $M(\mathfrak{a}_\mathfrak{p})=Z_K(\mathfrak{a}_\mathfrak{p})$.

COROLLARY. *Let* $\{\mathfrak{a}_{\mathfrak{p}i}\,|\,i\in I\}$ *be representatives of the* K_+*-conjugacy classes of* σ*-stable maximal abelian subspaces of* $\mathfrak{p}$. *Then there exists a one-to-one correspondence between the H-conjugacy classes of minimal parabolic subalgebras of* $\mathfrak{g}$ *and* $\bigcup_{i\in I} W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})$ *(disjoint union). The correspondence is given as follows. Fix a positive system* Σ_i^+ *of* $\Sigma(\mathfrak{a}_{\mathfrak{p}i})$ *for each* $i\in I$. *Then* $W(\mathfrak{a}_{\mathfrak{p}i}, K_+)w\in\bigcup_{i\in I} W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})$ *corresponds to the H-conjugacy class of minimal parabolic subalgebras of* $\mathfrak{g}$ *containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}i}, w\Sigma_i^+)$.

In §2 the K_+-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$ will be investigated. Let $\mathfrak{a}_\mathfrak{p}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_{\mathfrak{p}+}=\mathfrak{a}_\mathfrak{p}\cap\mathfrak{h}$ is maximal abelian in $\mathfrak{p}_+=\mathfrak{p}\cap\mathfrak{h}$. Put $\mathfrak{q}=\{X\in\mathfrak{g}\,|\,\sigma(X)=-X\}$, and $\Sigma(\mathfrak{a}_{\mathfrak{p}+})=\{\alpha\in\Sigma(\mathfrak{a}_\mathfrak{p})\,|\,H_\alpha\in\mathfrak{a}_{\mathfrak{p}+}\}$, where H_α is the unique element in $\mathfrak{a}_\mathfrak{p}$ such that $B(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{a}_\mathfrak{p}$ (B is the Killing form of $\mathfrak{g}$). Let α_i, $i=1, \cdots, k$ be elements of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ and X_{α_i} $i=1, \cdots, k$ be non-zero elements of $\mathfrak{g}_{\alpha_i}$. Then $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ is said to be a $\mathfrak{q}$-orthogonal system of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ if the following two conditions are satisfied:

(i) $X_{\alpha_i}\in\mathfrak{q}$ for $i=1, \cdots, k$,

(ii) $[X_{\alpha_i}, X_{\alpha_j}]=0$ and $[X_{\alpha_i}, \theta(X_{\alpha_j})]=0$ for $i, j=1, \cdots k$, $i\neq j$.

Two $\mathfrak{q}$-orthogonal systems $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ and $\{Y_{\beta_1}, \cdots, Y_{\beta_k}\}$ of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ are said to be conjugate under $W(\mathfrak{a}_\mathfrak{p}, K_+)$ if there is a $w\in W(\mathfrak{a}_\mathfrak{p}, K_+)$ such that $w\left(\sum_{i=1}^k \boldsymbol{R}H_{\alpha_i}\right)=\sum_{i=1}^k \boldsymbol{R}H_{\beta_i}$. Then the following theorem gives a complete characterization of the K_+-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$. This theorem includes Theorem 6 and Theorem 7 of Sugiura [**5**] which are the fundamental theorems for the classification of conjugacy classes of Cartan subalgebras of real semi-simple Lie algebras.

THEOREM 2. *Let* (G, H, σ) *be an affine symmetric space such that G is real semi-simple,* θ *a Cartan involution of* $\mathfrak{g}$ *commutative with* σ, *and* $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ *the corresponding Cartan decomposition of* $\mathfrak{g}$. *Let* $\mathfrak{a}_{\mathfrak{p}+}$ *be a maximal abelian subspace of* $\mathfrak{p}_+$ *and* $\mathfrak{a}_\mathfrak{p}$ *a maximal abelian subspace of* $\mathfrak{p}$ *containing* $\mathfrak{a}_{\mathfrak{p}+}$. *Then there exists a one-to-one correspondence between the* K_+*-conjugacy classes of* σ*-stable maximal abelian subspaces of* $\mathfrak{p}$ *and the* $W(\mathfrak{a}_\mathfrak{p}, K_+)$*-conjugacy classes of* $\mathfrak{q}$*-orthogonal systems of* $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. *The correspondence is given as follows. Let* $Q=\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ *be a*

$\mathfrak{q}$-*orthogonal system of* $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. *Put* $\mathfrak{r}=\sum_{i=1}^{k}\boldsymbol{R}H_{\alpha_i}$, $\mathfrak{a}'_{\mathfrak{p}+}=\{H\in\mathfrak{a}_{\mathfrak{p}+}\mid B(H,\mathfrak{r})=0\}$, $\mathfrak{a}'_{\mathfrak{p}-}=\mathfrak{a}_{\mathfrak{p}-}+\sum_{i=1}^{k}\boldsymbol{R}(X_{\alpha_i}-X_{-\alpha_i})$ $(\mathfrak{a}_{\mathfrak{p}-}=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{q})$, *and* $\mathfrak{a}'_{\mathfrak{p}}=\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}'_{\mathfrak{p}-}$. *Then the* $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$-*conjugacy class of* $\mathfrak{q}$-*orthogonal system of* $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ *containing* Q *corresponds to the* K_+-*conjugacy class of* σ-*stable maximal abelian subspace of* $\mathfrak{p}$ *containing* $\mathfrak{a}'_{\mathfrak{p}}$. *Moreover if* X_{α_i}, $i=1,\cdots,k$ *is normalized such that* $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, X_{-\alpha_i})=-1$, *then* $\mathfrak{a}'_{\mathfrak{p}}=\mathrm{Ad}(\exp(\pi/2)(X_{\alpha_1}+X_{-\alpha_1})\cdots\exp(\pi/2)(X_{\alpha_k}+X_{-\alpha_k}))\mathfrak{a}_{\mathfrak{p}}$ *where* $X_{-\alpha_i}=\theta(X_{\alpha_i})$.

As a consequence of Corollary 1 of Theorem 1 and Theorem 2, the following theorem gives explicitly the double coset decomposition $H\backslash G/P$. Finiteness of $H\backslash G/P$ is also clear (cf. [**9**]).

THEOREM 3. *Let* (G, H, σ) *be an affine symmetric space such that* G *is real semi-simple,* θ *a Cartan involution commutative with* σ, *and* $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ *the corresponding Cartan decomposition. Let* $\mathfrak{a}_{\mathfrak{p}}$ *be a maximal abelian subspace of* $\mathfrak{p}$ *such that* $\mathfrak{a}_{\mathfrak{p}+}$ *is maximal abelian in* $\mathfrak{p}_+$, *and* $\{Q_1,\cdots,Q_m\}$ *be representatives of* $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$-*conjugacy classes of* $\mathfrak{q}$-*orthogonal systems of* $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. *Suppose that* $Q_j=\{X_{\alpha_1},\cdots,X_{\alpha_k}\}$ *is normalized such that* $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, X_{-\alpha_i})=-1$, $i=1,\cdots,k$ *for each* $j=1,\cdots,m$. *Put* $c(Q_j)=\exp(\pi/2)(X_{\alpha_1}+X_{-\alpha_1})\cdots\exp(\pi/2)(X_{\alpha_k}+X_{-\alpha_k})$. *Then*

(i) *We have the following decomposition of* G.

$$G=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})}Hw_vc(Q_i)P\quad(\textit{disjoint union})$$

where $P=P(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$, Σ^+ *is a positive system of* $\Sigma(\mathfrak{a}_{\mathfrak{p}})$, $\mathfrak{a}_{\mathfrak{p}i}=\mathrm{Ad}(c(Q_i))\mathfrak{a}_{\mathfrak{p}}$, *and* w_v *is an element of* $M^*(\mathfrak{a}_{\mathfrak{p}i})$ *that represents an element of the left coset* $v\subset W(\mathfrak{a}_{\mathfrak{p}i})$.

(ii) *Put* $P_{i,w_v}=w_vc(Q_i)Pc(Q_i)^{-1}w_v^{-1}$. *Let* $h_1, h_2\in H$ *and* $p_1, p_2\in P$. *Then* $h_1w_vc(Q_i)p_1=h_2w_vc(Q_i)p_2$ *if and only if there exists an* $x\in H\cap P_{i,w_v}$ *such that* $h_2=h_1x$ *and that* $p_2=c(Q_i)^{-1}w_v^{-1}x^{-1}w_vc(Q_i)p_1$.

(iii) *Let* $P=P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=MA'_{\mathfrak{p}}N^+$ *be a minimal parabolic subgroup of* G *such that* $\mathfrak{a}'_{\mathfrak{p}}$ *is* σ-*stable. Then*

$$H\cap P=(K_+\cap M)A'_{\mathfrak{p}+}\exp(\mathfrak{h}\cap\mathfrak{n}^+\cap\sigma\mathfrak{n}^+)\,.$$

We call $(G, H', \sigma\theta)$ the affine symmetric space associated with (G, H, σ) if $H'=K_+\exp(\mathfrak{p}\cap\mathfrak{q})$ (Berger [2]). Then the following two corollaries hold (Corollary 2 of Theorem 1 and Corollary of Theorem 3).

COROLLARY. *There exists a one-to-one correspondence between the* H-*conjugacy classes of minimal parabolic subalgebras of* $\mathfrak{g}$ *and the* H'-*conjugacy classes of them. In this correspondence the* H-*conjugacy class containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ ($\mathfrak{a}_{\mathfrak{p}}$ *is a* σ-*stable maximal abelian subspace of* $\mathfrak{p}$) *corresponds to the* H'-*conjugacy class containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$.

COROLLARY. *Retain the notations given in Theorem* 3 *and let* $(G, H', \sigma\theta)$ *be the affine symmetric space associated with* (G, H, σ). *Then we have the following two decompositions of* G.

$$G=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}_i},K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}_i})}Hw_vc(Q_i)P\quad(disjoint\ union)$$

$$=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}_i},K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}_i})}H'w_vc(Q_i)P\quad(disjoint\ union).$$

In § 3 the open orbits and the closed orbits are determined. The results are as follows. A minimal parabolic subalgebra $\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ ($\mathfrak{a}_\mathfrak{p}$ is σ-stable) is contained in an open orbit if and only if the following two conditions are satisfied:

(i) $\mathfrak{a}_{\mathfrak{p}-}$ is maximal abelian in $\mathfrak{p}_-$,

(ii) Σ^+ is $\sigma\theta$-compatible (i.e. $\alpha\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}+})\Leftrightarrow\sigma\theta(\alpha)\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}+})$).

The number of open orbits is $|W_\sigma(\mathfrak{a}_\mathfrak{p})|/|W(\mathfrak{a}_\mathfrak{p}, K_+)|$, where $W_\sigma(\mathfrak{a}_\mathfrak{p})$ is the subgroup of the Weyl group $W(\mathfrak{a}_\mathfrak{p})$ defined by $W_\sigma(\mathfrak{a}_\mathfrak{p})=\{w\in W(\mathfrak{a}_\mathfrak{p})\mid w(\mathfrak{a}_{\mathfrak{p}+})=\mathfrak{a}_{\mathfrak{p}+}\}$. On the contrary, the closed orbits are characterized by the minimal parabolic subalgebras $\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ of $\mathfrak{g}$ such that $\mathfrak{a}_{\mathfrak{p}+}$ is maximal abelian in $\mathfrak{p}_+$ and that Σ^+ is σ-compatible. The number of closed orbits is $|W_\sigma(\mathfrak{a}_\mathfrak{p})|/|W(\mathfrak{a}_\mathfrak{p}, K_+)|$. In the correspondence given in Corollary 2 of Theorem 1, open orbits correspond to closed orbits and closed ones to open ones.

§ 1. H-conjugacy classes of minimal parabolic subalgebras.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and σ an automorphism of G. Then the following notations are used throughout this paper. Let S be a Lie subgroup of G. Then S_σ denotes the set of σ-fixed points of S, that is $S_\sigma=\{x\in S\mid\sigma(x)=x\}$. S_0 denotes the identity component of S. Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two subsets of $\mathfrak{g}$, and let S be a subset of G. Then $\mathfrak{z}_{\mathfrak{s}_1}(\mathfrak{s}_2)$, $Z_S(\mathfrak{s}_2)$, and $N_S(\mathfrak{s}_2)$ denote the centralizer of $\mathfrak{s}_2$ in $\mathfrak{s}_1$, the centralizer of $\mathfrak{s}_2$ in S, and the normalizer of $\mathfrak{s}_2$ in S, respectively. More precisely,

$$\mathfrak{z}_{\mathfrak{s}_1}(\mathfrak{s}_2)=\{X\in\mathfrak{s}_1\mid[X, Y]=0\quad\text{for all } Y\in\mathfrak{s}_2\},$$

$$Z_S(\mathfrak{s}_2)=\{x\in S\mid\mathrm{Ad}(x)Y=Y\quad\text{for all } Y\in\mathfrak{s}_2\},$$

and

$$N_S(\mathfrak{s}_2)=\{x\in S\mid\mathrm{Ad}(x)\mathfrak{s}_2=\mathfrak{s}_2\}.$$

An affine symmetric space is a triple (G, H, σ) consisting of a connected Lie group G, an involutive automorphism σ of G, and a closed subgroup H of G such that $(G_\sigma)_0\subset H\subset G_\sigma$. We assume in the following that G is real semisimple.

Let (G, H, σ) be an affine symmetric space such that G is real semi-simple. Then (G, H, σ) gives rise to a triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ in a natural manner, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, and the automorphism σ of $\mathfrak{g}$ is the one induced by the automorphism σ of G. Such a triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is called a symmetric Lie algebra. Put $\mathfrak{q}=\{X\in\mathfrak{g}\mid\sigma(X)=-X\}$. Then $\mathfrak{g}$ is decomposed to

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{q} \quad \text{(direct sum)}.$$

There exist Cartan involutions commutative with σ (Berger [2]). Let θ be one of them, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. Then we have

$$\mathfrak{g}=\mathfrak{k}_{+}+\mathfrak{k}_{-}+\mathfrak{p}_{+}+\mathfrak{p}_{-} \quad \text{(direct sum)},$$

where $\mathfrak{k}_{+}=\mathfrak{k}\cap\mathfrak{h}$, $\mathfrak{k}_{-}=\mathfrak{k}\cap\mathfrak{q}$, $\mathfrak{p}_{+}=\mathfrak{p}\cap\mathfrak{h}$, and $\mathfrak{p}_{-}=\mathfrak{p}\cap\mathfrak{q}$.

REMARK. The existence of Cartan involutions commutative with σ is also proved by Lemma 3 of this paper, and all such Cartan involutions are determined in Lemma 4.

Let K denote the analytic subgroup of G corresponding to $\mathfrak{k}$. Put $K_{+}=K\cap H$, $H'=K_{+}\exp\mathfrak{p}_{-}$, and $\mathfrak{h}'=\mathfrak{k}_{+}+\mathfrak{p}_{-}$. Then $(\mathfrak{g}, \mathfrak{h}', \sigma\theta)$ is a symmetric Lie algebra. In the Cartan decomposition $G=K\exp\mathfrak{p}$, K and $\exp\mathfrak{p}$ are σ-stable. So we have $G_{\sigma\theta}=K_{\sigma}\exp\mathfrak{p}_{-}$ and $(G_{\sigma\theta})_{0}=(K_{\sigma})_{0}\exp\mathfrak{p}_{-}$. Since K_{+} is a subgroup of K and $(K_{\sigma})_{0}\subset K_{+}\subset K_{\sigma}$, it follows that H' is a subgroup of G and $(G_{\sigma\theta})_{0}\subset H'\subset G_{\sigma\theta}$. Thus $(G, H', \sigma\theta)$ is an affine symmetric space. $(\mathfrak{g}, \mathfrak{h}', \sigma\theta)$ is called the symmetric Lie algebra associated with $(\mathfrak{g}, \mathfrak{h}, \sigma)$ and $(G, H', \sigma\theta)$ is called the affine symmetric space associated with (G, H, σ) ([2]).

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, $\mathfrak{a}_{\mathfrak{p}}$ a maximal abelian subspace of $\mathfrak{p}$, and Σ^{+} a positive system of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$. Then the subalgebra

$$\mathfrak{P}=\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^{+})=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}^{+}$$

is a minimal parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{m}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, $\mathfrak{n}^{+}=\sum_{\alpha\in\Sigma^{+}}\mathfrak{g}_{\alpha}$, and $\mathfrak{g}_{\alpha}=\{X\in\mathfrak{g}\mid[H, X]=\alpha(H)X \text{ for all } H\in\mathfrak{a}_{\mathfrak{p}}\}$. And the subgroup

$$P=P(\mathfrak{a}_{\mathfrak{p}}, \Sigma^{+})=MA_{\mathfrak{p}}N^{+}$$

is a minimal parabolic subgroup of G, where $A_{\mathfrak{p}}$ and N^{+} are the analytic subgroups of G corresponding to $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{n}^{+}$ respectively, and $M=Z_{K}(\mathfrak{a}_{\mathfrak{p}})$. Since all the minimal parabolic subalgebras are conjugate under $\mathrm{Ad}(G)$, they are obtained in this way. So are all the minimal parabolic subgroups.

Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and let $\mathfrak{P}=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}^{+}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ obtained as is stated above. Then a subspace $\mathfrak{a}$ of $\mathfrak{P}$ is called a split component of $\mathfrak{P}$ if $\mathfrak{a}=\mathrm{Ad}(n)\mathfrak{a}_{\mathfrak{p}}$ for some $n\in N^{+}$.

LEMMA 1. *Let $\mathfrak{a}$ be a split component of $\mathfrak{P}$. If $\mathfrak{a}$ is contained in another minimal parabolic subalgebra $\mathfrak{P}'$, then $\mathfrak{a}$ is a split component of $\mathfrak{P}'$.*

PROOF. For some Cartan decomposition $\mathfrak{g}=\mathfrak{k}'+\mathfrak{p}'$ of $\mathfrak{g}$, $\mathfrak{P}'$ can be written as $\mathfrak{P}'=\mathfrak{m}'+\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}^{+\prime}$. Let H_{1} be a regular element in $\mathfrak{a}$ (i.e. $\alpha(H_{1})\neq 0$ for all

$\alpha\in\Sigma(\mathfrak{a})$). Then $\mathfrak{a}$ is the set of all the elements X in $\mathfrak{z}(H_1)$ such that all the eigenvalues of ad$X$ are real numbers ([7], p. 57). Since the eigenvalues of adH_1 are all real, it follows that $H_1\in\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}^{+\prime}$ ([7], p. 57). Write $H_1=H_2+Y$ ($H_2\in\mathfrak{a}_{\mathfrak{p}'}$, $Y\in\mathfrak{n}^{+\prime}$). Then H_2 is regular. Thus there is an $n\in N^{+\prime}$ such that $H_1=\mathrm{Ad}(n)H_2$ ([4], p. 231). Since $\mathfrak{a}_{\mathfrak{p}'}$ is the set of all the elements X in $\mathfrak{z}(H_2)=\mathfrak{m}'+\mathfrak{a}_{\mathfrak{p}'}$ such that all the eigenvalues of ad$X$ are real, we have $\mathfrak{a}=\mathrm{Ad}(n)\mathfrak{a}_{\mathfrak{p}'}$. Thus $\mathfrak{a}$ is a split component of $\mathfrak{P}'$. q. e. d.

REMARK. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and $\mathfrak{a}_{\mathfrak{p}}$ a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{P}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{p}}$. Then it follows from Lemma 1 that $\mathfrak{P}=\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ for some positive system Σ^+ of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$.

Fix a minimal parabolic subgroup P of G and its Lie algebra $\mathfrak{P}$. For every element x of G there corresponds a minimal parabolic subalgebra $\mathrm{Ad}(x)\mathfrak{P}$. This gives a one-to-one correspondence between G/P and the set of minimal parabolic subalgebras of $\mathfrak{g}$ because P is the normalizer of $\mathfrak{P}$. Hence the problem is reduced to the characterization of H-conjugacy classes of minimal parabolic subalgebras of $\mathfrak{g}$. Then the following theorem holds.

THEOREM 1. *Let (G, H, σ) be an affine symmetric space such that G is a connected real semi-simple Lie group. Let θ be a Cartan involution commutative with σ, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. Then*

(i) *for every minimal parabolic subalgebra $\mathfrak{P}$ of $\mathfrak{g}$, there exist a σ-stable maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ and a positive system Σ^+ of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ such that $\mathfrak{P}$ is H_0-conjugate to $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$.*

(ii) *Let $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}'$ be σ-stable maximal abelian svbspaces of $\mathfrak{p}$, and let Σ^+ and Σ'^+ be positive systems of the root systems $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ and $\Sigma(\mathfrak{a}_{\mathfrak{p}}')$ respectively. Then if $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ and $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}', \Sigma'^+)$ are H-conjugate, $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}'$ are K_+-conjugate.*

To prove this theorem we prepare three lemmas.

LEMMA 2. *Every minimal parabolic subalgebra $\mathfrak{g}$ has a σ-stable split component.*

PROOF. Let $\mathfrak{P}$ be a minimal parabolic subalgebra of $\mathfrak{g}$ and P be the corresponding minimal parabolic subgroup of G. Let $\mathfrak{a}_{\mathfrak{p}'}$ be a split component of $\mathfrak{P}$ and let $\mathfrak{g}=\mathfrak{k}'+\mathfrak{p}'$ be a Cartan decomposition of $\mathfrak{g}$ such that $\mathfrak{a}_{\mathfrak{p}'}$ is a maximal abelian subspace of $\mathfrak{p}'$. Then we can write $\mathfrak{P}=\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}'}, \Sigma^+)=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}^+$ and $P=P(\mathfrak{a}_{\mathfrak{p}'}, \Sigma^+)=MA_{\mathfrak{p}'}N^+$. Since $\sigma\mathfrak{P}$ is also a minimal parabolic subalgebra of $\mathfrak{g}$, there is an $x\in G$ such that $\sigma\mathfrak{P}=\mathrm{Ad}(x)\mathfrak{P}$. By the Bruhat's lemma, x can be written as $x=nm_wp$ ($n\in N^+$, $p\in P$, $w\in W'=N_{K'}(\mathfrak{a}_{\mathfrak{p}'})/Z_{K'}(\mathfrak{a}_{\mathfrak{p}'})$, and m_w is an element of $N_{K'}(\mathfrak{a}_{\mathfrak{p}'})$ that represents w). Then

$$\sigma\mathfrak{P}\cap\mathfrak{P}=\mathrm{Ad}(x)\mathfrak{P}\cap\mathfrak{P}=\mathrm{Ad}(n)(\mathrm{Ad}(m_w)\mathfrak{P}\cap\mathfrak{P}).$$

If we set $\mathfrak{n}_w^+=\sum_{\alpha\in\Sigma^+\cap w\Sigma^+}\mathfrak{g}_\alpha$, then

$$\mathrm{Ad}(m_w)\mathfrak{P}\cap\mathfrak{P}=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}_w^+ .$$

Hence

$$\sigma\mathfrak{P}\cap\mathfrak{P}=\mathrm{Ad}(n)(\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}_w^+) .$$

Let H_1 be a regular element of $\mathfrak{a}_{\mathfrak{p}'}$. Then the mapping $f: n\to\mathrm{Ad}(n)H_1-H_1$ is an analytic diffeomorphism of N^+ onto $\mathfrak{n}^+$ ([4], p. 231), and $N_w^+=\exp\mathfrak{n}_w^+$ is mapped onto $\mathfrak{n}_w^+$ by this mapping. In fact, if $X\in\mathfrak{n}_w^+$, then

$$f(\exp X)=\mathrm{Ad}(\exp X)H_1-H_1=[X, H_1]+\frac{1}{2}[X, [X, H_1]]+\cdots\in\mathfrak{n}_w^+ .$$

Conversely let $X=\sum_{\alpha\in\Sigma^+}X_\alpha$ be an element of $\mathfrak{n}^+$ which is not contained in $\mathfrak{n}_w^+$, and let β be the lowest root of $\Sigma^+\cap w\Sigma^-$ such that $X_\beta\neq 0$. Then

$$f(\exp X)\equiv[X_\beta, H_1]\not\equiv 0 \qquad \mathrm{mod} \sum_{\alpha\in\Sigma^+-\{\beta\}}\mathfrak{g}_\alpha ,$$

so $f(\exp X)\notin\mathfrak{n}_w^+$. Let $n'\in N^+$. Then $\mathrm{Ad}(n')H_1$ is contained in $\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}'}+\mathfrak{n}_w^+$ if and only if $n'\in N_w^+$. It follows that every split component of $\mathfrak{P}$ which is contained in $\sigma\mathfrak{P}\cap\mathfrak{P}$ is of the form $\mathrm{Ad}(nn')\mathfrak{a}_{\mathfrak{p}'}$, $n'\in N_w^+$.

Let $\mathfrak{a}=\mathrm{Ad}(nn')\mathfrak{a}_{\mathfrak{p}'}$ be one of them. Since $\sigma(\mathfrak{a})$ is a split component of $\sigma\mathfrak{P}$ and is contained in $\sigma\mathfrak{P}\cap\mathfrak{P}$, $\sigma(\mathfrak{a})$ is a split component of $\mathfrak{P}$. (Lemma 1). Thus there is a unique element $n''\in N_w^+$ such that $\sigma(\mathfrak{a})=\mathrm{Ad}(nn'')\mathfrak{a}_{\mathfrak{p}'}$. The mapping $n'\to n''$ is a continuous involutive mapping of N_w^+ onto itself, so it has a fixed point. In fact, if it has no fixed point, we have a two fold covering $N_w^+\to N_w^+/\sim$ by the equivalence relation $n'\sim n''$. This is impossible since the Euler characteristic number of $N_w^+\cong\boldsymbol{R}^k$ is one. Let n_1 be a fixed point. Then $\mathfrak{a}_1=\mathrm{Ad}(nn_1)\mathfrak{a}_{\mathfrak{p}'}$ is a σ-stable split component contained in $\mathfrak{P}$. q. e. d.

LEMMA 3. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a symmetric Lie algebra such that $\mathfrak{g}$ is real semisimple, θ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{t}$ a $\sigma\theta$-stable subspace of $\mathfrak{g}$. Then there exists an $x\in\mathrm{Ad}(G)$ such that $x\theta x^{-1}$ commutes with σ and that $x(\mathfrak{t})=\mathfrak{t}$.*

PROOF. This lemma is proved in the same way as in p. 156 of [4] as follows. It is shown as in [4] that $\sigma\theta$ is a self-adjoint transformation of $\mathfrak{g}$ with respect to the positive definite inner product B_θ ($B_\theta(X, Y)=-B(X, \theta Y)$ for X, $Y\in\mathfrak{g}$). Since $\mathfrak{t}$ is $\sigma\theta$-stable, we can take an orthonormal basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ such that $\{X_1, \cdots, X_m\}$ is a basis of $\mathfrak{t}$ and that $\sigma\theta$ is represented by a diagonal matrix with respect to this basis. Put $\tau=(\sigma\theta)^2$ and define τ^s ($s\in\boldsymbol{R}$) as in [4]. Then $\tau^{1/4}\theta\tau^{-1/4}$ commutes with σ, $\tau^{1/4}\in\mathrm{Ad}(G)$, and $\tau^{1/4}\mathfrak{t}=\mathfrak{t}$. q. e. d.

LEMMA 4. *Let θ_1 and θ_2 be Cartan involutions commutative with σ. Then there exists an $h\in\mathrm{Ad}(H_0)$ such that $h\theta_1h^{-1}=\theta_2$.*

PROOF. Put $\tau'=\theta_2\theta_1$. Since σ commutes with θ_1, $\mathfrak{h}$ and $\mathfrak{q}$ are orthogonal with respect to B_{θ_1}. On the other hand τ' commutes with σ, so $\tau'(\mathfrak{h})=\mathfrak{h}$ and $\tau'(\mathfrak{q})=\mathfrak{q}$. Put $\tau=(\tau')^2$ and define τ^s ($s\in\boldsymbol{R}$) as in [4]. Then $\tau^{1/4}\theta_1\tau^{-1/4}$ commutes with θ_2. Then it is easily shown that $\tau^{1/4}\theta_1\tau^{-1/4}=\theta_2$ ([4], p. 158). On the other hand we have $\tau^s=\mathrm{Ad}(\exp sX)$ for some $X\in\mathfrak{g}$. Remark that $[X, \mathfrak{h}]\subset\mathfrak{h}$, and $[X, \mathfrak{q}]\subset\mathfrak{q}$. Then if we write $X=X_1+X_2$ ($X_1\in\mathfrak{h}$, $X_2\in\mathfrak{q}$), it follows that

$$[X_2, \mathfrak{h}]\subset\mathfrak{h}\cap\mathfrak{q}=\{0\} \quad \text{and} \quad [X_2, \mathfrak{q}]\subset\mathfrak{q}\cap\mathfrak{h}=\{0\} .$$

Thus $[X_2, \mathfrak{g}]=\{0\}$. Since $\mathfrak{g}$ is semi-simple, this implies $X_2=0$. Hence $\tau^{1/4}=\exp((1/4)X_1)\in\mathrm{Ad}(H_0)$. q. e. d.

PROOF OF THEOREM 1. (i) A σ-stable split component $\mathfrak{a}$ which is obtained in Lemma 2 is a maximal abelian subspace of $\mathfrak{p}'$ for some Cartan decomposition $\mathfrak{g}=\mathfrak{k}'+\mathfrak{p}'$. Let θ' be the corresponding Cartan involution. Then Lemma 3 implies that there exists a Cartan involution θ'' commutative with σ such that $\mathfrak{a}\subset\mathfrak{p}''$ ($\mathfrak{g}=\mathfrak{k}''+\mathfrak{p}''$ is the Cartan decomposition corresponding to θ''). By Lemma 4 there is an $h\in H_0$ such that $\mathfrak{a}_\mathfrak{p}=\mathrm{Ad}(h)\mathfrak{a}$ is contained in $\mathfrak{p}$. It is clear that $\mathfrak{a}_\mathfrak{p}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$. Thus there exists a positive system Σ^+ of $\Sigma(\mathfrak{a}_\mathfrak{p})$ such that $\mathrm{Ad}(h)\mathfrak{P}=\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ (cf. the Remark following Lemma 1).

(ii) Let $\mathfrak{P}=\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ and $\mathfrak{P}'=\mathfrak{P}(\mathfrak{a}'_\mathfrak{p}, \Sigma'^+)$ be H-conjugate. Then there is an $h\in H$ such that $\mathrm{Ad}(h)\mathfrak{a}'_\mathfrak{p}$ is a split component of $\mathfrak{P}$. Since $\mathfrak{a}_\mathfrak{p}$ is σ-stable so is $\mathfrak{m}$. Then

$$\sigma\mathfrak{P}=\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\sigma\mathfrak{n}^+=\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+w\mathfrak{n}^+$$

for some element w in the Weyl group W, and

$$\mathfrak{P}\cap\sigma\mathfrak{P}=\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}_w^+ .$$

Thus we have $\mathrm{Ad}(h)\mathfrak{a}'_\mathfrak{p}=\mathrm{Ad}(n_1)\mathfrak{a}_\mathfrak{p}$ for some $n_1\in N_w^+$. Since $\mathrm{Ad}(h)\mathfrak{a}'_\mathfrak{p}$ is σ-stable, it follows that

$$\mathrm{Ad}(n_1)\mathfrak{a}_\mathfrak{p}=\sigma(\mathrm{Ad}(n_1)\mathfrak{a}_\mathfrak{p})=\mathrm{Ad}(\sigma n_1)\mathfrak{a}_\mathfrak{p} ,$$

hence $n_1=\sigma n_1$. Let $n_1=\exp X$, $X\in\mathfrak{n}_w^+$. Then $\exp\sigma X=\exp X$, so $\sigma X=X$, $X\in\mathfrak{h}$, and $n_1\in H_0$. Put $n_1{}^{-1}h=h'=k\exp X$ ($k\in K$, $X\in\mathfrak{p}$). Then $k\in K_+$ and $\mathrm{Ad}(k\exp X)\mathfrak{a}'_\mathfrak{p}=\mathfrak{a}_\mathfrak{p}$. Since $\mathrm{Ad}(\exp X)H=H$ for every $H\in\mathfrak{a}'_\mathfrak{p}$ ([7], p. 28), we have $\mathrm{Ad}(k)\mathfrak{a}'_\mathfrak{p}=\mathfrak{a}_\mathfrak{p}$. Thus $\mathfrak{a}'_\mathfrak{p}$ is K_+-conjugate to $\mathfrak{a}_\mathfrak{p}$. q. e. d.

REMARK. As is proved above, two maximal abelian subspaces $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}'_\mathfrak{p}$ of $\mathfrak{p}$ are H-conjugate if and only if they are K_+-conjugate.

Define a subgroup $W(\mathfrak{a}_\mathfrak{p}, K_+)$ of the Weyl group $W(\mathfrak{a}_\mathfrak{p})$ by

$$W(\mathfrak{a}_\mathfrak{p}, K_+)=(M^*(\mathfrak{a}_\mathfrak{p})\cap K_+)/(M(\mathfrak{a}_\mathfrak{p})\cap K_+)=N_{K_+}(\mathfrak{a}_\mathfrak{p})/Z_{K_+}(\mathfrak{a}_\mathfrak{p}) ,$$

where $M(\mathfrak{a}_\mathfrak{p})=Z_K(\mathfrak{a}_\mathfrak{p})$ and $M^*(\mathfrak{a}_\mathfrak{p})=N_K(\mathfrak{a}_\mathfrak{p})$. Then the following corollary holds.

COROLLARY 1. *Let $\{\mathfrak{a}_{\mathfrak{p}i}\mid i\in I\}$ be a set of representatives of K_+-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$. Then there exists a one-to-one correspondence between the H-conjugacy classes of minimal parabolic subalgebras of $\mathfrak{g}$ and $\bigcup_{i\in I} W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})$ (disjoint union). The correspondence is given as* *follows. Fix a positive system* Σ_i^+ *of* $\Sigma(\mathfrak{a}_{\mathfrak{p}i})$ *for each* $i\in I$. *Then* $W(\mathfrak{a}_{\mathfrak{p}i}, K_+)w\in \bigcup_{i\in I} W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})$ *corresponds to the H-conjugacy class of minimal parabolic subalgebras of* $\mathfrak{g}$ *containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}i}, w\Sigma_i^+)$.

PROOF. It follows from (i) of Theorem 1 that every minimal parabolic subalgebra is H-conjugate to one of $\{\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}i}, w\Sigma_i^+)\,|\,i\in I,\ w\in W(\mathfrak{a}_{\mathfrak{p}i})\}$. On the other hand $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}i}, w\Sigma_i^+)$ is H-conjugate to $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}j}, w'\Sigma_j^+)$ if and only if $i=j$ and $w=w''w'$ for some $w''\in W(\mathfrak{a}_{\mathfrak{p}i}, K_+)$ in view of (ii) of Theorem 1. Thus the given correspondence is a bijection between $\bigcup_{i\in I} W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})$ and the H-conjugacy classes of minimal parabolic subalgebras of $\mathfrak{g}$. q.e.d.

The following corollary follows easily from Corollary 1.

COROLLARY 2. *There exists a one-to-one correspondence between the H-conjugacy classes of minimal parabolic subalgebras of* $\mathfrak{g}$ *and the H'-conjugacy classes of them. In this correspondence the H-conjugacy class containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ ($\mathfrak{a}_{\mathfrak{p}}$ *is a* σ*-stable maximal abelian subspace of* $\mathfrak{p}$) *corresponds to the H'-conjugacy class containing* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$.

EXAMPLE 1. Let G_C be a connected complex semi-simple Lie group and G a connected real form of G_C. Then (G_C, G, σ) is an affine symmetric space, where σ is the conjugation of G_C with respect to G. Let $\mathfrak{g}_C$ and $\mathfrak{g}$ be Lie algebras of G_C and G respectively. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and put $\mathfrak{u}=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. Then $\mathfrak{g}_C=\mathfrak{u}+\sqrt{-1}\,\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}_C$, and we have $\mathfrak{g}_C=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}+\mathfrak{p}+\sqrt{-1}\,\mathfrak{k}$. Let θ be the corresponding Cartan involution. Let $\mathfrak{h}'=\mathfrak{h}_{\mathfrak{p}}+\sqrt{-1}\,\mathfrak{h}_{\mathfrak{k}}$ be a σ-stable maximal abelian subspace of $\sqrt{-1}\,\mathfrak{u}$ ($\mathfrak{h}_{\mathfrak{p}}\subset\mathfrak{p}$, $\sqrt{-1}\,\mathfrak{h}_{\mathfrak{k}}\subset\sqrt{-1}\,\mathfrak{k}$). Then $\mathfrak{h}=\mathfrak{h}_{\mathfrak{p}}+\mathfrak{h}_{\mathfrak{k}}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$. Thus in this case the problem is reduced to the study of conjugacy of Cartan subalgebras of $\mathfrak{g}$ (cf. [5]). A minimal parabolic subgroup is a Borel subgroup in this case. For some examples, $G\backslash G_C/B$ is determined in [1]. Open orbits and closed orbits are determined in [8] (see also §3). The associated affine symmetric space is $(G_C, K_C, \sigma\theta)$ (K_C is the analytic subgroup of G_C corresponding to $\mathfrak{k}_C$).

EXAMPLE 2. Consider an affine symmetric space $(G\times G, \Delta G, \sigma)$, where G is a connected real semi-simple Lie group, ΔG is the diagonal of $G\times G$, and σ is the mapping $(x, y)\to(y, x)$. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Then the mapping $\theta+\theta\colon \mathfrak{g}+\mathfrak{g}\to\mathfrak{g}+\mathfrak{g}$ ($(\theta+\theta)(X, Y)=(\theta X, \theta Y)$ for $X, Y\in\mathfrak{g}$) is a Cartan involution of $\mathfrak{g}+\mathfrak{g}$ commutative with σ. Then $\mathfrak{g}+\mathfrak{g}=\Delta\mathfrak{k}+\tilde{\mathfrak{k}}+\Delta\mathfrak{p}+\tilde{\mathfrak{p}}$, where $\tilde{\mathfrak{k}}=\{(X, -X)\,|\,X\in\mathfrak{k}\}$ and $\tilde{\mathfrak{p}}=\{(Y, -Y)\,|\,Y\in\mathfrak{p}\}$. A subspace $\mathfrak{a}_{\mathfrak{p}}+\mathfrak{a}_{\mathfrak{p}}$ ($\mathfrak{a}_{\mathfrak{p}}$ is a maximal abelian subspace of $\mathfrak{p}$) is a σ-stable maximal abelian subspace of $\mathfrak{p}+\mathfrak{p}$. In this case all the σ-stable maximal abelian subspaces of $\mathfrak{p}+\mathfrak{p}$ are ΔK-conjugate. If we identify $W(\mathfrak{a}_{\mathfrak{p}}+\mathfrak{a}_{\mathfrak{p}})$ with $W(\mathfrak{a}_{\mathfrak{p}})+W(\mathfrak{a}_{\mathfrak{p}})$, then

$$W(\mathfrak{a}_{\mathfrak{p}}+\mathfrak{a}_{\mathfrak{p}}, \Delta K)=\Delta W(\mathfrak{a}_{\mathfrak{p}}).$$

§ 2. K_+-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$.

The results of this section are obtained by following the process in [5] (or [7], p. 88~p. 95). The following lemma seems to be familiar. But since this is frequently used in this section, we state it with a proof for the sake of completeness.

LEMMA 5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, θ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{h}$ be a θ-stable semi-simple subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{h}\cap\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{h}$.*

PROOF. Let $B_{\mathfrak{g}}$ and $B_{\mathfrak{h}}$ be the Killing forms of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then $B_{\mathfrak{g}}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Let $\{X_1, \cdots, X_k\}$ be an orthonormal basis of $\mathfrak{k}$ with respect to $-B_{\mathfrak{g}}$ such that $\{X_1, \cdots, X_s\}$ is a basis of $\mathfrak{h}\cap\mathfrak{k}$. Let $\{Y_1, \cdots, Y_m\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to $B_{\mathfrak{g}}$ such that $\{Y_1, \cdots, Y_t\}$ is a basis of $\mathfrak{h}\cap\mathfrak{p}$. Then if $X\in\mathfrak{h}\cap\mathfrak{k}$, $\operatorname{ad}X$ is represented by a skew-symmetric matrix with respect to the basis $\{X_1, \cdots, X_k, Y_1, \cdots, Y_m\}$ of $\mathfrak{g}$ ([4], p. 215). So is $\operatorname{ad}X|_{\mathfrak{h}}$, the restriction of $\operatorname{ad}X$ to $\mathfrak{h}$. In the same way if $Y\in\mathfrak{h}\cap\mathfrak{p}$, then $\operatorname{ad}Y|_{\mathfrak{h}}$ is represented by a symmetric matrix. Put $\operatorname{ad}X|_{\mathfrak{h}}=(a_{ij})$ and $\operatorname{ad}Y|_{\mathfrak{h}}=(b_{ij})$. Then

$$B_{\mathfrak{h}}(X, X)=\operatorname{tr}(\operatorname{ad}X|_{\mathfrak{h}}{}^2)=\sum_{i,j=1}^{s+t}a_{ij}a_{ji}=-\sum_{i,j=1}^{s+t}a_{ij}{}^2$$

and
$$B_{\mathfrak{h}}(Y, Y)=\sum_{i,j=1}^{s+t}b_{ij}{}^2 .$$

If $X\neq 0$, then $\operatorname{ad}X|_{\mathfrak{h}}\neq 0$ since $\mathfrak{h}$ is semi-simple, so $B_{\mathfrak{h}}(X, X)<0$. If $Y\neq 0$, then $B_{\mathfrak{h}}(Y, Y)>0$. This implies that $\mathfrak{h}=\mathfrak{h}\cap\mathfrak{k}+\mathfrak{h}\cap\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{h}$.

q. e. d.

The fundamental tool of this section is the following.

MAXIMUM PRINCIPLE ([7], p. 90, and [5], Theorem 1). Let $\mathfrak{g}$ be a semi-simple Lie algebra, and let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be subalgebras of $\mathfrak{g}$ generating tori T_1 and T_2 in $\operatorname{Int}(\mathfrak{g})$. If there exists a compact subgroup L with Lie algebra $\mathfrak{l}$ in $\operatorname{Int}(\mathfrak{g})$ such that

$$[k(\mathfrak{t}_1), \mathfrak{t}_2]\subset\mathfrak{l}$$

for all $k\in L$, then there exists an element $k_0\in L$ such that

$$[k_0(\mathfrak{t}_1), \mathfrak{t}_2]=0 .$$

Let (G, H, σ) be an affine symmetric space such that G is real semi-simple, and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let θ be a Cartan involution of $\mathfrak{g}$ commutative with σ, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition, and K the analytic subgroup of G for $\mathfrak{k}$. Put $K_+=K\cap H$ and $K'_+=\operatorname{Ad}(K_+)_0$.

Let $\mathfrak{a}_{\mathfrak{p}}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$. Write $\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}_{\mathfrak{p}+}+\mathfrak{a}_{\mathfrak{p}-}$ ($\mathfrak{a}_{\mathfrak{p}+}=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}$, $\mathfrak{a}_{\mathfrak{p}-}=\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{q}$).

LEMMA 6. *$\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}}$, $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}$ and $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}$ generate tori in* $\mathrm{Int}(\mathfrak{g}_C)$.

PROOF. Let T, T_+ and T_- be the analytic subgroups of $\mathrm{Int}(\mathfrak{g}_C)$ corresponding to $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}}$, $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}$ and $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}$, respectively. We have only to prove that T, T_+ and T_- are closed subgroups of $\mathrm{Int}(\mathfrak{g}_C)$ since the analytic subgroup U of $\mathrm{Int}(\mathfrak{g}_C)$ corresponding to $\mathfrak{u}=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$ is compact. It is proved for T in [5]. Extend σ to the automorphism of $\mathrm{Int}(\mathfrak{g}_C)$. Then T is σ-stable. Put $T'_+=\{x\in T\,|\,\sigma(x)=x\}$. Since T'_+ is a closed subgroup of T and coincides to T_+ in a neighborhood of the origin, T_+ is also closed. In the same way it is proved that T_- is closed. q. e. d.

LEMMA 7. *All the σ-stable maximal abelian subspaces of $\mathfrak{p}$ such that the $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$)-parts are maximal abelian in $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) are mutually K'_+-conjugate.*

PROOF. Since $\mathfrak{h}=\mathfrak{k}_++\mathfrak{p}_+$ is θ-stable, $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$ ([5], p. 42). So $\bar{\mathfrak{h}}=[\mathfrak{h},\mathfrak{h}]$ is semi-simple and θ-stable. Then $\bar{\mathfrak{h}}=\bar{\mathfrak{h}}\cap\mathfrak{k}+\bar{\mathfrak{h}}\cap\mathfrak{p}=\bar{\mathfrak{k}}+\bar{\mathfrak{p}}$ is a Cartan decomposition of $\bar{\mathfrak{h}}$ (Lemma 5). Let $\mathfrak{c}$ be the center of $\mathfrak{h}$. $\mathfrak{c}$ is θ-stable and is written as $\mathfrak{c}=\mathfrak{c}_{\mathfrak{k}}+\mathfrak{c}_{\mathfrak{p}}$. Then every maximal abelian subspace of $\mathfrak{p}_+$ is the sum of $\mathfrak{c}_{\mathfrak{p}}$ and a maximal abelian subspace of $\bar{\mathfrak{p}}$. Since all the maximal abelian subspaces in $\bar{\mathfrak{p}}$ are conjugate under $\bar{K}$ (the analytic subgroup of $\mathrm{Int}(\mathfrak{g}_C)$ corresponding to $\bar{\mathfrak{k}}$), so are those in $\mathfrak{p}_+$. Hence all the maximal abelian subspaces in $\mathfrak{p}_+$ are conjugate under K'_+.

Given $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}'_{\mathfrak{p}}$ which satisfy the condition of the Lemma, we can thus assume that $\mathfrak{a}_{\mathfrak{p}+}=\mathfrak{a}'_{\mathfrak{p}+}$. $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}$ and $\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-}$ generate tori in $\mathrm{Int}(\mathfrak{g}_C)$ (Lemma 6) and $Z_{K'_+}(\mathfrak{a}_{\mathfrak{p}+})$ is a compact subgroup of $\mathrm{Int}(\mathfrak{g}_c)$. If $k\in Z_{K'_+}(\mathfrak{a}_{\mathfrak{p}+})$, then

$$[\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-},\ k(\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-})]\subset\mathfrak{k}_+\,,$$

and

$$\begin{aligned}&[\mathfrak{a}_{\mathfrak{p}+},\ [\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-},\ k(\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-})]]\\ &=[[\mathfrak{a}_{\mathfrak{p}+},\ \mathfrak{a}_{\mathfrak{p}-}],\ k(\mathfrak{a}'_{\mathfrak{p}-})]+[\mathfrak{a}_{\mathfrak{p}-},\ [\mathfrak{a}_{\mathfrak{p}+},\ k(\mathfrak{a}'_{\mathfrak{p}-})]]\\ &=\{0\}\,.\end{aligned}$$

Therefore $[\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-},\ k(\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-})]\subset\mathfrak{z}_{\mathfrak{k}_+}(\mathfrak{a}_{\mathfrak{p}+})$. Applying the Maximum principle, there exists a $k_0\in Z_{K'_+}(\mathfrak{a}_{\mathfrak{p}+})$ such that

$$[\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-},\ k_0(\sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-})]=[\mathfrak{a}_{\mathfrak{p}-},\ k_0(\mathfrak{a}'_{\mathfrak{p}-})]=\{0\}\,.$$

On the other hand $[\mathfrak{a}_{\mathfrak{p}+},\ k_0(\mathfrak{a}'_{\mathfrak{p}-})]=\{0\}$. It follows that $k_0(\mathfrak{a}'_{\mathfrak{p}-})=\mathfrak{a}_{\mathfrak{p}-}$ because $\mathfrak{a}_{\mathfrak{p}}$ is maximal abelian in $\mathfrak{p}$. Thus $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}'_{\mathfrak{p}}$ are K'_+-conjugate. q. e. d.

REMARK. If $\mathfrak{a}_{\mathfrak{p}}$ is a maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{p}_+$ (resp. $\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{p}_-$) is maximal abelian in $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$), then it is easily shown that $\mathfrak{a}_{\mathfrak{p}}$ is σ-stable.

LEMMA 8. *Let* $\mathfrak{a}_\mathfrak{p}$ *and* $\mathfrak{a}'_\mathfrak{p}$ *be* σ*-stable maximal abelian subspaces of* $\mathfrak{p}$. *If* $\mathfrak{a}'_{\mathfrak{p}+}\subset\mathfrak{a}_{\mathfrak{p}+}$, *then there exists an element* $k\in Z_{K'_+}(\mathfrak{a}'_{\mathfrak{p}+})$ *such that* $k(\mathfrak{a}_{\mathfrak{p}-})\subset\mathfrak{a}'_{\mathfrak{p}-}$. *Moreover if* $\mathfrak{a}'_{\mathfrak{p}+}=\mathfrak{a}_{\mathfrak{p}+}$, *then* $k(\mathfrak{a}_{\mathfrak{p}-})=\mathfrak{a}'_{\mathfrak{p}-}$.

PROOF. If $k\in Z_{K'_+}(\mathfrak{a}'_{\mathfrak{p}+})$, then

$$[k(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}),\ \sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-}]\subset\mathfrak{k}_+\,,$$

and

$$\begin{aligned}&[\mathfrak{a}'_{\mathfrak{p}+},\ [k(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}),\ \sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-}]]\\&=[[\mathfrak{a}'_{\mathfrak{p}+},\ k(\mathfrak{a}_{\mathfrak{p}-})],\ \mathfrak{a}'_{\mathfrak{p}-}]+[k(\mathfrak{a}_{\mathfrak{p}-}),\ [\mathfrak{a}'_{\mathfrak{p}+},\ \mathfrak{a}'_{\mathfrak{p}-}]]\\&=\{0\}\,.\end{aligned}$$

Thus $[k(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}-}),\ \sqrt{-1}\,\mathfrak{a}'_{\mathfrak{p}-}]\subset\mathfrak{z}_{\mathfrak{k}_+}(\mathfrak{a}'_{\mathfrak{p}+})$. Applying the Maximum principle, there exists a $k_0\in Z_{K'_+}(\mathfrak{a}'_{\mathfrak{p}+})$ such that

$$[k_0(\mathfrak{a}_{\mathfrak{p}-}),\ \mathfrak{a}'_{\mathfrak{p}-}]=\{0\}\,.$$

On the other hand $[k_0(\mathfrak{a}_{\mathfrak{p}-}),\ \mathfrak{a}'_{\mathfrak{p}+}]=\{0\}$, hence we have

$$k_0(\mathfrak{a}_{\mathfrak{p}-})\subset\mathfrak{a}'_{\mathfrak{p}-}\,. \qquad \text{q.e.d.}$$

DEFINITION. Fix a σ-stable maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$ such that $\mathfrak{a}_{\mathfrak{p}+}$ is maximal abelian in $\mathfrak{p}_+$. Then a σ-stable maximal abelian subspace $\mathfrak{a}'_\mathfrak{p}$ of $\mathfrak{p}$ is said to be standard if $\mathfrak{a}'_{\mathfrak{p}+}\subset\mathfrak{a}_{\mathfrak{p}+}$ and $\mathfrak{a}'_{\mathfrak{p}-}\supset\mathfrak{a}_{\mathfrak{p}-}$.

It follows from Lemma 7 and Lemma 8 that every K_+-conjugacy class of σ-stable maximal abelian subspaces of $\mathfrak{p}$ contains a standard one.

LEMMA 9. *Let* $\mathfrak{a}_\mathfrak{p}$, $\mathfrak{a}'_\mathfrak{p}$ *and* $\mathfrak{a}''_\mathfrak{p}$ *be* σ*-stable maximal abelian subspaces of* $\mathfrak{p}$ *where* $\mathfrak{a}_{\mathfrak{p}+}$ *is maximal abelian in* $\mathfrak{p}_+$. *Suppose that* $\mathfrak{a}'_{\mathfrak{p}+}\subset\mathfrak{a}_{\mathfrak{p}+}$, $\mathfrak{a}''_{\mathfrak{p}+}\subset\mathfrak{a}_{\mathfrak{p}+}$, $\mathfrak{a}_{\mathfrak{p}-}\subset\mathfrak{a}'_{\mathfrak{p}-}$, *and* $\mathfrak{a}_{\mathfrak{p}-}\subset\mathfrak{a}''_{\mathfrak{p}-}$. *Then* $\mathfrak{a}'_\mathfrak{p}$ *and* $\mathfrak{a}''_\mathfrak{p}$ *are* K_+*-conjugate if and only if* $\mathfrak{a}'_{\mathfrak{p}+}$ *and* $\mathfrak{a}''_{\mathfrak{p}+}$ *are conjugate under* $W(\mathfrak{a}_\mathfrak{p}, K_+)$.

PROOF. If $\mathfrak{a}'_{\mathfrak{p}+}$ and $\mathfrak{a}''_{\mathfrak{p}+}$ are conjugate under $W(\mathfrak{a}_\mathfrak{p}, K_+)$, it follows from Lemma 8 that $\mathfrak{a}'_\mathfrak{p}$ and $\mathfrak{a}''_\mathfrak{p}$ are K_+-conjugate.

We will prove the coverse assertion. Suppose $k'\mathfrak{a}'_\mathfrak{p}=\mathfrak{a}''_\mathfrak{p}$, $k'\in\mathrm{Ad}\,(K_+)$. Then $k'\mathfrak{a}'_{\mathfrak{p}+}=\mathfrak{a}''_{\mathfrak{p}+}$. Let k_1 be an element of $Z_{K'_+}(\mathfrak{a}''_{\mathfrak{p}+})$. Then

$$[k_1k'(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}),\ \sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}]\subset\mathfrak{k}_+\,,$$

and

$$\begin{aligned}&[\mathfrak{a}''_{\mathfrak{p}+},\ [k_1k'(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}),\ \sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}]]\\&=[[\mathfrak{a}''_{\mathfrak{p}+},\ k_1k'(\mathfrak{a}_{\mathfrak{p}+})],\ \mathfrak{a}_{\mathfrak{p}+}]+[k_1k'(\mathfrak{a}_{\mathfrak{p}+}),\ [\mathfrak{a}''_{\mathfrak{p}+},\ \mathfrak{a}_{\mathfrak{p}+}]]\\&=\{0\}\,.\end{aligned}$$

Thus $[k_1k'(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}),\ \sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}]\subset\mathfrak{z}_{\mathfrak{k}_+}(\mathfrak{a}''_{\mathfrak{p}+})$. Applying the Maximum principle, there exists a $k_0\in Z_{K'_+}(\mathfrak{a}''_{\mathfrak{p}+})$ such that

$$[k_0k'(\sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}),\ \sqrt{-1}\,\mathfrak{a}_{\mathfrak{p}+}]=\{0\}\,.$$

Hence $k_0k'(\mathfrak{a}_{\mathfrak{p}+})=\mathfrak{a}_{\mathfrak{p}+}$ because $\mathfrak{a}_{\mathfrak{p}+}$ is maximal abelian in $\mathfrak{p}_+$. On the other hand, we have

$$k_0k'(\mathfrak{a}'_{\mathfrak{p}+})=\mathfrak{a}''_{\mathfrak{p}+}.$$

It follows from Lemma 8 that there is a $k_2\in Z_{K'_+}(\mathfrak{a}_{\mathfrak{p}+})$ such that

$$k_2k_0k'(\mathfrak{a}_{\mathfrak{p}})=\mathfrak{a}_{\mathfrak{p}},$$

so $k_2k_0k'\in\mathrm{Ad}(M^*(\mathfrak{a}_{\mathfrak{p}})\cap K_+)$. Since $k_2k_0k'(\mathfrak{a}'_{\mathfrak{p}+})=\mathfrak{a}''_{\mathfrak{p}+}$, $\mathfrak{a}'_{\mathfrak{p}+}$ and $\mathfrak{a}''_{\mathfrak{p}+}$ are conjugate under $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$. q.e.d.

We will show in the latter half of this section when a subspace of a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}+}$ of $\mathfrak{p}_+$ is the $\mathfrak{p}_+$-part of some σ-stable maximal abelian subspace of $\mathfrak{p}$. Fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}+}$ of $\mathfrak{p}_+$ and a σ-stable maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ containing $\mathfrak{a}_{\mathfrak{p}+}$, Let H_α be the unique element of $\mathfrak{a}_{\mathfrak{p}}$ such that $B(H_\alpha, H)=\alpha(H)$ for all $H\in\mathfrak{a}_{\mathfrak{p}}$.

LEMMA 10. *Let $\mathfrak{a}'_{\mathfrak{p}}$ be a standard σ-stable maximal abelian subspace of $\mathfrak{p}$. Put* $\mathfrak{r}=\{H\in\mathfrak{a}_{\mathfrak{p}+}\mid B(H,\mathfrak{a}'_{\mathfrak{p}+})=0\}$ *and* $\Sigma(\mathfrak{r})=\{\alpha\in\Sigma\mid H_\alpha\in\mathfrak{r}\}$. *Then* $\mathfrak{r}=\sum_{\alpha\in\Sigma(\mathfrak{r})}\boldsymbol{R}H_\alpha$.

PROOF. Note that

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}_{\mathfrak{p}-})=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\sum_{\alpha\in\Sigma(\mathfrak{r})}\mathfrak{g}_\alpha,$$

and $\mathfrak{c}'_{\mathfrak{p}}\subset\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}_{\mathfrak{p}-})$. If $\mathfrak{r}\supsetneqq\sum_{\alpha\in\Sigma(\mathfrak{r})}\boldsymbol{R}H_\alpha$, then there is a non-zero element $H_1\in\mathfrak{r}$ such that $\alpha(H_1)=0$ for all $\alpha\in\Sigma(\mathfrak{r})$. Hence

$$[H_1, \mathfrak{a}'_{\mathfrak{p}}]\subset[H_1, \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}_{\mathfrak{p}-})]=\{0\}.$$

This implies that $H_1\in\mathfrak{a}'_{\mathfrak{p}}$. Then we have $H_1\in\mathfrak{a}'_{\mathfrak{p}}\cap\mathfrak{r}=\{0\}$, a contradiction.

q.e.d.

DEFINITION. For a subspace $\mathfrak{r}$ of $\mathfrak{a}_{\mathfrak{p}+}$, put $\Sigma(\mathfrak{r})=\{\alpha\in\Sigma\mid H_\alpha\in\mathfrak{r}\}$. Then a subspace $\mathfrak{r}$ of $\mathfrak{a}_{\mathfrak{p}+}$ satisfying $\mathfrak{r}=\sum_{\alpha\in\Sigma(\mathfrak{r})}\boldsymbol{R}H_\alpha$ is called a root space of $\mathfrak{a}_{\mathfrak{p}+}$.

Let $\mathfrak{r}\subset\mathfrak{a}_{\mathfrak{p}+}$ be a root space and put $\mathfrak{r}'=\{X\in\mathfrak{a}_{\mathfrak{p}}\mid B(X,\mathfrak{r})=0\}$. Let $\mathfrak{g}(\mathfrak{r})$ be the subalgebra of $\mathfrak{g}$ generated by $\sum_{\alpha\in\Sigma(\mathfrak{r})}\mathfrak{g}_\alpha$, and $\mathfrak{c}$ the center of $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}')$. Then

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{r}')=\mathfrak{c}+\mathfrak{g}(\mathfrak{r})\quad\text{(direct sum)},$$

where $\mathfrak{g}(\mathfrak{r})$ is semi-simple ([7], p. 66). Since $\mathfrak{g}(\mathfrak{r})$ is σ-stable and θ-stable, we have

$$\mathfrak{g}(\mathfrak{r})=\mathfrak{k}(\mathfrak{r})+\mathfrak{p}(\mathfrak{r})=\mathfrak{k}_+(\mathfrak{r})+\mathfrak{k}_-(\mathfrak{r})+\mathfrak{p}_+(\mathfrak{r})+\mathfrak{p}_-(\mathfrak{r})$$

where $\mathfrak{k}(\mathfrak{r})=\mathfrak{k}\cap\mathfrak{g}(\mathfrak{r})$, and so on. Since $\mathfrak{r}$ is a maximal abelian subspace of $\mathfrak{p}(\mathfrak{r})$ contained in $\mathfrak{p}_+(\mathfrak{r})$ ([7], p. 67), and since $\mathfrak{g}(\mathfrak{r})=\mathfrak{k}(\mathfrak{r})+\mathfrak{p}(\mathfrak{r})$ is a Cartan decomposition of $\mathfrak{g}(\mathfrak{r})$ (Lemma 5), we have

$$\mathfrak{g}(\mathfrak{r})=\mathfrak{m}(\mathfrak{r})+\mathfrak{r}+\sum_{\alpha\in\Sigma(\mathfrak{r})}\mathfrak{g}_{\alpha}\,,$$

where $\mathfrak{m}(\mathfrak{r})=\mathfrak{z}_{\cdot(\mathfrak{r})}(\mathfrak{r})$.

Let $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Then $\sigma\mathfrak{g}_{\alpha}=\mathfrak{g}_{\alpha}$, so we can write $\mathfrak{g}_{\alpha}=\mathfrak{g}_{\alpha+}+\mathfrak{g}_{\alpha-}$, $\mathfrak{g}_{\alpha+}=\mathfrak{g}_{\alpha}\cap\mathfrak{h}$, $\mathfrak{g}_{\alpha-}=\mathfrak{g}_{\alpha}\cap\mathfrak{q}$.

DEFINITION. Let $\{X_{\alpha_1},\cdots,X_{\alpha_k}\}$ be a set of non-zero root vectors of $\mathfrak{g}$ such that $X_{\alpha_i}\in\mathfrak{g}_{\alpha_i}$ and $\alpha_i\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Then $\{X_{\alpha_1},\cdots,X_{\alpha_k}\}$ is said to be a $\mathfrak{q}$-orthogonal system of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ if the following two conditions are satisfied:

(i) $X_{\alpha_i}\in\mathfrak{g}_{\alpha_i-}$ for $i=1,\cdots,k$,

(ii) $[X_{\alpha_i},X_{\alpha_j}]=0$, $[X_{\alpha_i},\theta(X_{\alpha_j})]=0$ for $i,j=1,\cdots,k$, $i\neq j$.

LEMMA 11. *Let Σ be a root system of the pair $(\mathfrak{g},\mathfrak{a}_{\mathfrak{p}})$. Let $X_{\alpha}\in\mathfrak{g}_{\alpha}$, $X_{\beta}\in\mathfrak{g}_{\beta}$, and $X_{-\alpha}=\theta(X_{\alpha})$ $(\alpha,\beta\in\Sigma)$. Suppose that $X_{\alpha}\neq0$, $X_{\beta}\neq0$,*

$$(\mathrm{ad}X_{\alpha})^{k}X_{\beta}\neq0 \quad \text{for} \quad k=1,\cdots,s,$$

$$(\mathrm{ad}X_{\alpha})^{s+1}X_{\beta}=0,$$

$$(\mathrm{ad}X_{-\alpha})^{k}X_{\beta}\neq0 \quad \text{for} \quad k=1,\cdots,-r,$$

and

$$(\mathrm{ad}X_{-\alpha})^{-r+1}X_{\beta}=0.$$

Then

(i) $[X_{-\alpha},[X_{\alpha},X_{\beta}]]=\dfrac{s(1-r)}{2}\alpha(H_{\alpha})B(X_{\alpha},X_{-\alpha})X_{\beta}$.

(ii) $-2\dfrac{\beta(H_{\alpha})}{\alpha(H_{\alpha})}=r+s$.

Especially,

(iii) *If $(\alpha,\beta)<0$, then $[X_{\alpha},X_{\beta}]\neq0$.*

(iv) *If $(\alpha,\beta)=0$ and $[X_{\alpha},X_{\beta}]\neq0$, then $[X_{-\alpha},X_{\beta}]\neq0$ (where $(\alpha,\beta)=B(H_{\alpha},H_{\beta})=\beta(H_{\alpha})$).*

PROOF. Note that $[X_{\alpha},X_{-\alpha}]=B(X_{\alpha},X_{-\alpha})H_{\alpha}$ ([7], p. 66). Then the proofs of (i) and (ii) are the same as those of theorems on complex semi-simple Lie algebras ([4], p. 143~145). (iii) and (iv) follows from (ii). q.e.d.

It follows from (iii) of Lemma 11 that if $\{X_{\alpha_1},\cdots,X_{\alpha_k}\}$ is $\mathfrak{q}$-orthogonal then $(\alpha_i,\alpha_j)=0$ for $i\neq j$.

LEMMA 12. *Let $(\mathfrak{g},\mathfrak{h},\sigma)$ be a symmetric Lie algebra such that $\mathfrak{g}$ is semi-simple, θ a Cartan involution commutative with σ, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Suppose that there is a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ contained in $\mathfrak{p}_{+}$. Then the following two conditions are equivalent:*

(i) *There exists a maximal abelian subspace of $\mathfrak{p}$ contained in $\mathfrak{p}_{-}$;*

(ii) *There exists a $\mathfrak{q}$-orthogonal system $\{X_{\alpha_1},\cdots,X_{\alpha_k}\}$ of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ $(k=\dim\mathfrak{a}_{\mathfrak{p}})$.*

PROOF. (ii) ⇒ (i). Since $X_{\alpha_i}-X_{-\alpha_i}\in\mathfrak{p}_{-}$ and

$$[X_{\alpha_i}-X_{-\alpha_i}, X_{\alpha_j}-X_{-\alpha_j}]=0 \qquad \text{for all } i, j=1, \cdots, k,$$

the subspace of $\mathfrak{p}_-$ spanned by $\{X_{\alpha_i}-X_{-\alpha_i}|i=1, \cdots, k\}$ is a desired one.

(i) $\Rightarrow$ (ii). For every positive integer m, define a proposition (P_m) as follows.

(P_m). If $\mathfrak{p}_-$ contains an abelian subspace of dimension m, then there exists a $\mathfrak{q}$-orthogonal system $\{X_{\alpha_1}, \cdots, X_{\alpha_m}\}$ of $\Sigma(\mathfrak{a}_\mathfrak{p})$.

For every $\mathfrak{q}$-orthogonal system $Q=\{X_{\alpha_1}, \cdots, X_{\alpha_m}\}$ of $\Sigma(\mathfrak{a}_\mathfrak{p})$, define a real number $f(Q)$ by $f(Q)=\sum_{i=1}^{m}(\alpha_i, \alpha_i)$. We will prove (P_m) $(m\leqq k)$ by induction.

(P_1). As $\mathfrak{p}_-\neq\{0\}$, there is an $\alpha\in\Sigma$ such that $\mathfrak{g}_{\alpha-}\neq\{0\}$. Take a non-zero element $X_\alpha\in\mathfrak{g}_{\alpha-}$. Then $\{X_\alpha\}$ is a $\mathfrak{q}$-orthogonal system.

$(\mathrm{P}_{m-1})\Rightarrow(\mathrm{P}_m)$. Consider the set of all the $\mathfrak{q}$-orthogonal systems of $\Sigma(\mathfrak{a}_\mathfrak{p})$ consisting of $m-1$ root vectors. Then we can choose a $\mathfrak{q}$-orthogonal system $Q=\{X_{\alpha_1}, \cdots, X_{\alpha_{m-1}}\}$ such that $f(Q)$ attains the maximum value.

As is proved in Lemma 7 all the maximal abelian subspaces of $\mathfrak{p}_-$ have the same dimension. So we can take an m-dimensional abelian subspace $\mathfrak{t}$ of $\mathfrak{p}_-$ containing $\{X_{\alpha_i}-X_{-\alpha_i}|i=1, \cdots, m-1\}$. Every element Y of $\mathfrak{t}$ is written as $Y=\sum_{\beta>0}(Y_\beta-Y_{-\beta})$, $Y_\beta\in\mathfrak{g}_{\beta-}$, $Y_{-\beta}=\theta(Y_\beta)$. Then we have

$$\text{(A)} \qquad 0=[X_{\alpha_i}-X_{-\alpha_i}, Y]=\sum_{\beta>0}([X_{\alpha_i}, Y_\beta]-[X_{-\alpha_i}, Y_\beta]$$
$$-[X_{\alpha_i}, Y_{-\beta}]+[X_{-\alpha_i}, Y_{-\beta}])$$

for every $i=1, \cdots, m-1$. Put $\Sigma_1=\{\alpha\in\Sigma|H_\alpha\in\sum_{i=1}^{m-1}\boldsymbol{R}H_{\alpha_i}\}$. As the rank of the semi-simple subalgebra generated by $\sum_{\alpha\in\Sigma_1}\mathfrak{g}_\alpha$ is $m-1$ ([7], p. 67), $\mathfrak{t}$ is not contained in $\sum_{\alpha\in\Sigma_1}\mathfrak{g}_\alpha$. Thus there is an element $Y\in\mathfrak{t}$ such that $Y_\beta\neq0$ for some $\beta\in\Sigma^+-\Sigma_1^+$.

If $[X_{\alpha_i}, Y_\beta]=0$ and $[X_{\alpha_i}, Y_{-\beta}]=0$ for all $i=1, \cdots, m-1$, then $\{X_{\alpha_1}, \cdots, X_{\alpha_{m-1}}, Y_\beta\}$ is $\mathfrak{q}$-orthogonal. This implies (P_m). Thus we can assume that $[X_{\alpha_i}, Y_\beta]\neq0$ or $[X_{\alpha_i}, Y_{-\beta}]\neq0$ for some i. Changing the order of the roots, we have only to consider the case $[X_{\alpha_i}, Y_{-\beta}]\neq0$. In the right hand side of (A), the only non-trivial terms contained in $\mathfrak{g}_{\alpha_i-\beta}$ are $-[X_{\alpha_i}, Y_{-\beta}]$ and $-[X_{-\alpha_i}, Y_{\beta'}]$ if $\beta'=2\alpha_i-\beta>0$ ($-[X_{\alpha_i}, Y_{-\beta}]$ and $[X_{-\alpha_i}, Y_{\beta'}]$ if $\beta'<0$). Thus changing the order of the roots if necessary, we can assume $[X_{\alpha_i}, Y_{-\beta}]=-[X_{-\alpha_i}, Y_{\beta'}]$. In particular, $\beta'\in\Sigma$.

It is easily seen that β and β' are not proportional. If $(\beta, \beta')>0$, then $\beta'-\beta\in\Sigma$. If we set $\gamma=\beta'-\alpha_i$, then $\beta'-\beta=2\gamma$. (Fig. 1). Then we have $\angle\beta0\gamma>\frac{\pi}{2}$ or $\angle\beta'0\gamma<\frac{\pi}{2}$. Suppose that $\angle\beta'0\gamma<\frac{\pi}{2}$. Since $\frac{2(\beta', 2\gamma)}{(2\gamma, 2\gamma)}$ and $\frac{2(\gamma, \beta')}{(\beta', \beta')}$ are integers and since

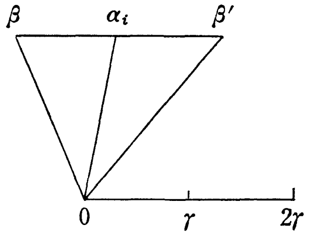

Fig. 1

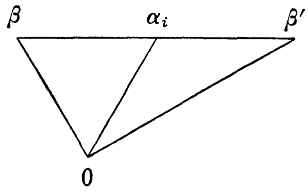

Fig. 2

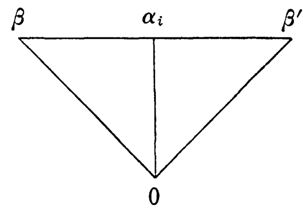

Fig. 3

$$\frac{2(\beta', 2\gamma)}{(2\gamma, 2\gamma)}\frac{2(\gamma, \beta')}{(\beta', \beta')}=\frac{2(\beta', \gamma)^2}{(\beta', \beta')(\gamma, \gamma)}<2,$$

it follows that

$$\frac{2(\beta', 2\gamma)}{(2\gamma, 2\gamma)}=1 \quad \text{and} \quad \frac{2(\gamma, \beta')}{(\beta', \beta')}=1.$$

Thus $\angle\beta' 0\gamma=\frac{\pi}{4}$ and $|\beta'|=\sqrt{2}|\gamma|$. Similarly if $\angle\beta 0\gamma>\frac{\pi}{2}$, then $\angle\beta 0\gamma=\frac{3}{4}\pi$ and $|\beta|=\sqrt{2}|\gamma|$. In both cases we have $(\beta, \beta')=0$, a contradiction. We also have $(\beta, \beta')\not<0$ in the same way. Hence $(\beta, \beta')=0$. Then the two cases are possible (Fig. 2 and Fig. 3).

The root system of Fig. 2 only occurs in the root system of the type G_2. Thus the simple ideal of $\mathfrak{g}$ containing X_{α_i} is either the complex simple Lie algebra of the type G_2 or the normal real form of the type G_2. Since $\beta+\alpha_i\in\Sigma$, $[X_{\alpha_i}, Y_\beta]\neq 0$ in both cases. It follows from (A) that there exists a $\beta''\in\Sigma$ such that $[X_{\alpha_i}, Y_\beta]=[X_{-\alpha_i}, Y_{\beta''}]$. But $\beta''=\beta+2\alpha_i$ cannot be a root, hence this is a contradiction.

Therefore we have only to consider the case of Fig. 3. If $[X_{\alpha_j}, Y_{-\beta}]\neq 0$ for some $j\neq i$, then $\angle\beta 0\alpha_j=\frac{\pi}{4}$ as is stated above. Since $\angle\alpha_i 0\alpha_j=\frac{\pi}{2}$, it follows that 0, α_i, α_j and β lies on the same plane. This is impossible because $\beta\notin\Sigma_1$. Hence $[X_{\alpha_j}, Y_{-\beta}]=0$ and similarly $[X_{\alpha_j}, Y_\beta]=0$ for $j\neq i$. Therefore $Q'=\{X_{\alpha_1}, \cdots, \hat{X}_{\alpha_i}, \cdots X_{\alpha_{m-1}}, Y_\beta\}$ (X_{α_i} is excluded) is $\mathfrak{q}$-orthogonal and $f(Q')>f(Q)$. This contradicts the assumption that $f(Q)$ attains the maximum value. Thus we have proved (P_m). q.e.d.

DEFINITION. Two $\mathfrak{q}$-orthogonal systems $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ and $\{Y_{\beta_1}, \cdots, Y_{\beta_k}\}$ of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ are said to be conjugate under $W(\mathfrak{a}_\mathfrak{p}, K_+)$ if there exists an element $w\in W(\mathfrak{a}_\mathfrak{p}, K_+)$ such that

$$w\Big(\sum_{i=1}^k \boldsymbol{R}H_{\alpha_i}\Big)=\sum_{i=1}^k \boldsymbol{R}H_{\beta_i}.$$

The following theorem includes Theorem 6 and Theorem 7 of [5].

THEOREM 2. *Let (G, H, σ) be an affine symmetric space such that G is real semi-simple, θ a Cartan involution of $\mathfrak{g}$ commutative with σ, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}_{\mathfrak{p}+}$ be a maximal abelian subspace of $\mathfrak{p}_+$ and $\mathfrak{a}_{\mathfrak{p}}$ a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_{\mathfrak{p}+}$. Then there exists a one-to-one correspondence between the K_+-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$ and the $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$-conjugacy classes of $\mathfrak{q}$-orthogonal systems of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. The correspondence is given as follows. Let $Q=\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ be a $\mathfrak{q}$-orthogonal systems of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Put $\mathfrak{r}=\sum_{i=1}^{k}\boldsymbol{R}H_{\alpha_i}$, $\mathfrak{a}'_{\mathfrak{p}+}=\{H\in\mathfrak{a}_{\mathfrak{p}+}\mid B(H, \mathfrak{r})=0\}$, $\mathfrak{a}'_{\mathfrak{p}-}=\mathfrak{a}_{\mathfrak{p}-}+\sum_{i=1}^{k}\boldsymbol{R}(X_{\alpha_i}-X_{-\alpha_i})$, and $\mathfrak{a}'_{\mathfrak{p}}=\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}'_{\mathfrak{p}-}$. Then the $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$-conjugacy class of $\mathfrak{q}$-orthogonal system of $\Sigma(\mathfrak{a}_{\mathfrak{p}}{}^+)$ containing Q corresponds to the K_+-conjugacy class of σ-stable maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}'_{\mathfrak{p}}$. Moreover if X_{α_i}, $i=1, \cdots, k$ is normalized such that $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, X_{-\alpha_i})=-1$, then $\mathfrak{a}'_{\mathfrak{p}}=\mathrm{Ad}\,(\exp\frac{\pi}{2}(X_{\alpha_1}+X_{-\alpha_1})\cdots\exp\frac{\pi}{2}(X_{\alpha_k}+X_{-\alpha_k}))\mathfrak{a}_{\mathfrak{p}}$.*

PROOF. Let $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ be a $\mathfrak{q}$-orthogonal system of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Then the $\mathfrak{a}'_{\mathfrak{p}}$ given in the statement of Theorem 2 is a standard σ-stable maximal abelian subspace of $\mathfrak{p}$ (Lemma 12).

Conversely let $\mathfrak{a}'_{\mathfrak{p}}$ be a standard σ-stable maximal abelian subspace of $\mathfrak{p}$. Put $\mathfrak{r}=\{H\in\mathfrak{a}_{\mathfrak{p}+}\mid B(H, \mathfrak{a}'_{\mathfrak{p}+})=0\}$. Since $\mathfrak{a}'_{\mathfrak{p}-}\subset\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}'_{\mathfrak{p}+}+\mathfrak{a}_{\mathfrak{p}-})\cap\mathfrak{p}_-=\mathfrak{c}\cap\mathfrak{p}_-+\mathfrak{p}_-(\mathfrak{r})$, and since $\mathfrak{c}\cap\mathfrak{p}_-=\mathfrak{a}_{\mathfrak{p}-}$, $\mathfrak{g}(\mathfrak{r})$ satisfies the condition (i) of Lemma 12. It follows from (ii) that there exists a $\mathfrak{q}$-orthogonal system $\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ such that $\mathfrak{r}=\sum_{i=1}^{k}\boldsymbol{R}H_{\alpha_i}$. Then the first statement follows from Lemma 9. The last statement is clear ([7], p. 29). q. e. d.

As a consequence of Corollary 1 of Theorem 1 and Theorem 2, the following theorem gives explicitly the double coset decomposition $H\backslash G/P$.

THEOREM 3. *Let (G, H, σ) be an affine symmetric space such that G is real semi-simple, θ a Cartan involution commutative with σ, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_{\mathfrak{p}+}$ is maximal abelian in $\mathfrak{p}_+$, and $\{Q_1, \cdots, Q_m\}$ be representatives of $W(\mathfrak{a}_{\mathfrak{p}}, K_+)$-conjugacy classes of $\mathfrak{q}$-orthogonal systems of $\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Suppose that $Q_j=\{X_{\alpha_1}, \cdots, X_{\alpha_k}\}$ is normalized such that $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, X_{-\alpha_i})=-1$, $i=1, \cdots, k$ for each $j=1, \cdots, m$. Put $c(Q_j)=\exp\frac{\pi}{2}(X_{\alpha_1}+X_{-\alpha_1})\cdots\exp\frac{\pi}{2}(X_{\alpha_k}+X_{-\alpha_k})$. Then*

(i) *we have the following decomposition of G*

$$G=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})}Hw_v c(Q_i)P \quad (\textit{disjoint union})$$

where $P=P(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$, Σ^+ is a positive system of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$, $\mathfrak{a}_{\mathfrak{p}i}=\mathrm{Ad}\,(c(Q_i))\mathfrak{a}_{\mathfrak{p}}$, and w_v is an element of $M^*(\mathfrak{a}_{\mathfrak{p}i})$ that represents an element of the left coset $v\subset W(\mathfrak{a}_{\mathfrak{p}i})$.

(ii) *Put* $P_{i,w_v}=w_vc(Q_i)Pc(Q_i)^{-1}w_v^{-1}$. *Let* $h_1, h_2\in H$ *and* $p_1, p_2\in P$. *Then* $h_1w_vc(Q_i)p_1=h_2w_vc(Q_i)p_2$ *if and only if there exists an* $x\in H\cap P_{i,w_v}$ *such that* $h_2=h_1x$ *and that* $p_2=c(Q_i)^{-1}w_v^{-1}x^{-1}w_vc(Q_i)p_1$.

(iii) *Let* $P=P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=MA'_{\mathfrak{p}}N^+$ *be a minimal parabolic subgroup of* G *such that* $\mathfrak{a}'_{\mathfrak{p}}$ *is* σ*-stable. Then*

$$H\cap P=(K_+\cap M)A'_{\mathfrak{p}+}\exp(\mathfrak{h}\cap\mathfrak{n}^+\cap\sigma\mathfrak{n}^+).$$

PROOF. The statement (i) follows easily from Corollary 1 of Theorem 1 and Theorem 2. (ii) is clear. (iii) is proved as follows.

Note that

$$G_\sigma\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=G_\sigma\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)\cap\sigma P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+).$$

Since M and $A'_{\mathfrak{p}}$ are σ-stable, we have

$$P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)\cap\sigma P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=MA'_{\mathfrak{p}}(N^+\cap\sigma N^+).$$

Let $m\in M$, $a\in A'_{\mathfrak{p}}$, $n\in N^+\cap\sigma N^+$. If $\sigma(man)=man$, then $\sigma(m)=m$, $\sigma(a)=a$ and $\sigma(n)=n$. Hence

$$G_\sigma\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=M_\sigma A'_{\mathfrak{p}+}(N^+\cap\sigma N^+\cap G_\sigma).$$

Since $\sigma P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=P(\mathfrak{a}'_{\mathfrak{p}}, w\Sigma^+)$ for some $w\in W(\mathfrak{a}_{\mathfrak{p}})$,

$$N^+\cap\sigma N^+\cap G_\sigma=N_w^+\cap G_\sigma=\exp(\mathfrak{n}_w^+\cap\mathfrak{h})$$

as is shown in the proof of (ii) of Theorem 1. Hence

$$G_\sigma\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=M_\sigma A'_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h}).$$

Since $H\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)$ is a union of some connected components of this, there is a subgroup M'_σ of M_σ such that $(M_\sigma)_0\subset M'_\sigma\subset M_\sigma$ and that

$$H\cap P(\mathfrak{a}'_{\mathfrak{p}}, \Sigma^+)=M'_\sigma A'_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h}).$$

It is easy to see that $M'_\sigma=M\cap K_+$. q. e. d.

COROLLARY. *Retain the notations given in Theorem* 3 *and let* $(G, H', \sigma\theta)$ *be the affine symmetric space associated with* (G, H, σ). *Then we have the following two decompositions of* G.

$$G=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})}Hw_vc(Q_i)P \quad (\textit{disjoint union})$$

$$=\bigcup_{i=1}^{m}\bigcup_{v\in W(\mathfrak{a}_{\mathfrak{p}i}, K_+)\backslash W(\mathfrak{a}_{\mathfrak{p}i})}H'w_vc(Q_i)P \quad (\textit{disjoint union}).$$

LEMMA 13. *Let* $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ *be a minimal parabolic subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{a}_{\mathfrak{p}}$ *is* σ*-stable. Then*

(i) $\mathfrak{h}+\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$

$$=\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\sum_{\alpha\in\Sigma^+\cup\sigma\Sigma^+}\mathfrak{g}_\alpha+\sum_{\alpha\in(\Sigma^-\cap\sigma\Sigma^-)-\Sigma(\mathfrak{a}_{\mathfrak{p}+})^-}\mathfrak{h}\cap(\mathfrak{g}_\alpha+\sigma\mathfrak{g}_\alpha)+\sum_{\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})^-}\mathfrak{g}_{\alpha+}.$$

(ii) $\dim\mathfrak{g}-\dim(\mathfrak{h}+\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+))$

$$=\frac{1}{2}\sum_{\alpha\in(\Sigma^+\cap\sigma\Sigma^+)-\Sigma(\mathfrak{a}_{\mathfrak{p}+})^+}\dim\mathfrak{g}_\alpha+\sum_{\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})^+}\dim\mathfrak{g}_{\alpha-}.$$

PROOF. Note that

$$\mathfrak{h}+\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)=\mathfrak{h}+\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)+\sigma\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$$
$$=\mathfrak{h}+\mathfrak{m}+\mathfrak{a}_\mathfrak{p}+\mathfrak{n}^++\sigma\mathfrak{n}^+.$$

Since $\mathfrak{m}$, $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{g}_\alpha+\sigma\mathfrak{g}_\alpha$ are σ-stable,

$$\mathfrak{h}=\mathfrak{h}\cap\mathfrak{m}+\mathfrak{h}\cap\mathfrak{a}_\mathfrak{p}+\sum_{\alpha\in\Sigma-\Sigma(\mathfrak{a}_{\mathfrak{p}+})}\mathfrak{h}\cap(\mathfrak{g}_\alpha+\sigma\mathfrak{g}_\alpha)+\sum_{\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})}\mathfrak{g}_{\alpha+}.$$

Then the statement (i) follows from those two equations. (ii) follows from (i).
q. e. d.

EXAMPLE 3. Let

$$G=SU(p, 2)=\{X\in SL(n, \boldsymbol{C})\,|\,{}^t\bar{X}EX=E\},$$

where $n=p+2$ and $E=\begin{bmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & 0 & & -1 & \\ & & & & -1 \end{bmatrix}\begin{matrix} \\ p \\ \\ \\ \\ \end{matrix}$.

Then the Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g}=\{X\in M(n, \boldsymbol{C})\,|\,{}^t\bar{X}E+EX=0,\ \mathrm{tr}\,X=0\}$$
$$=\left\{\begin{bmatrix} A & B \\ {}^t\bar{B} & D\end{bmatrix}\,\middle|\,A\in\mathfrak{u}(p),\ D\in\mathfrak{u}(2),\ B\in M(p, 2, \boldsymbol{C}),\ \mathrm{tr}\,A+\mathrm{tr}\,D=0\right\}.$$

Put

$$\mathfrak{k}=\left\{\begin{bmatrix} A & 0 \\ 0 & D\end{bmatrix}\,\middle|\,A\in\mathfrak{u}(p),\ D\in\mathfrak{u}(2),\ \mathrm{tr}\,A+\mathrm{tr}\,D=0\right\},$$

and

$$\mathfrak{p}=\left\{\begin{bmatrix} 0 & B \\ {}^t\bar{B} & 0\end{bmatrix}\,\middle|\,B\in M(p, 2, \boldsymbol{C})\right\}.$$

Then $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition. Suppose $p\geqq 4$. Then

$$\mathfrak{a}_{\mathfrak{p}}=\left\{\left(\begin{array}{cccc} \multicolumn{2}{c}{0} & h_1 & \\ & & & h_2 \\ h_1 & & \multicolumn{2}{c}{0} \\ & h_2 & & \end{array}\right)\middle|\, h_1,\ h_2\in \boldsymbol{R}\right\}$$

is a maximal abelian subspace of $\mathfrak{p}$. Let α_i denote the linear form on $\mathfrak{a}_{\mathfrak{p}}$ defined by

$$\alpha_i:\left(\begin{array}{cccc} \multicolumn{2}{c}{0} & h_1 & \\ & & & h_2 \\ h_1 & & \multicolumn{2}{c}{0} \\ & h_2 & & \end{array}\right)\longrightarrow h_i\,.$$

Then the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ is

$$\Sigma=\{\pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_1\pm\alpha_2,\ \pm 2\alpha_1,\ \pm 2\alpha_2\}\,.$$

And the root spaces are

$$\mathfrak{g}_{\alpha_1}=\left\{\left(\begin{array}{ccccc} & 0 & \begin{smallmatrix} {}^tD \\ 0 \end{smallmatrix} & 0 & \\ -\bar{D} & 0 & 0 & \bar{D} & 0 \\ & 0 & \begin{smallmatrix} {}^tD \\ 0 \end{smallmatrix} & 0 & \end{array}\right)\middle|\, D\in M(p-2,\ 1,\ \boldsymbol{C})\right\},$$

$$\mathfrak{g}_{\alpha_2}=\left\{\left(\begin{array}{ccccc} 0 & & \begin{smallmatrix} 0 \\ {}^tF \end{smallmatrix} & & 0 \\ 0 & -\bar{F} & 0 & 0 & \bar{F} \\ 0 & & \begin{smallmatrix} 0 \\ {}^tF \end{smallmatrix} & & 0 \end{array}\right)\middle|\, F\in M(p-2, 1,\ \boldsymbol{C})\right\},$$

$$\mathfrak{g}_{\alpha_1+\alpha_2}=\left\{X_{\alpha_1+\alpha_2,\,x}=\left(\begin{array}{cccc} & x & & -x \\ -\bar{x} & & \bar{x} & \\ & \multicolumn{2}{c}{0} & \\ & x & & -x \\ -\bar{x} & & \bar{x} & \end{array}\right)\middle|\, x\in \boldsymbol{C}\right\},$$

$$\mathfrak{g}_{\alpha_1-\alpha_2}=\left\{X_{\alpha_1-\alpha_2,\,y}=\left(\begin{array}{cccc} & y & & y \\ -\bar{y} & & \bar{y} & \\ & \multicolumn{2}{c}{0} & \\ & y & & y \\ \bar{y} & & -\bar{y} & \end{array}\right)\middle|\, y\in \boldsymbol{C}\right\},$$

$$\mathfrak{g}_{2\alpha_1}=\left\{\begin{pmatrix} t & & -t & \\ & 0 & & 0 \\ & & 0 & \\ t & & -t & \\ & 0 & & 0\end{pmatrix} \;\middle|\; t\in\sqrt{-1}\boldsymbol{R}\right\},$$

and so on.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{h}=\left\{\begin{array}{c} \begin{array}{cccc} \overbrace{\quad}^{1} & \overbrace{\qquad}^{m+1} & \qquad & \overbrace{\quad}^{1} \end{array} \\ \left(\begin{array}{c|c|c|c} * & 0 & * & 0 \\ \hline 0 & * & 0 & * \\ \hline * & 0 & * & 0 \\ \hline 0 & * & 0 & * \end{array}\right)\in\mathfrak{g} \end{array}\right\} \quad \text{where} \quad 0<m<p-2,$$

and $\mathfrak{q}$ a subspace of $\mathfrak{g}$ defined by

$$\mathfrak{q}=\left\{\begin{array}{c} \begin{array}{cccc} \overbrace{\quad}^{1} & \overbrace{\qquad}^{m+1} & \qquad & \overbrace{\quad}^{1} \end{array} \\ \left(\begin{array}{c|c|c|c} 0 & * & 0 & * \\ \hline * & 0 & * & 0 \\ \hline 0 & * & 0 & * \\ \hline * & 0 & * & 0 \end{array}\right)\in\mathfrak{g} \end{array}\right\}.$$

Let σ denote the mapping $X+Y\rightarrow X-Y$ ($X\in\mathfrak{h}$, $Y\in\mathfrak{q}$) of $\mathfrak{g}$ onto $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is a symmetric Lie algebra, and $\mathfrak{a}_{\mathfrak{p}}$ is contained in $\mathfrak{h}$. Put

$$X_{\alpha_1, k}=\begin{array}{c} \overbrace{\qquad\qquad}^{k+2} \qquad\quad \\ \begin{pmatrix} & & 1 & & \\ & & & & \\ & -1 & & 1 & \\ & & 1 & & \\ & & & & \end{pmatrix} \end{array}\in\mathfrak{g}_{\alpha_1},$$

and

$$X_{\alpha_2, k}=\begin{array}{c} \overbrace{\qquad\qquad}^{k+2} \qquad\quad \\ \begin{pmatrix} & & & & \\ & & 1 & & \\ & -1 & & 1 & \\ & & & & \\ & & 1 & & \end{pmatrix} \end{array}\in\mathfrak{g}_{\alpha_2}.$$

Then $\{X_{\alpha_1, i}, X_{\alpha_2, j}\}$ is a $\mathfrak{q}$-orthogonal system of $\Sigma(\mathfrak{a}_{\mathfrak{p}})$ for $0<i\leqq m<j\leqq p-2$, and $\{X_{\alpha_1+\alpha_2, x}, X_{\alpha_1-\alpha_2, x}\}$ is also a $\mathfrak{q}$-orthogonal system for a non-zero complex number x.

It is easily shown that

$$M(\mathfrak{a}_p)=\left\{\begin{pmatrix} u_1 & & & & 0 \\ & u_2 & & & \\ & & A & & \\ & & & u_1 & \\ 0 & & & & u_2 \end{pmatrix}\,\middle|\, A\in U(p-2),\ u_1,\ u_2\in U(1),\ u_1^2u_2^2\det A=1\right\},$$

and representatives of $W(\mathfrak{a}_p)$ in $M^*(\mathfrak{a}_p)$ are given by

$$\left\{I_n,\ \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & I_{p-2} & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix},\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & I_{p-2} & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix},\ \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & I_{p-2} & & \\ & & & & -1 \\ & & & -1 & \end{pmatrix},\right.$$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & E_{p-2} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix},\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & E_{p-2} & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix},\ \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & E_{p-2} & & \\ & & & & 1 \\ & & & -1 & \end{pmatrix},$$

$$\left.\begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & E_{p-2} & & \\ & & & & -1 \\ & & & 1 & \end{pmatrix}\right\}$$

where

$$E_{p-2}=\begin{array}{c} \overbrace{}^{p-3} \\ \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \end{array}.$$

On the other hand, we have

$$G_\sigma=\left\{\begin{array}{c} \qquad\overbrace{}^{m+2} \\ \left(\begin{array}{c|c|c|c} \ast & 0 & \ast & 0 \\ \hline 0 & \ast & 0 & \ast \\ \hline \ast & 0 & \ast & 0 \\ \hline 0 & \ast & 0 & \ast \end{array}\right) \end{array}\in G\right\}$$

$$\cong\{(A,\ B)\in U(p-m-1,\ 1)\times U(m+1,\ 1)\,|\det A\det B=1\}\ .$$

Since G_σ is connected, H must be equal to G_σ. Then (G, H, σ) is an affine symmetric space. It is easily shown that representatives of $W(\mathfrak{a}_\mathfrak{p}, K_+)$ in $M^*(\mathfrak{a}_\mathfrak{p})\cap K_+$ are given by

$$\left\{ I_n,\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & I_{p-2} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & E_{p-2} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix},\ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & E_{p-2} & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \right\}.$$

Hence

$$\{\emptyset,\ \{X_{\alpha_1, i}\},\ \{X_{\alpha_2, j}\},\ \{X_{\alpha_1+\alpha_2, x}\},\ \{X_{\alpha_1, i},\ X_{\alpha_2, j}\}\}$$

are representatives of $W(\mathfrak{a}_\mathfrak{p}, K_+)$-conjugacy classes of $\mathfrak{q}$-orthogonal systems of $\Sigma(\mathfrak{a}_\mathfrak{p})$, where i is an integer such that $0<i\leqq m$, j is an integer such that $m<j\leqq p-2$, and x is a non-zero complex number. It follows from Theorem 2 that there are 5 H-conjugacy classes of σ-stable maximal abelian subspaces of $\mathfrak{p}$. Calculating matrices, it is shown that $|W(\mathfrak{a}'_\mathfrak{p}, K_+)|=4$ for every σ-stable maximal abelian subspace $\mathfrak{a}'_\mathfrak{p}$ of $\mathfrak{p}$. Since $|W(\mathfrak{a}'_\mathfrak{p})|=8$, thus there are exactly 10 H-orbits on G/P (Corollary 1 of Theorem 1).

Let $Q_1=\{\emptyset\}$, $Q_2=\left\{\frac{1}{2}X_{\alpha_1, 1}\right\}$, $Q_3=\left\{\frac{1}{2}X_{\alpha_2, m+1}\right\}$, $Q_4=\left\{\frac{1}{4}X_{\alpha_1+\alpha_2, 1}\right\}$, and $Q_5=\left\{\frac{1}{2}X_{\alpha_1, 1},\ \frac{1}{2}X_{\alpha_2, m+1}\right\}$. Then $c(Q_1)=I_n$,

$$c(Q_2)=\begin{array}{c} \overbrace{\hphantom{0000}}^{p-1} \\ \begin{pmatrix} 0 & & 1 & 0 & & \\ & 1 & & & & \\ -1 & & 0 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array},\quad c(Q_3)=\begin{array}{c} \overbrace{\hphantom{000000000}}^{m+3} \\ \begin{pmatrix} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ -1 & & & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & 0 & & & & & 1 \end{pmatrix} \end{array}$$

$$c(Q_4)=\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & & & & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & & & & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},\quad \text{and}\quad c(Q_5)=c(Q_2)c(Q_3).$$

Let $N^+=\exp(\mathfrak{g}_{\alpha_1}+\mathfrak{g}_{\alpha_2}+\mathfrak{g}_{\alpha_1+\alpha_2}+\mathfrak{g}_{\alpha_1-\alpha_2}+\mathfrak{g}_{2\alpha_1}+\mathfrak{g}_{2\alpha_2})$, and $P=MA_{\mathfrak{p}}N^+$. Then it follows from Theorem 3 that G is decomposed to

$$G=HP\cup Hw_1P\cup Hc(Q_2)P\cup Hc(Q_2)w_1P\cup Hc(Q_3)P\cup Hc(Q_3)w_1P$$
$$\cup Hc(Q_4)P\cup Hc(Q_4)w_2P\cup Hc(Q_5)P\cup Hc(Q_5)w_1P$$

where $w_1=\begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & I_{p-2} & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}$ and $w_2=\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & E_{p-2} & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$.

We can easily calculate $H\cap c(Q_i)wPw^{-1}c(Q_i)^{-1}$ ($i=1,\cdots 5$, $w\in W(\mathfrak{a}_{\mathfrak{p}})$) by (iii) of Theorem 3. Codimensions of orbits are as follows (Lemma 13).

representative	codimension	
I_n	$2p$	closed
w_1	$2p$	
$c(Q_2)$	$2(p-m-1)$	$(0<m\leqq p-2)$
$c(Q_2)w_1$	$2(p-m)$	
$c(Q_3)$	$2(m+2)$	$(0\leqq m<p-2)$
$c(Q_3)w_1$	$2(m+1)$	
$c(Q_4)$	1	
$c(Q_4)w_2$	$2(p-1)$	
$c(Q_5)$	0	open
$c(Q_5)w_1$	0	

§ 3. Open orbits and closed orbits.

Note that there is a one-to-one correspondence $xP\to \mathrm{Ad}(x)\mathfrak{P}$ between G/P and the set of all the minimal parabolic subalgebras of $\mathfrak{g}$ (see § 1). Let $\mathfrak{a}_{\mathfrak{p}}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$. Define a subgroup $W_\sigma(\mathfrak{a}_{\mathfrak{p}})$ of the Weyl group $W(\mathfrak{a}_{\mathfrak{p}})$ by

$$W_\sigma(\mathfrak{a}_{\mathfrak{p}})=\{w\in W(\mathfrak{a}_{\mathfrak{p}})\mid w(\mathfrak{a}_{\mathfrak{p}+})=\mathfrak{a}_{\mathfrak{p}+}\}\,.$$

PROPOSITION 1 (cf. [8]). *A minimal parabolic subalgebra $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ which is identified with a point of G/P is contained in an open H-orbit if and only if the following two conditions are satisfied:*

(i) $\mathfrak{a}_{\mathfrak{p}-}$ *is maximal abelian in* $\mathfrak{p}_-$,

(ii) Σ^+ *is $\sigma\theta$-compatible* (i.e. $\alpha\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}+})\Rightarrow\sigma\theta(\alpha)\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}+})$).

The number of open orbits is $|W_\sigma(\mathfrak{a}_{\mathfrak{p}})|/|W(\mathfrak{a}_{\mathfrak{p}}, K^+)|$.

PROOF. It follows from Lemma 13 that $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^+)$ is contained in an open orbit if and only if the following two conditions are satisfied:

(iii) $(\Sigma^+\cap\sigma\Sigma^+)-\Sigma(\mathfrak{a}_{\mathfrak{p}+})^+=\emptyset$,

(iv) $\mathfrak{g}_{\alpha-}=\{0\}$ for all $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$.

Clearly (iii) is equivalent to (ii). We will prove that (iv) is equivalent to (i). Since

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{\mathfrak{p}-})=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\sum_{\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})}\mathfrak{g}_\alpha\,,$$

if $\mathfrak{g}_{\alpha-}=\{0\}$ for all $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ then $\mathfrak{z}_{\mathfrak{p}-}(\mathfrak{a}_{\mathfrak{p}-})=\mathfrak{a}_{\mathfrak{p}-}$. Hence $\mathfrak{a}_{\mathfrak{p}-}$ is maximal abelian in $\mathfrak{p}_-$. Conversely suppose that $\mathfrak{g}_{\alpha-}\neq\{0\}$ for some $\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. Let Y_α be a non-zero element of $\mathfrak{g}_{\alpha-}$. Then $\mathfrak{a}_{\mathfrak{p}-}+\boldsymbol{R}(Y_\alpha-\theta Y_\alpha)$ is an abelian subspace of $\mathfrak{p}_-$ properly containing $\mathfrak{a}_{\mathfrak{p}-}$. Thus $\mathfrak{a}_{\mathfrak{p}-}$ is not maximal abelian in $\mathfrak{p}_-$. Hence the first assertion is proved.

The closure of the Weyl chamber corresponding to a positive system Σ^+ contains a regular element H_1 of $\mathfrak{a}_{\mathfrak{p}-}$ (i.e. $\alpha(H_1)\neq0$ for all $\alpha\in\Sigma-\Sigma(\mathfrak{a}_{\mathfrak{p}+})$) if and only if Σ^+ is $\sigma\theta$-compatible. Let $\tilde{\alpha}$ be the projection of $\alpha\in\Sigma$ to $\mathfrak{a}_{\mathfrak{p}-}$ ($\alpha\in\Sigma$ is identified with $H_\alpha\in\mathfrak{a}_{\mathfrak{p}}$). Put $W_\sigma=W_\sigma(\mathfrak{a}_{\mathfrak{p}})$. We will show that $w_{\tilde{\alpha}}\in W_\sigma|_{\mathfrak{a}_{\mathfrak{p}-}}$ for all $\alpha\in\Sigma-\Sigma(\mathfrak{a}_{\mathfrak{p}+})$ as in [7] p. 24, where $w_{\tilde{\alpha}}$ is the reflection of $\mathfrak{a}_{\mathfrak{p}-}$ with respect to $\tilde{\alpha}$. It is true if $\alpha=\tilde{\alpha}$. If $(\alpha, \sigma\theta(\alpha))<0$, then $\alpha+\sigma\theta(\alpha)=2\tilde{\alpha}\in\Sigma$, so $w_{\tilde{\alpha}}\in W_\sigma|_{\mathfrak{a}_{\mathfrak{p}-}}$. If $(\alpha, \sigma\theta(\alpha))>0$, then $\alpha-\sigma\theta(\alpha)=\alpha+\sigma(\alpha)\in\Sigma$. Let X_α be a non-zero element of $\mathfrak{g}_\alpha$. Then $[X_\alpha, \sigma X_\alpha]\neq0$ (Lemma 11) and $\sigma[X_\alpha, \sigma X_\alpha]=-[X_\alpha, \sigma X_\alpha]$, so $[X_\alpha, \sigma X_\alpha]\in\mathfrak{g}_{\beta-}$ where $\beta=\alpha+\sigma(\alpha)\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})$. As is proved before this is impossible since $\mathfrak{a}_{\mathfrak{p}-}$ is maximal abelian in $\mathfrak{p}_-$. Thus $(\alpha, \sigma\theta(\alpha))\not>0$. If $(\alpha, \sigma\theta(\alpha))=0$, then $w_{\tilde{\alpha}}=w_\alpha w_{\sigma\theta(\alpha)}|_{\mathfrak{a}_{\mathfrak{p}-}}\in W_\sigma|_{\mathfrak{a}_{\mathfrak{p}-}}$. Hence the assertion is proved.

It is known that the group generated by $\{w_{\tilde{\alpha}}\,|\,\alpha\in\Sigma\}$ acts simply transitively on the Weyl chambers of $\mathfrak{a}_{\mathfrak{p}-}$ ([3], p. 73). Let Δ and Δ' be two Weyl chambers the closures of which contain regular elements of $\mathfrak{a}_{\mathfrak{p}-}$. There is a $w\in W_\sigma$ such that $w(\mathfrak{a}_{\mathfrak{p}-}\cap\bar{\Delta})=\mathfrak{a}_{\mathfrak{p}-}\cap\bar{\Delta}'$ because $\mathfrak{a}_{\mathfrak{p}-}\cap\bar{\Delta}$ and $\mathfrak{a}_{\mathfrak{p}-}\cap\bar{\Delta}'$ are closures of Weyl chambers of $\mathfrak{a}_{\mathfrak{p}-}$. Since the Weyl chambers of $\mathfrak{a}_{\mathfrak{p}}$ the closures of which contain $\mathfrak{a}_{\mathfrak{p}-}\cap\bar{\Delta}'$ are transitive under the group generated by $\{w_\alpha\,|\,\alpha\in\Sigma(\mathfrak{a}_{\mathfrak{p}+})\}$ $(\subset W_\sigma)$, there is a $w'\in W_\sigma$ such that $w'\Delta=\Delta'$. Conversely it is clear that every element of W_σ maps a $\sigma\theta$-compatible order to a $\sigma\theta$-compatible order. Therefore it follows from Corollary 1 of Theorem 1 that the number of open orbits is $|W_\sigma(\mathfrak{a}_{\mathfrak{p}})|/|W(\mathfrak{a}_{\mathfrak{p}}, K_+)|$. q.e.d.

PROPOSITION 2 (cf. [8]). *A minimal parabolic subalgebra* $\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ *is contained in a closed H-orbit if and only if the following two conditions are satisfied:*

(i) $\mathfrak{a}_{\mathfrak{p}+}$ *is maximal abelian in* $\mathfrak{p}_+$.

(ii) Σ^+ *is* σ*-compatible* (*i.e.* $\alpha\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}-})\Rightarrow\sigma(\alpha)\in\Sigma^+-\Sigma^+(\mathfrak{a}_{\mathfrak{p}-})$).

The number of closed orbits is $|W_\sigma(\mathfrak{a}_\mathfrak{p})|/|W(\mathfrak{a}_\mathfrak{p}, K_+)|$.

PROOF. Since G/P is compact, $\mathfrak{P}(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ is contained in a closed orbit if and only if $H/H\cap P(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ is compact. Suppose that $H/H\cap P(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ is compact. Note that $H\cap P(\mathfrak{a}_\mathfrak{p}, \Sigma^+)=(M\cap K_+)A_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})$ ((iii) of Theorem 3). Let $\Sigma^{+\prime}$ be a σ-compatible positive system of Σ containing $\Sigma^+\cap\sigma\Sigma^+$. Let $\mathfrak{a}'_{\mathfrak{p}+}$ be a maximal abelian subspace of $\mathfrak{p}_+$ containing $\mathfrak{a}_{\mathfrak{p}+}$. Define a positive system $\bar{\Sigma}^+$ of the root system $\bar{\Sigma}$ of the pair $(\mathfrak{h}, \mathfrak{a}'_{\mathfrak{p}+})$ such that $\mathfrak{n}^{+\prime}\cap\sigma\mathfrak{n}^{+\prime}\cap\mathfrak{h}\subset\bar{\mathfrak{n}}^+$, where $\mathfrak{n}^{+\prime}=\sum_{\alpha\in\Sigma^{+\prime}}\mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}}^+=\sum_{\lambda\in\bar{\Sigma}^+}\mathfrak{h}_\lambda$. Then we have the Iwasawa decomposition of H,

$$H=K_+A'_{\mathfrak{p}+}\bar{N}^+,$$

and natural projections

$$f:\ H/A_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})\to H/H\cap P(\mathfrak{a}_\mathfrak{p}, \Sigma^+),$$

and

$$g:\ H/A_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})\to H/A_{\mathfrak{p}+}\bar{N}^+\cong K_+\times A'_{\mathfrak{p}+}/A_{\mathfrak{p}+}.$$

Since the fibres of f which are diffeomorphic to $M\cap K_+$ are compact, $H/A_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})$ and $H/A_{\mathfrak{p}+}\bar{N}^+$ are compact. Hence $A'_{\mathfrak{p}+}=A_{\mathfrak{p}+}$ and $\mathfrak{a}_{\mathfrak{p}+}$ is maximal abelian in $\mathfrak{p}$. On the other hand we have

$$\begin{aligned}H/A_{\mathfrak{p}+}\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})&=H/\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})A_{\mathfrak{p}+}\\&\cong K_+\bar{N}^+/\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h}).\end{aligned}$$

If Σ^+ is not σ-compatible, then there is a minimal root $\beta\in(\Sigma^{+\prime}\cap\sigma\Sigma^{+\prime})-(\Sigma^+\cap\sigma\Sigma^+)$ with respect to the order of $\Sigma^{+\prime}$, and then $\beta\neq\sigma\beta$. Put $\Phi=(\Sigma^{+\prime}\cap\Sigma^{+\prime})-\{\beta, \sigma\beta\}$. Then $\mathfrak{t}=\sum_{\alpha\in\Phi}(\mathfrak{g}_\alpha+\sigma(\mathfrak{g}_\alpha))\cap\mathfrak{h}$ is an ideal of $\bar{\mathfrak{n}}^+$, and $\bar{\mathfrak{n}}^+=\mathfrak{t}+\{(\mathfrak{g}_\beta+\sigma\mathfrak{g}_\beta)\cap\mathfrak{h}\}$. We have a projection

$$h:\ \bar{N}^+/\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})\to\bar{N}^+/\exp\mathfrak{t}.$$

Since $\exp(X+Y)\in\exp X\exp\mathfrak{t}$ for $X\in\bar{\mathfrak{n}}^+$, $Y\in\mathfrak{t}$,

$$\bar{N}^+/\exp\mathfrak{t}\cong\bar{\mathfrak{n}}/\mathfrak{t}\cong\boldsymbol{R}^k$$

for some $k>0$. Hence $\bar{N}^+/\exp(\mathfrak{n}^+\cap\sigma\mathfrak{n}^+\cap\mathfrak{h})$ is not compact, so $H/H\cap P(\mathfrak{a}_\mathfrak{p}, \Sigma^+)$ is not compact. Thus Σ^+ is σ-compatible. The converse assertion follows easily from the above consideration, seeing that if $\mathfrak{a}'_{\mathfrak{p}+}=\mathfrak{a}_{\mathfrak{p}+}$, then $\mathfrak{n}^{+\prime}\cap\sigma\mathfrak{n}^{+\prime}\cap\mathfrak{h}=\bar{\mathfrak{n}}^+$. The proof of the second assertion is the same as that of Proposition 1.

q. e. d.

COROLLARY. *In the correspondence between the H-orbits on G/P and the H'-orbits on G/P given in Corollary* 2 *of Theorem* 1 (*see also Corollary of Theorem* 3), *open orbits correspond to closed orbits and closed ones to open ones.*

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Added in proof. Recently W. Rossmann announced without proof the same results as our Theorem 1 and Proposition 1 in [**11**] W. Rossmann, The structure of semisimple symmetric spaces, Queen's Paper Pure Appl. Math., **48** (1978), 513-520.