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Note on the computation of Bessel functions through recurrence formula

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§ 1. Introduction. The problem.

As is well-known, Bessel function of the first kind $J_n(x)$ satisfies the recurrence formula

$$J_{n+1}(x)-\frac{2n}{x}J_n(x)+J_{n-1}(x)=0\,. \tag{1}$$

For a fixed value x, we may compute the values of $J_2(x), J_3(x), \cdots$ through the formula (1), if we know the values of $J_0(x)$ and $J_1(x)$. However, this method does not fit to the practice, because the formula (1) implies serious *unstability*[1]. Even if the initial values of $J_0(x)$ and $J_1(x)$ have a little error, the successive values of $J_n(x)$ given by (1) will, in general, tend to $+\infty$ or $-\infty$, although the true values of $J_n(x)$ must tend to 0 when $n\to+\infty$.

On the other hand, it is useful to apply the formula (1) from large values of n to the smaller ones. Precisely speaking, the value of $J_n(x)$ is computed by the following algorism.

1°. Choose sufficiently large N, which will be discussed later, and put

$$j^*_{N+1}=0, \qquad j^*_N=\varepsilon\,.$$

Here ε is usually taken as the smallest positive number admissible in the computor, viz., 10^{-10} or 2^{-128}, etc.

2°. Compute j^*_n $(n=N-1, N-2, \cdots, 1, 0)$ by the recurrence formula

$$j^*_{n-1}=\frac{2n}{x}j^*_n-j^*_{n+1}\,. \tag{1'}$$

3°. Noting the relation

$$J_0(x)+2\sum_{n=1}^{\infty}J_{2n}(x)=1\,,$$

the values of $J_n(x)$ are obtained by

$$J_n(x)=\frac{1}{K}j^*_n \tag{2}$$

where we have put

1) This unstability is well known; e.g. see Uno [4].

$$K = j_0^* + 2\sum_n j_{2n}^*; \quad 1 \leqq n, \quad 2n \leqq N. \tag{3}$$

The difficulty of this method lies in the *choice of the starting value N.* If N is chosen too small, the formula (2) does not give the correct values of $J_n(x)$. While, if N is chosen too large, we need considerable amount of computations, and further, overflow may occur during the computations of j_n^*.

As for the starting value N, Mr. I. Uoki[2] (Toyo Kogyo Co., Hiroshima) has given the following empirical formula, if the accuracy of 10^{-10} for the values of $J_n(x)$ is needed.

$$\left.\begin{array}{lll} N=7x+6 & \text{for} & 0.1 \leqq x < 1 \\ N=2x+10 & \text{for} & 1 \leqq x < 10 \\ N=1.3x+16 & \text{for} & 10 \leqq x \leqq 100 \end{array}\right\}. \tag{4}$$

This is very convenient for practical application.

He further tried much experiments to determine the limit of starting value N for which the overflow will not occur. We shall call this limit "*the overflow limit*" in the followings.

The purpose of the present paper is to determine theoretically the overflow limit N. The main result is the formula (17) in § 2, and as we shall show in § 3, this is fairly nice approximation of the overflow limit.

The author is deeply grateful to Prof. S. Moriguti (Univ. of Tokyo) and Prof. T. Uno (Nihon Univ.) for their valuable suggestions during the preparation of the paper.

§ 2. Theoretical consideration on the overflow limit.

Since the values of j_n^* $(n=0, 1, \cdots)$ are proportional to those of $J_n(x)$, we may consider j_n^* as the true value of $J_n(x)$ as far as the mutual ratios are concerned.

If x is sufficiently large, Nicolson's formula[3] gives the approximate values of $J_n(x)$ as follows.

$$J_n(x) \doteqdot \frac{t^{1/3}}{3^{2/3}x^{1/3}}\left[I_{-\frac{1}{3}}(t) - I_{\frac{1}{3}}(t)\right] \quad \text{for} \quad n \geqq x, \tag{5}$$

$$J_n(x) \doteqdot \frac{t^{1/3}}{3^{2/3}x^{1/3}}\left[J_{\frac{1}{3}}(t) + J_{-\frac{1}{3}}(t)\right] \quad \text{for} \quad n \leqq x \tag{6}$$

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2) The result is published in Uno [**5**]. In his original formula, $N=5$ for $x<0.1$, but in the author's opinion, this method must not be used for such small values of x.

3) Jahnke-Emde [**2**], p. 142, Watson [**6**], p. 249. The author is grateful to Prof. Moriguti for his kindness to indicate this formula to the author. The outline of the proof is due to [**3**].

where

$$t=\frac{2^{3/2}}{3}x\left|1-\frac{n}{x}\right|^{3/2}. \tag{7}$$

In Watson's book [**6**], the formulas (5) and (6) are proved by integral representation. Here we shall give the outline of an alternative proof.

Putting $v=n/x-1, J_n(x)=g(v)$, the recurrence formula (1) reads

$$\delta^2 g(v)/\delta v^2 = 2x^2 v g(v),$$

where the left hand side means the second difference. Tending $\delta v \to 0$, we have the following differential equation

$$g''(v)=2x^2 v g(v). \tag{8}$$

The equation (8) is transformed into a Bessel's differential equation of order 1/3, changing the independent variable from v to $t=xv^{3/2}$. The formulas (5) and (6) follow if we solve it under the initial conditions

$$g(v)\to 0 \quad \text{for} \quad v\to+\infty,$$

$$g(0)=J_x(x)\doteqdot\frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi x^{1/3}}, \tag{9}$$ [4]

and

$$g'_+(0)=g'_-(0).$$

We further remark an asymptotic formula of $J_N(x)$ for $N\gg x$, whose first term is

$$J_N(x)\doteqdot\left(\frac{x}{2}\right)^N\frac{e^N}{N^{N+1/2}\sqrt{2\pi}}. \tag{10}$$ [5]

The formulas (5) and (6) show that starting from a sufficiently large value $N\gg x$, and computing j_n^* for $n=N-1, N-2, \cdots$ successively, the value j_n^* will increase monotonously for decreasing n when $n\geqq x$, and j_n^* will take its maximum at $n=n_0$, n_0 being near x and $n_0<x$. Precisely, the function

$$\varphi(t)=t^{1/3}[J_{\frac{1}{3}}(t)+J_{-\frac{1}{3}}(t)]$$

is oscillating in $t\geqq 0$, and attains its maximum at the first zero point t_0 of the function

$$\varphi'(t)=t^{1/3}[J_{-\frac{2}{3}}(t)-J_{\frac{2}{3}}(t)],$$

which lies at

$$t=t_0=0.685546. \tag{11}$$

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4) E. g., Watson [**6**], p. 231, known as Cauchy's formula.

5) E. g., Watson [**6**], 225-227. As Horn mentioned, the formula (10) is nothing but the first term of the Taylor expansion of $J_N(x)$, where $N!$ is replaced by the Stirling formula.

We put

$$C_1=\varphi(t_0)/\varphi(0)=1.403819/0.930437=1.50877\,. \tag{12}$$[6]

Now it is plausible to assume that *overflow does not occur if the maximum of* j_n^*, *say* j_{n0}^*, *is less than the upper limit of the admissible values* in the computor. We put

$10^P=$ *the ratio between the largest and the smallest positive numbers admissible in the computor.*

Choosing ε as small as possible, the above condition is given by the inequality

$$10^P>j_{n0}^*/\varepsilon=j_{n0}^*/j_N^*\,. \tag{13}$$

However, we must consider one more remark. Even the values j_{n0}^* and

$$j_{n0-1}^*=\frac{2n_0}{x}j_{n0}^*-j_{n0+1}^*$$

do not exceed the upper limit, the value $(2n_0/x)j_{n0}^*$ *may overflow,* since the factor $2n_0/x$ is approximately 2 in the present case. Therefore it will be better to replace the inequality (13) by the formula

$$10^P>2j_{n0}^*/j_N^*\,. \tag{13'}$$

Now we have

$$j_{n0}^*/j_N^*=J_{n0}(x)/J_N(x)\doteqdot C_1J_x(x)/J_N(x)\,.$$

Using the asymptotic formulas (9), (10) and (12), the condition (13′) reads

$$2C_1\frac{\Gamma(1/3)2^NN^{N+1/2}\sqrt{2\pi}}{2^{2/3}3^{1/6}\pi x^{N+1/3}e^N}$$

$$=C_2\left(\frac{2N}{ex}\right)^{N+1/3}N^{1/6}<10^P \tag{14}$$

or

$$\left(N+\frac{1}{3}\right)\log_{10}\frac{2N}{ex}+\frac{1}{6}\log_{10}N<P-C_3\,. \tag{15}$$

Here we have put

$$C_2=\frac{\sqrt{2}\,C_1\Gamma(1/3)e^{1/3}}{3^{1/6}\sqrt{\pi}}=3.7477\,,$$

$$C_3=\log_{10}C_2=0.57377\,. \tag{16}$$

When x is about $1\sim200$, N is approximately $10\sim500$, and hence the term

$$\frac{1}{6}\log_{10}N<0.45$$

is much less than the first term in (15). Thus we may replace this term by 0.45, so that the overflow limit N will be given by the formula

6) The values of (11) and (12) are computed from the table [**1**].

$$\left(N+\frac{1}{3}\right)\log_{10}\frac{2N}{ex}=P-1\,. \tag{17}$$

Up to here, we have implicitely assumed that x *is large enough.* But the formula (17) itself is also a good estimation of N for rather smaller value of x. We shall give a short consideration of N when x is small.

If x is small, the maximum of $J_n(x)$ will be taken at $n=n_0=0$. In fact, this is surely so if $x\leqq x_0=1.4207$. In this case, the value j_n^* increases rather rapidly when n decreases from N to 0, so that roughly estimating, we may assume that

$$j_n^*\doteqdot\frac{2(n+1)}{x}\frac{2(n+2)}{x}\cdots\frac{2N}{x}\,.$$

Hence the condition (13′) reads

$$10^P>2j_0^*/\varepsilon\doteqdot 2^{N+1}N!/x^N\,. \tag{18}$$

If (18) is satisfied, the value K in (3) also does not exceed the upper limit, and then, no overflow may occur.

Using Stirling formula, (18) is replaced by

$$10^P>\frac{N^{N+1/2}2^{N+1}\sqrt{2\pi}}{(ex)^N}=2\sqrt{2\pi}\left(\frac{ex}{2}\right)^{1/3}\left(\frac{2N}{ex}\right)^{N+1/3}N^{1/6}\,,$$

i. e.,

$$\left(N+\frac{1}{3}\right)\log_{10}\frac{2N}{ex}<P-\frac{1}{6}\log_{10}N-C_4\,, \tag{19}$$

where

$$C_4=\log_{10}2\sqrt{2\pi}\left(\frac{ex_0}{2}\right)^{1/3}=0.79537\,.$$

In our present case, N being approximately less than 50, we may assume

$$\frac{1}{6}\log_{10}N\doteqdot 0.3\,,$$

so that the right hand side of (19) may be replaced by $P-1$. The overflow limit N will be given by the equation replacing $<$ in (19) by $=$, which reduces just the formula (17) itself. Therefore, we may use the formula (17) to determine N also for smaller value of x.

§ 3. Comparison with the experiments.

First we shall compare the value n_0 with the true value. From (6), n_0 is determined approximately by

$$x-n_0\doteqdot\frac{x^{1/3}}{2}(3t_0)^{2/3}=0.8086\,x^{1/3}\,. \tag{20}$$

The comparison is as follows.

x	$x-n_0$ from (20)	$x-n_0$ (true value)[7]	$J_{n_0}(x)/J_x(x)$
1	0.81	1	1.73889
2	1.02	1	1.63455
5	1.38	1	1.49817
10	1.74	2	1.53193
15	1.99	2	1.53726
20	2.19	2	1.52409
25	2.36	2	1.50787
30	2.51	3	1.49613
50	2.98	3	1.52112
70	3.34	3	1.51005
100	3.75	4	1.51327
150	4.30	4	1.51057
200	4.73	5	1.51109
			$C_1=1.50877$

Finally, we shall give the comparison of the overflow limit N computed theoretically from the formula (17) with empirical values[8].

	$P=39$		$P=100$	
x	N from (17)	N (empirical)	N from (17)	N (empirical)
1	28	28	60	60
10	60	60	110	110
30	99	96	162	160
50	131	130	205	200
100	207	$\sim$200	294	290

As the conclusion, the formula (17) is a fairly nice approximation of the overflow limit, which will be useful for practical estimation of the overflow limit N with given x and P.

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7) The true values of $x-n_0$ and $J_{n_0}(x)/J_x(x)$ are due to the computations of the values of j_n^* using HIPAC 101 in St. Paul's Univ. by the author.

8) For given values of x and P, the value N is determined in solving the equation (17) by successive approximation. The empirical values of N are due to Mr. I. Uoki and Miss H. Nagasaka (Nihon Univ.) in an unpublished note.

References

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