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On Gaussian sums attached to the general linear groups over finite fields

by Takeshi KONDO

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Introduction. Let $F(q)$ $(=F_q)$ be the finite field with q elements, q being a power of a prime number p. We denote by $M_n(F_q)$ the total matric ring of degree n over F_q, and by $GL(n, q)$ the group of regular elements of $M_n(F_q)$. To any irreducible representation ξ of $GL(n, q)$ by complex matrices, we can attach *Gaussian sums* $W(\xi, A)$ as follows. For every positive integer d, and for every $\alpha \in F(q^d)$, put

$$e_d[\alpha] = \exp\left[\frac{2\pi\sqrt{-1}}{p} \operatorname{Tr}_{F(q^d)|F(p)}(\alpha)\right].$$

Then, for every $A \in M_n(F_q)$, we define $W(\xi, A)$ by

$$W(\xi, A) = \sum_{X \in G(n,q)} \xi(X) e_1[\operatorname{tr}(AX)].$$

The matrices of this kind were investigated in E. Lamprecht [1] for the multicative groups of more general finite rings, but the explicit values of these matrices were not obtained.

The purpose of the present paper is to determine explicitly $W(\xi, A)$ for non-singular A and any irreducible representation ξ of $GL(n, q)$. To explain our result, first we note that, if $A \in GL(n, q)$,

$$W(\xi, A) = \xi(A)^{-1} W(\xi, 1_n),$$

where 1_n denotes the identity element of $M_n(F_q)$. Moreover, we see easily that $W(\xi, 1_n)$ is a scalar matrix. Then define a complex number $w(\xi)$ by

$$W(\xi, 1_n) = w(\xi)\xi(1_n).$$

Fix once and for all an isomorphism θ of the multiplicative group of $F(q^{n!})$ into the multiplicative group of complex numbers. Further fix a generator ε of the multiplicative group of $F(q^{n!})$, and put, for every integer d such that $1 \leqq d \leqq n$,

$$\varepsilon_d = \varepsilon^{\kappa}, \qquad \kappa = \frac{q^{n!}-1}{q^d-1}.$$

Then ε_d is a generator of the multiplicative group of $F(q^d)$. For every irreducible polynomial g of degree d with coefficients in $F(q)$, we define the usual

Gaussian sum $\tau(g)$ in the following way: taking a root ε_d^k of g, put

$$\tau(g)=\sum_{\alpha\in F(q^d)}\theta(\alpha)^k e_d[\alpha].$$

It is easily verified that $\tau(g)$ does not depend on the choice of k. Now, by J. A. Green [1], we can obtain all the irreducible characters of $GL(n,q)$. In view of his result, we can speak of the type $(\cdots g^{\nu(g)}\cdots)$ of every irreducible character ξ, where the g are irreducible polynomials with coefficients in $F(q)$ and $\nu(g)$ is a certain partition of a non-negative integer (cf. Notation and § 1.1). Then our principal result is stated as follows.

THEOREM. *If ξ is an irreducible representation of $GL(n,q)$ of type $(\cdots g^{\nu(g)}\cdots)$, then*

$$w(\xi)=(-1)^{n-\Sigma|\nu(g)|}q^{\frac{n(n-1)}{2}}\prod_{g\in P}\tau(g)^{|\nu(g)|},$$

where P is the set of polynomials defined in § 1.1 *and the* $|\nu(g)|$ *are non-negative integers defined in Notation.*

In particular, the absolute value of $w(\xi)$ is $q^{\frac{n^2-k}{2}}$, if ξ is of type $(\cdots(X-1)^{\kappa}\cdots)$ and $k=|\kappa|$.

In § 1, we recall the structure of the characters of $GL(n,q)$ given by J. A. Green, and in § 2.1, we consider the character sums attached to certain characters $B^{\rho}(h)$ and prove a property of polynomials $Q_\rho^\lambda(q)$ which appear in the calculation of the characters of $GL(n,q)$. Using this property of $Q_\rho^\lambda(q)$, the character sums attached to $B^{\rho}(h)$ can be expressed by the product of the usual Gaussian sums attached to finite fields. Then, in § 2.2, the above theorem is proved by virtue of this fact. In § 2.3, we make some remarks in the case where A is a singular matrix and explain the relation between the results of E. Lamprecht [2] and the ones of this paper.

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NOTATION. The notation used in the introduction, $F(q)$, $M_n(F_q)$, $GL(n,q)$, $e_d[\alpha]$, $\tau(g)$, θ, ε, ε_d, $W(\xi,A)$, $w(\xi)$ will be preserved throughout the paper. By a *partition* of a positive integer n, we mean, as usual, the expression of n as a sum of positive integers. If ρ is the partition of n defined by $n=\sum_{d=1}^{n} ds_d$ $(1\leqq d\leqq n,\ 0\leqq s_d)$, we write

$$\rho=(1^{s_1},2^{s_2},\cdots).$$

For $\rho=(1^{s_1},2^{s_2},\cdots)$, we put

$$z_\rho=1^{s_1}s_1!\,2^{s_2}s_2!\cdots=\prod_{d=1}^{n}d^{s_d}s_d!.$$

If ρ is a partition of a positive integer m, we put

$$|\rho|=m.$$

In addition to this, we consider the partition of 0, which will be denoted by 0, and put

$$|0|=0,$$

$$z_0=1.$$

For a polynomial $F=X^t+aX^{t-1}+\cdots$, we denote by $d(F)$ the degree of F, and put $\operatorname{tr}(F)=-a$.

1. The character of $GL(n, q)$

In this section, we recall the structure of the characters of $GL(n, q)$ which are given in J. A. Green [**1**][1].

1.1. Symbol $(\cdots f^{\nu(f)}\cdots)$. Let P be the set of all irreducible polynomials f with coefficients in $F(q)$ such that

1) $1\leqq d(f)\leqq n$,
2) the leading coefficient of f is 1,
3) f has not zero as its root.

Let $\nu(f)$ be a function on P which assigns a partition $\nu(f)$ to every $f\in P$, such that

$$\sum_{f\in P}|\nu(f)|\,d(f)=n\,.$$

For such a function $\nu(f)$, we define a symbol

$$(\cdots f^{\nu(f)}\cdots),$$

where f ranges over all elements of P. We call the polynomial $F=\prod f^{|\nu(f)|}$ *the characteristic polynomial* of the symbol $(\cdots f^{\nu(f)}\cdots)$.

DEFINITION[2]. Let $\rho=(1^{s_1}, 2^{s_2}, \cdots)$ be a partition of n and F be a polynomial of degree n with coefficients in $F(q)$ whose leading coefficient is 1 and which has not zero as its root. By a *ρ-decomposition* r of F, we mean a decomposition into several polynomials which is given in the following two steps;

1) $F=\prod_{d=1}^{n}F_d$, $d(F_d)=ds_d$,
2) $F_d=\prod f^{k_f(d)}$ and $\dfrac{d}{d(f)}\mid k_f(d)$,

where the second product runs over all the elements f of P such that $d(f)$

1) For details we refer the reader to J. A. Green [**1**].

2) Instead of the notion "mode from ρ-variables X^ρ into the symbol $(\cdots f^{\nu(f)}\cdots)$", which is used in [**1**, Definition 4.10, p. 423], we use "the ρ-decomposition of the characteristic polynomial of the symbol $(\cdots f^{\nu(f)}\cdots)$" for convenience of the calculation of Gaussian sums. It is easy to see that every mode from ρ-variables X^ρ to the symbol $(\cdots f^{\nu(f)}\cdots)$ is canonically in one-to-one correspondence with every ρ-decomposition of the characteristic polynomial of the symbol $(\cdots f^{\nu(f)}\cdots)$.

divides d.

For given F and ρ, it may happen that there is no ρ-decomposition of F. Further, a ρ-decomposition of F is not necessarily unique. For a ρ-decomposition r of F, and for an element f of P which appears in the decomposition r as above, we denote by $\rho(r,f)$ the partition of the integer $\sum_{d=1}^{n} k_f(d)$,

$$(1^{t_1}, 2^{t_2}, \cdots), \qquad t_i = \frac{k(id(f))}{i} \quad (1 \leqq i \leqq n).$$

Let H be a polynomial of degree ds_d which is decomposed into the product $H = \prod f^{k_f(d)}$ of the elements f of P such that $d(f)$ divides d. Assume that $\frac{d}{d(f)} \mid k_f(d)$. Then we define a positive integer $z_d(H)$ by

$$z_d(H) = \prod_{d(f)\mid d} \left(\frac{d}{d(f)}\right)^{\frac{d(f)k_f(d)}{d}} \left(\frac{d(f)k_f(d)}{d}\right)!,$$

the product being over all elements f of P such that $d(f) \mid d$. It is easy to see that, if F is the characteristic polynomial of a symbol $(\cdots f^{\nu(f)} \cdots)$ and r is a ρ-decomposition of F, then

(1) $$|\nu(f)| = |\rho(r,f)| \text{ for every element } f \in P,$$

(2) $$\prod_{f\in P} z_{\rho(r,f)} = \prod_{d=1}^{n} z_d(F_d) \quad \text{if} \quad F = \prod F_d .$$

We note that every symbol $(\cdots f^{\nu(f)} \cdots)$ is canonically in one-to-one correspondence with every conjugate class of $GL(n, q)$[3]. Later, in § 1.3, we shall construct an irreducible representation of $GL(n, q)$ for every symbol $(\cdots f^{\nu(f)} \cdots)$.

1.2. The character $B^{\rho}(h)$. If d is a positive integer and h is an arbitrary integer, we denote by $s_d(h; x)$ the function on $F(q^d)$ defined by

$$s_d(h; x) = \theta(x)^h + \theta(x)^{hq} + \cdots + \theta(x)^{hq^{d-1}} .$$

If f is an element of P such that $d(f)$ divides d and if α is a root of f, $s_d(h; \alpha)$ is independent of the choice of α. We can therefore write $s_d(h; \alpha) = s_d(h; f)$.

Let $\rho = (1^{s_1}, 2^{s_2}, \cdots)$ be a partition of n and $h_{11}, h_{12}, \cdots, h_{1s_1}; h_{21}, \cdots, h_{2s_2}; \cdots$ be $\sum_{d=1}^{n} s_d$ integers. Then we define, for each d, a function $S_d(h_{d1}, \cdots, h_{ds_d}; x_{d1}, \cdots, x_{ds_d})$ on the product $F(q^d) \times \cdots \times F(q^d)$ of s_d copies of $F(q^d)$ by

(♯) $$S_d(h_{d1}, \cdots, h_{ds_d}; x_{d1}, \cdots, x_{ds_d}) = \sum_{1', 2', \cdots, s'_d} s_d(h_{d1'}; x_{d1}) s_d(h_{d2'}; x_{d2}) \cdots s_d(h_{ds'_d}; x_{ds_d})$$

the summation being over all permutations $1', 2', \cdots, s'_d$ of $1, 2, \cdots, s_d$.

Let H_d be a polynomial of degree ds_d such that

3) Cf. [1, p. 406].

$$H_d=\prod f^{k_f(d)}, \quad \frac{d}{d(f)} \mid k_f(d).$$

For each f appearing in the product $\prod f^{k_f(d)}$, choose $\frac{d(f)k_f(d)}{d}$ variables among the x_{di} $(1 \leqq i \leqq s_d)$ in such a way that their join for all f becomes the whole $\{x_{d1}, \cdots, x_{ds_d}\}$. Substitute f for x_{di} in the expression (♯) if the variables x_{di} corresponds to f by our choice. Then we get from (♯) a complex number

$$(\natural) \qquad \sum_{1',2',\cdots,s'_d} (\cdots s_d(h_{d*}\,; f) \cdots s_d(h_{d*}\,; f) \cdots)$$

Since $S_d(h_{d1}, \cdots, h_{ds_d}\,; x_{d1}, \cdots, x_{ds_d})$ is symmetric in $x_{d1}, \cdots, x_{ds_d}$, this number is determined only by $h_{d1}, \cdots, h_{ds_d}$ and H_d; it is independent of the choice of variables among the x_{di} corresponding to f. We denote by $S_d(h_{d1}, \cdots, h_{ds_d}\,; H_d)$ the complex number (♮).

Let $r_1, r_2, \cdots, r_t$ be all distinct ρ-decompositions of the characteristic polynomial $F=\prod f^{|\nu(f)|}$ of a conjugate class $c=(\cdots f^{\nu(f)} \cdots)$, and write

$$r_i: \quad F=\prod_{d=1}^{n} F_d^{(i)}, \quad F_d^{(i)}=\prod_{d(f)\mid d} f^{k_f^{(i)}(d)}.$$

Put

$$(3) \qquad B_\rho(h\,; r_i)=\prod_d S_d(h_{d1}, \cdots, h_{ds_d}\,; F_d^{(i)}).$$

Now, in [**1**], the polynomials $Q_\rho^\lambda(q)$ in q are defined[4], where λ, ρ are any two partitions of a non-negative integer. Using these polynomials $Q_\rho^\lambda(q)$, put

$$(4) \qquad Q(r_i\,; c)=\prod_{f\in P} \frac{Q^{\nu(f)}_{\rho(r_i,f)}(q^{d(f)})}{z_{\rho(r_i,f)}}.$$

(Remark that, by (1), $|\nu(f)|=|\rho(r,f)|$ for every $f\in P$.) Then there exists a character $B^\rho(h)$[5] whose value at the conjugate class $c=(\cdots f^{\nu(f)} \cdots)$ is

$$(5) \qquad B^\rho(h)(c)=\sum_{i=1}^{t} Q(r_i\,; c) B_\rho(h\,; r_i).$$

This character is fundamental for the calculation of irreducible characters of $GL(n, q)$ and Gaussian sums.

1.3. The irreducible character of type $(\cdots g^{\nu(g)} \cdots)$. If a symbol $e=(\cdots g^{\nu(g)} \cdots)$ is given, we can construct an irreducible character of $GL(n, q)$ in the following way.

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4) For the definition of $Q_\rho^\lambda(q)$, we refer the reader to [**1**, Definition 4.1, p. 420]. The polynomials $Q_\rho^\lambda(q)$ have some interesting properties, but, later, we shall use only one property of $Q_\rho^\lambda(q)$ except for some properties which are easily seen from the definition. (Cf. footnote 10)).

5) Cf. [**1**, Definition 6.2, p. 433]. $B^\rho(h)$ is not necessarily the character of a matrix representation, but $(-1)^{n-\Sigma s_i} B^\rho(h)$ is so.

Let $\rho=(1^{s_1}, 2^{s_2}, \cdots)$ be a partition of n and r be a ρ-decomposition of the characteristic polynomial G of the symbol $e=(\cdots g^{\nu(g)}\cdots)$; and write

$$r:\quad G=\prod G_d,\quad G_d=\prod g^{k_g(d)}.$$

For the ρ-decomposition r of G, we determine in the following way the integers h_{di}, which appear in the definition of $B^\rho(h)$.

For a moment we regard the h_{di} as variables. Consider a fixed $d\,(1\leqq d\leqq n)$. Let $\varepsilon_{d(g)}^{cg}$ be a fixed root of every element g of P such that $d(g)$ divides d. Put

$$n_g=c_g\frac{q^d-1}{q^{d(g)}-1}:$$

For each g appearing in the product $G_d=\prod g^{k_g(d)}$, choose $\frac{d(g)k_g(d)}{d}$ variables among the h_{di} in such a way that their join for all g becomes $\{h_{d1},\cdots,h_{ds_d}\}$. Then, if h_{di} corresponds to g, we put

$$h_{di}=n_g\,.$$

Since $\sum_{g\in P}\frac{d(g)k_g(d)}{d}=s_d$, s_d integers h_{di} have been determined. The function $S_d(h_{d1},\cdots,h_{ds_d};x_{d1},\cdots,x_{ds_d})$ with these values as the h_{di} $(1\leqq i\leqq s_d)$ is, as easily seen, determined only by G_d; it is independent of the choice of a root $\varepsilon_{d(g)}^{cg}$ of g and variables h_{di} corresponding to g. Thus the symbol $(\cdots g^{\nu(g)}\cdots)$ and the ρ-decomposition r of its characteristic polynomial being given, the character $B^\rho(h)$ with these values as the h_{di} can be uniquely determined by the above process. We denote by $B^\rho(rh)$ the character $B^\rho(h)$ with these values as the h_{di}.

Then the irreducible character of type $e=(\cdots g^{\nu(g)}\cdots)$ is

(6) $$(-1)^{n-\Sigma|\nu(g)|}\sum_\rho\sum_r\chi(r,e)B^\rho(rh)\,,$$

where the first summation is over all partitions of n, the second one over all ρ-decompositions of the characteristic polynomial of the symbol $e=(\cdots g^{\nu(g)}\cdots)$, and $\chi(r,e)$ is the constant determined by the symbol $e=(\cdots g^{\nu(g)}\cdots)$ and the ρ-decomposition r of its characteristic polynomial[6]. All irreducible characters of $GL(n,q)$ are obtained in this way [**1**, Th. 14].

2. Gaussian sum $W(\xi, 1_n)$

2.1. Before calculating $W(\xi,1_n)$, we consider the character sum attached to the character $B^\rho(h)$,

$$W(B^\rho)=\sum_{X\in GL(n,q)}B^\rho(X)e_1[\mathrm{tr}(X)]\,,$$

6) We need not the explicit formula of $\chi(r,e)$. For this formula we refer the reader to [**1**, Lemma 8.2, p. 441].

where the h_{di} are dropped for simplicity. Put

$$\psi_n(q)=\prod_{i=1}^{n}(q^i-1),$$

$$c_\rho(q)=\prod_{i=1}^{n}(q^i-1)^{s_i} \quad \text{if} \quad \rho=(1^{s_1}, 2^{s_2}, \cdots),$$

$$\tau_d(h_{di})=\sum_{\alpha\in F(q^d)}\theta(\alpha)^{h_{di}}e_d[\alpha].$$

Then we have the following

LEMMA 1.

$$W(B^\rho)=q^{\frac{n(n-1)}{2}}\frac{\psi_n(q)}{c_\rho(q)}\prod_{d,i}\tau_d(h_{di}).$$

PROOF. If c is the conjugate class which corresponds to the symbol $(\cdots f^{\nu(f)}\cdots)$[7], then the centralizer in $GL(n,q)$ of an element of c is of order

$$a_c(q)=\prod_{f\in P}a_{\nu(f)}(q^{d(f)})\text{[8]}.$$

Therefore the number of elements of c is

$$\frac{q^{\frac{n(n-1)}{2}}\psi_n(q)}{\prod_{f\in P}a_{\nu(f)}(q^{d(f)})},$$

since $q^{\frac{n(n-1)}{2}}\psi_n(q)$ is the order of $GL(n,q)$. Then we have

$$W(B^\rho)=q^{\frac{n(n-1)}{2}}\psi_n(q)\sum_c\frac{B^\rho(c)}{a_c(q)}e_1[\mathrm{tr}(F_c)],$$

the summation being over all conjugate classes of $GL(n,q)$, $B^\rho(c)$ the value at the conjugate class c of the character $B^\rho(h)$, and F_c the characteristic polynomial of the conjugate class c. By (3), (4) and (5),

$$B^\rho(c)=\sum_i Q(r_i,c)B^\rho(h;r_i) \tag{7}$$

$$=\sum_i\prod_{f\in P}\frac{Q^{\nu(f)}_{\rho(r_i,f)}(q^{d(f)})}{z_{\rho(r_i,f)}}\prod_{d=1}^{n}S_d(h_{d1},\cdots,h_{ds_d};F^{(i)}_d).$$

If F is a polynomial of degree n, we consider the sum $\sum_{F=F_c}\frac{B^\rho(c)}{a_c(q)}$, extended over all conjugate classes whose characteristic polynomials are F. Then we have by (2) and (7)

$$\sum_{F_c=F}\frac{B^\rho(c)}{a_c(q)}=\sum_{|\nu(f)|}\sum_{r_i}\prod_f\frac{Q^{\nu(f)}_{\rho(r_i,f)}(q^{d(f)})}{a_{\nu(f)}(q^{d(f)})}\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F^{(i)}_d)}{z_d(F^{(i)}_d)},$$

7) Cf. footnote 3).

8) Cf. [**1**, p. 409 and Lemma 2.4, p. 410]. $a_\lambda(q)$ is a polynomial in q which is defined for every partition λ of a non-negative integer.

the summation $\sum_{|\nu(f)|}$ being over all combinations of partitions of $|\nu(f)|$ for every $f \in P$.

Now we need a property of polynomials $Q_\rho^\lambda(q)$ which is not given in Green's paper.

LEMMA 2.

$$\sum_\lambda \frac{Q_\rho^\lambda(q)}{a_\lambda(q)} = \frac{1}{c_\rho(q)}, \tag{8}$$

the summation being over all partitions λ of n.

The proof will be given later.

We return to the proof of Lemma 1.

Since

$$\begin{aligned}
&\sum_{|\nu(f)|} \prod_f \frac{Q_{\rho(r,f)}^{\nu(f)}(q^{d(f)})}{a_{\nu(f)}(q^{d(f)})} \\
&= \prod_f \sum_{\lambda = |\nu(f)|} \frac{Q_{\rho(r,f)}^{\lambda}(q^{d(f)})}{a_\lambda(q^{d(f)})} \\
&= \prod_f \frac{1}{c_{\rho(r,f)}} && \text{(by Lemma 2)} \\
&= \frac{1}{c_\rho(q)} && \text{(by the definition of } \rho(r,f))
\end{aligned}$$

we have

$$\sum_{F_c = F} \frac{B^\rho(c)}{a_c(q)} = \frac{1}{c_\rho(q)} \sum_{r_i} \prod_d \frac{S_d(h_{d1}, \cdots, h_{ds_d}; F_d^{(i)})}{z_d(F_d^{(i)})}. \tag{9}$$

Therefore we have

$$W(B^\rho) = \frac{q^{\frac{n(n-1)}{2}} \psi_n(q)}{c_\rho(q)} \sum_F \sum_r \prod_d \frac{S_d(h_{d1}, \cdots, h_{ds_d}; F_d)}{z_d(F_d)} e_1[\operatorname{tr}(F)], \tag{10}$$

the first summation being over all polynomials F with coefficients in $F(q)$ of degree n such that the leading coefficient of F is 1 and F has not zero as its root, the second one over all ρ-decompositions r of F $(r: F = \prod_d F_d)$. On the other hand, it is obvious that

$$\tau_d(h_{di}) = \sum_{d(f)|d} \frac{d(f)}{d} s_d(h_{di}; f) e_1\left[\frac{d}{d(f)} \operatorname{tr}(f)\right],$$

the summation being over all elements f of P such that $d(f)$ divides d. A direct computation shows

$$\prod_{i=1}^{sd} \tau_d(h_{di}) = \sum_{F_d} \frac{e_1[\operatorname{tr}(F_d)]}{z_d(F_d)} S_d(h_{d1}, \cdots, h_{ds_d}; F_d),$$

the summation being over all polynomials F_d of degree ds_d such that

$F_d=\prod_{d(f)|d} f^{k_f(d)}$ and $\frac{d}{d(f)}$ divides $k_f(d)$. By the definition of ρ-decomposition, we have

$$\prod_{d,i}\tau_d(h_{di})=\sum_F e_1[\mathrm{tr}(F)]\sum_r\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F_d)}{z_d(F_d)},\tag{11}$$

where the ρ-decomposition r of F is $F=\prod_{d=1}^n F_d$. By (10) and (11), we obtain

$$W(B^\rho)=\frac{q^{\frac{n(n-1)}{2}}\psi_n(q)}{c_\rho(q)}\prod_{d,i}\tau_d(h_{di}).$$

This completes the proof of Lemma 1.

Proof[9] of Lemma 2. This proceeds by induction on n. In the proof of the above Lemma 1, if $|\nu(f)|<n$, we may assume that (8) holds and, therefore, so does (9). We note that $|\nu(f)|=n$ can occur if and only if $F=l^n$ where l is a linear polynomial. If we put

$$Y_\rho(q)=\sum_\lambda\frac{Q_\rho^\lambda(q)}{a_\lambda(q)},$$

we have by (9)

$$\sum_{X\in GL(n,q)}B^\rho(X)=q^{\frac{n(n-1)}{2}}\psi_n(q)\Big(Y_\rho(q)\sum_{F=l^n}\sum_r\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F_d)}{z_d(F_d)}+\frac{1}{c_\rho(q)}\sum_{F\neq l^n}\sum_r\prod_d\frac{S_d(\cdots;F_d)}{z_d(F_d)}\Big).\tag{12}$$

On the other hand, if we put $s_d(h_{di})=\sum_{\alpha\in F(q^d)}\theta(\alpha)^{h_{di}}$, we have a formula analogous to (11),

$$\prod_{d,i}s_d(h_{di})=\sum_F\sum_r\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F_d)}{z_d(F_d)}.\tag{13}$$

If we choose integers h_{di} so that $h_{di}\not\equiv 0 \bmod q^d-1$ for some d and i, we have obviously

$$\prod_{d,i}s_d(h_{di})=0.\tag{14}$$

Then we have by (13) and (14)

$$\begin{aligned}\sum_{F\neq l^n}\sum_r\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F_d)}{z_d(F_d)}&=-\sum_{F=l^n}\sum_r\prod_d\frac{S_d(h_{d1},\cdots,h_{ds_d};F_d)}{z_d(F_d)}\\&=\sum_{\alpha\in F(q)}\theta(\alpha)^{\sum_{d,i}h_{di}}\end{aligned}$$

where the last equality follows from the definition of $S_d(h_{d1},\cdots,h_{ds_d};F_d)$ and $z_d(F_d)$. Therefore by (12),

$$\sum_{X\in GL(n,q)}B^\rho(X)=q^{\frac{n(n-1)}{2}}\psi_n(q)\Big(Y_\rho(q)-\frac{1}{c_\rho(q)}\Big)\Big(\sum_{\alpha\in F(q)}\theta(\alpha)^{\sum_{d,i}h_{di}}\Big).\tag{15}$$

9) The proof is similar to that of [1, Theorem 10, p. 431].

We have imposed on the integers h_{di} the condition $h_{di}\not\equiv 0 \bmod q^d-1$ for some d and i. We can take the h_{di} so that they satisfy one more condition $\sum_{d,i} h_{di}\equiv 0 \bmod q-1$. Then it follows from (15) that $\left(Y_\rho(q)-\frac{1}{c_\rho(q)}\right)(q-1)$ is always an integer for each prime power q since $\sum_{X\in GL(n,q)} B^\rho(X)$ is an integer divisible by the order $(=q^{\frac{n(n-1)}{2}}\psi_n(q))$ of $GL(n,q)$ on account of an elementary property of group character. On the other hand, $\left(Y_\rho(q)-\frac{1}{c_\rho(q)}\right)(q-1)$ is a rational function in q whose numerator is of smaller degree than the denominator[10], provided that $n>1$. This means that it must be identically zero, i.e.

$$Y_\rho(q)=\frac{1}{c_\rho(q)}.$$

This completes the proof of Lemma 2.

2.2. Proof of the Theorem. Let ξ be an irreducible character of type $\underset{\cdot}{e}=(\cdots g^{\nu(g)}\cdots)$ whose structure is described in § 1.3. Then we shall prove that the character sum attached to $B^\rho(rh)$ is

$$(-1)^{\sum_g|\nu(g)|-\sum s_i}q^{\frac{n(n-1)}{2}}\psi_n(q)\prod_g\tau(g)^{|\nu(g)|},$$

if $\rho=(1^{s_1},2^{s_2},\cdots)$. In fact, each h_{di} in $B^\rho(rh)$ is of the form $n_g=c_g\frac{q^d-1}{q^{d(g)}-1}$, where $\varepsilon^{c_g}_{d(g)}$ is a root of an element g of P such that $d(g)$ divides d. Then by a well known property of usual Gaussian sums attached to finite fields[11], and the definition of $\tau(g)$, we have

$$\begin{aligned}&\tau_d\left(c_g\frac{q^d-1}{q^{d(g)}-1}\right)\\&=\sum_{\alpha\in F(q^d)}\theta(\alpha)^{n_g}e_d[\alpha]\\&=\sum_{\alpha\in F(q^d)}\theta(\mathrm{N}(\alpha))^{c_g}e_{d(g)}[\mathrm{Tr}(\alpha)]\\&=(-1)^{\frac{d}{d(g)}-1}\tau(g)^{\frac{d}{d(g)}}{}^{11)},\end{aligned}$$

where $\mathrm{N}(\alpha)$, $\mathrm{Tr}(\alpha)$ are norm and trace from $F(q^d)$ to $F(q^{d(g)})$ respectively. Therefore, we have, by the above Lemma 1 and the definition of $B^\rho(rh)$

$$W(B^\rho(rh))=(-1)^{\sum_g|\nu(g)|-\sum s_i}q^{\frac{n(n-1)}{2}}\psi_n(q)\prod_g\tau(g)^{|\nu(g)|}. \tag{16}$$

Then, by (6) and (16), we have the matrix character of $W(\xi,1_n)$

10) Cf. [**1**, Lemma 2.4 and Lemma 4.3].

11) Cf. E. Lamprecht [**2**] S. 41, or, for example, A. Weil. Number of solutions of equations in finite fields. Bull. Amer. Math. Soc., **55** (1949).

$$=(-1)^{n-\Sigma|\nu(g)|}q^{\frac{n(n-1)}{2}}(\sum_{\rho,r}\chi(r,e)\frac{\psi_n(q)}{c_\rho(q)}(-1)^{\Sigma|\nu(g)|-\Sigma s_i})\prod_{g\in P}\tau(g)^{|\nu(g)|}.$$

Since $\sum_{\rho,r}(-1)^{\Sigma|\nu(g)|-\Sigma s_i}\frac{\psi_n(q)}{c_\rho(q)}\chi(r,e)$ is the degree of the irreducible character of type $e=(\cdots g^{\nu(g)}\cdots)$[12], we have, by the definition of $w(\xi)$,

$$w(\xi)=(-1)^{n-\Sigma|\nu(g)|}q^{\frac{n(n-1)}{2}}\prod_{g\in P}\tau(g)^{|\nu(g)|}.$$

If $g=X-1$, by the definition of $\tau(g)$, we have $\tau(g)=-1$. Further, it is well known that, if $g\neq X-1$, the absolute value of $\tau(g)$ is $q^{\frac{d(g)}{2}}$. Therefore, if ξ is of type $(\cdots(X-1)^{\kappa}\cdots)$ and $|\kappa|=k$, the absolute value of $w(\xi)$ is $q^{\frac{n^2-k}{2}}$. This completes the proof of the theorem.

2.3. In [**2**], E. Lamprecht introduced some notions "vollkommen", "echt", "eigentlich", "quasi-echt", in order to explain the properties of Gaussian sums attached to finite rings. In the case of $M_n(F_q)$, it is easy to see that

(i) the additive character $e_1[\mathrm{tr}(AX)]$ is "echt", if and only if A is non singular;

(ii) if $A(\neq 0)$ is a singular matrix, $e_1[\mathrm{tr}(AX)]$ is "quasi-echt";

(iii) if ξ is not a trivial representation, ξ is "eigentlich".

Moreover, if ξ is of type $(\cdots g^{\nu(g)}\cdots)$, ξ is "vollkommen" if and only if the characteristic polynomial of the symbol $(\cdots g^{\nu(g)}\cdots)$ is not divisible by the polynomial $X-1$[13].

Let A be nonsingular. Then our theorem solves completely the case where multiplicative representation is arbitrary ("vollkommen" or "non-vollkommen") and additive character is "echt"[14]. However, if A is singular, Kor. 2 to Satz 3 of [**2**] says that $W(\xi,A)$ is a zero matrix if ξ is "vollkommen", while, if ξ is not "vollkommen", Kor. 1 to Satz 3 of [**2**] says only that the determinant of $W(\xi,A)$ is zero. In this case where A is singular and ξ is not "vollkommen", examples[15] show that $W(\xi,A)$ is not necessarily a zero matrix, but the author has been unable to obtain the numerical value of this matrix.

2.4. Finally, we note that *the* $W(\xi,1_n)$ *has a property analogous to that of the usual Gaussian sums attached to finite fields;*

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12) This follows from (6) and the fact that the degree of $B^{\rho}(h)$ is $(-1)^{n-\Sigma s_i}\frac{\psi_n(q)}{c_\rho(q)}$. Cf. [**1**, p. 437].

13) Cf. [**1**, Theorem 13], and [**2**].

14) Cf. [**2**, Satz 4 and Satz 4 Kor. 2]. In the case where the finite ring is $M_n(F_q)$, our theorem implies Satz 4 of [**2**], and, if ξ is not "vollkommen", it is more precise than Kor. 2 to Satz 4 of [**2**].

15) Cf. [**2**, S. 43-44].

(17) $$W(\xi, 1_n)W(\bar{\xi}, 1_n)=\xi(-1_n)q^{n^2-k},$$

where $\bar{\xi}$ is the irreducible representation of $GL(n, q)$ which is complex conjugate to ξ.

This follows easily from the fact that the absolute value of $w(\xi)$ is $q^{(n^2-k)/2}$.

The College of General Education,
University of Tokyo

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