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On decomposable symmetric affine spaces.

By Atsuo FUJIMOTO

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§ 1. Decomposable spaces

Consider two affinely connected spaces without torsion A_{p} and A_{n-p} of the dimension p and $n-p$ respectively. Denote by $\Gamma_{j^{1}k^{1}}^{i^{1}}(x^{l^{1}})$ and $\Gamma_{j^{2}k^{2}}^{i^{2}}(x^{l^{2}})$ the connections, $(x^{i^{1}})$ and $(x^{i^{2}})$ the coordinates on A_{p} and A_{n-p} respectively. As to the ranges of indices we shall adopt the following convention $i, j, k, l=1,\cdots, n$; $i^{1}, j^{1}, k^{1}, l^{1}$ (indices of the first kind) $=1,\cdots,p$; $i^{2}, j^{2}, k^{2}, l^{2}$ (indices of the second kind) $=p+1,\cdots, n$.

The n-dimensional affinely connected space A_{n} with coordinates $(x^{i^{1}}, x^{i^{2}})$ and the connection $\tilde{\Gamma}_{jk}^{i}$ will be called the *product space* of A_{p} and A_{n-p}, if the components of the connection with the indices of different kind vanish and $\tilde{\Gamma}_{j^{1}k^{1}}^{i^{1}}=\Gamma_{j^{1}k^{1}}^{i^{1}}(x^{l^{1}})$, $\tilde{\Gamma}_{j^{2}k^{2}}^{i^{2}}=\Gamma_{j^{2}k^{2}}^{i^{2}}(x^{l^{2}})$. In this case A_{n} is said to be *decomposable*, and the coordinates $(x^{i^{1}}, x^{i^{2}})$ are called a code. When $(y^{i^{1}})$ and $(y^{i^{2}})$ are normal coordinates on A_{p} and A_{n-p} respectively, then $(y^{i^{1}}, y^{i^{2}})$ is a normal code on A_{n} ([1]).

An object defined on a decomposable A_{n} is said to be *breakable* if its components with the indices of different kind are all zero with respect to a code. If an object is breakable and its components with indices of the same kind depend, in any code, only on the variables of that kind, then the object is called a *product object*.

§ 2. Symmetric affine space

An n-dimensional affinely connected space A_{n} without torsion is said to be *symmetric* in Cartan's sense if the reflexion about any point in A_{n} is an affine collineation. An A_{n} with connexion Γ_{jk}^{i} is symmetric if and only if the first covariant derivative of the curvature tensor vanishes, i. e.

$$R_{jkl;m}^{i}=0 ,$$

where

$$R_{jkl}^{i}=\Gamma_{jk,l}^{i}-\Gamma_{jl,k}^{i}+\Gamma_{jk}^{h}\Gamma_{hl}^{i}-\Gamma_{jl}^{h}\Gamma_{hk}^{i} ;$$

and we denote by a semi-colon the covariant differentiation, while by a comma the partial differentiation.

A symmetric A_n admits always a transitive group of affine collineations cosisting of transvections and of isotropic subgroup.

The generators ξ^i_a $(a=1,\cdots,n)$ of the transvections along the geodesics given by

$$y^i=0$$

$$y^a=s \quad (i\neq a)$$

in the normal coordinates (y^i) at 0 are given as the solutions of the differential equations

$$\xi^i_{;j;k}+R^{*i}_{jks}\xi^s=0$$

satisfying the initial conditions

$$(\xi^i_a)_0=\delta^i_a\,, \qquad \left(\frac{\partial\xi^i_a}{\partial y_j}\right)_0=0\,,$$

where R^{*i}_{jkl} are the components of the curvature tensor for A_n with respect to the above normal coordinate system.

The generators η^i_α $(\alpha=n+1,\cdots,n+r)$ of the isotropic subgroup fixing the point 0 are given, in the normal coordinates (y^i) at 0, by

$$\eta^i_\alpha=C^i_{j\alpha}y^j\,,$$

where $C^i_{j\alpha}$ are the complete solutions of the following equations with the unknowns a^i_j

$$a^i_sB^s_{jkl}-a^s_jB^i_{skl}-a^s_kB^i_{jsl}-a^s_lB^i_{jks}=0$$

B^i_{jkl} being the components of the curvature tensor evaluated at the point 0.

In this case, putting

$$X_if=\xi^j_i\frac{\partial f}{\partial y^j}$$

$$Y_\alpha f=\eta^i_\alpha\frac{\partial f}{\partial y^i} \quad (i,j=1,\cdots,n\,;\ \alpha=n+1,\cdots,n+r)\,,$$

we can write the structural equations for the complete group of affine collineations of symmetric A_n in the following form

$$[X_i,X_j]=C^\alpha_{ij}Y_\alpha$$

$$[X_i, Y_\alpha]=C^j_{i\alpha}X_j$$
$$[Y_\alpha, Y_\beta]=C^\gamma_{\alpha\beta}Y_\gamma \quad (i,j=1,\cdots,n\,;\ \alpha,\beta,\gamma=n+1,\cdots,n+r)\,.$$

Moreover, if the generators are taken as above, then we obtain

$$B^i_{jkl}=C^i_{\alpha j}C^\alpha_{kl}\,.$$

§ 3. The group G_{n+r}

We consider an $(n+r)$-parameter continuous transformation group G_{n+r} of which structural equations are given by

$$\begin{aligned}&[X_i, X_j]=C^\alpha_{ij}Y_\alpha\\ (3.1)\qquad &[X_i, Y_\alpha]=C^j_{i\alpha}X_j\\ &[Y_\alpha, Y_\beta]=C^\gamma_{\alpha\beta}Y_\gamma \quad (i,j=1,\cdots,n\,;\ \alpha,\beta,\gamma=n+1,\cdots,n+r)^{1)}.\end{aligned}$$

In this case we can define an involutive automorphism σ of G_{n+r}. We shall call the subgroup generated by Y_α which is invariant under σ an ***isotropic subgroup*** and denote it by H_r. We shall call X_i $(i=1,\cdots,n)$ the generators of the *transvections* of G_{n+r}. In the following, we shall call the group having the above structure (3.1) merely ***group*** G_{n+r} for the sake of simplicity.

The group G_{n+r} is said to be effective if H_r does not contain any invariant subgroup of G_{n+r}.

As for the effectiveness of G_{n+r} we have the following

LEMMA 1. *The group G_{n+r} is effective if and only if the matrix*

$$C=||C^j_{\alpha i}||$$

is of rank r, where α denotes the rows and i and j the colomns.

PROOF. Suppose that the rank of C is $r-s$ $(s>0)$, then the set of equations

$$e^\alpha C^j_{\alpha i}=0 \tag{3.2}$$

has s independent solutions

$$e^\alpha=u^\alpha_\lambda \qquad (\lambda=1,\cdots,s)\,.$$

If we define new generators Z_λ by

$$Z_\lambda=u^\alpha_\lambda Y_\alpha \qquad (\lambda=1,\cdots,s)\,,$$

1) We assume hereafter that the indices i,j,k,l run from 1 to n and α,β,γ run from $n+1$ to $n+r$ unless otherwise stated.

we get

$$[Z_\lambda, X_i] = u^\alpha_\lambda C^j_{\alpha i} X_j = 0$$

because u^α_λ are solutions of (3.2).

Furthermore making use of the Jacobi relations

$$C^\gamma_{\alpha\beta} C^j_{\gamma i} + C^k_{\beta i} C^j_{k\alpha} + C^k_{i\alpha} C^j_{k\beta} = 0 ,$$

we get

$$u^\alpha_\lambda C^\gamma_{\alpha\beta} C^j_{\gamma i} = u^\alpha_\lambda C^j_{\alpha k} C^k_{\beta i} + u^\alpha_\lambda C^k_{\alpha i} C^j_{k\beta} = 0 .$$

that is, $u^\alpha_\lambda C^\gamma_{\alpha\beta}$ are again solutions of (3.2). Hence we can put

$$u^\alpha_\lambda C^\gamma_{\alpha\beta} = A^\mu_{\lambda\beta} u^\gamma_\mu$$

with some constants A's and we obtain

$$[Z_\lambda, Y_\beta] = A^\mu_{\lambda\beta} u^\gamma_\mu Y_\gamma = A^\mu_{\lambda\beta} Z_\mu .$$

Consequently Z_λ $(\lambda = 1, \cdots, s)$ generate an invariant subgroup of G_{n+r}.

Conversely, let us assume that H_r contains an invariant subgroup of G_{n+r} and $Z_\lambda = u^\alpha_\lambda Y_\alpha$ $(\lambda = 1, \cdots, s)$ are its symbols, then the matrix C is of rank $< r$ because we have

$$[Z_\lambda, X_i] = u^\alpha_\lambda C^j_{\alpha i} X_j = 0 .$$

§ 4. The symmetric A_n determined by G_{n+r}

THEOREM 1. *If an effective group G_{n+r} is given, then there always exists an n-dimensional symmetric affine space A_n whose complete group of affine collineations contains the subgroup isomorphic to G_{n+r}.*

PROOF. First we shall define the symmetric A_n. Let L_{n+r} be the group space with (0)-connexion of the group G_{n+r}. The canonical parameters e^A $(A = 1, \cdots, n+r)$ give a normal coordinate system at the identity. Let L_n be the subspace of L_{n+r} which consists of the transvections. Then L_n is given by

$$e^\alpha = 0 \quad (\alpha = n+1, \cdots, n+r) .$$

It is well known that L_n is totally geodesic, and is a symmetric affine space ([3]).

If we define A_n with normal coordinates (y^i) from L_n with normal coordinates (e^i) by the transformation

$$e^i = 2y^i ,$$

then A_n is a symmetric affine space and the components in (y^i) of the curvature tensor of A_n are given at the origin 0 by

$$B^i_{jkl}=C^i_{\alpha j}C^\alpha_{kl}\,. \tag{4.1}$$

Now we determine the structural equations for the complete group $\mathfrak{G}$ of affine collineations of A_n.

From the Jacobi relations for G_{n+r}

$$C^s_{\alpha k}C^r_{sl}+C^s_{\alpha l}C^r_{ks}+C^\beta_{kl}C^r_{\beta\alpha}=0$$

$$C^s_{\alpha j}C^i_{rs}+C^s_{jr}C^i_{\alpha s}+C^\beta_{r\alpha}C^i_{j\beta}=0$$

and from (4.1) we see that $C^j_{i\alpha}$ are r solutions of the equations with the unknowns a^i_j

$$a^i_sB^s_{jkl}-a^s_jB^i_{skl}-a^s_kB^i_{jsl}-a^s_lB^i_{jks}=0\,. \tag{4.2}$$

Moreover these $C^j_{i\alpha}$ are r independent solutions of (4.2) by the assumption of effectiveness.

Let $E^j_{i\lambda}$ $(\lambda=n+1,\cdots,n+s\,;\ s\geqq r)$ be the complete solutions of (4.2). We can assume without loss of generality that

$$E^j_{i\alpha}=C^j_{i\alpha}\,.$$

According to § 2, the generators $\tilde{\eta}^j_\lambda$ of isotropic subgroup fixing the point 0 are

$$\tilde{\eta}^j_\lambda=E^j_{i\lambda}y^i \quad (\lambda=n+1,\cdots,n+s)\,.$$

Let $\tilde{\xi}^i_a$ be the generators of the transvections along the geodesics

$$y^i=0$$

$$y^a=s \qquad (i\neq a)$$

in the normal coordinates (y^i) at 0.

We can write the structural equations for $\mathfrak{G}$ in the following form (§ 2)

$$\begin{aligned}
&[\tilde{X}_i,\tilde{Y}_j]=D^\alpha_{ij}\tilde{Y}_\alpha+D^{\alpha'}_{ij}\tilde{X}_{\alpha'}\\
&[\tilde{X}_i,\tilde{Y}_\alpha]=D^j_{i\alpha}\tilde{X}_j\\
&[\tilde{X}_i,\tilde{Y}_{\alpha'}]=D^j_{i\alpha'}\tilde{X}_j\\
&[\tilde{Y}_\alpha,\tilde{Y}_\beta]=D^r_{\alpha\beta}\tilde{Y}_r+D^{r'}_{\alpha\beta}\tilde{Y}_{r'}\\
&[\tilde{Y}_\alpha,\tilde{Y}_{\beta'}]=D^r_{\alpha\beta'}\tilde{Y}_r+D^{r'}_{\alpha\beta'}\tilde{Y}_{r'}\\
&[\tilde{Y}_{\alpha'},\tilde{Y}_{\beta'}]=D^r_{\alpha'\beta'}\tilde{Y}_r+D^{r'}_{\alpha'\beta'}\tilde{Y}_{r'}
\end{aligned}
\qquad
\begin{pmatrix}
i,j=1,\cdots,n\\
\alpha,\beta,r=n+1,\cdots,n+r\\
\alpha',\beta',r'=n+r+1,\cdots,n+s
\end{pmatrix} \tag{4.3}$$

where we have put

$$\tilde{X}_i f = \tilde{\xi}^j_i \frac{\partial f}{\partial y^j}$$

$$\tilde{Y}_\lambda f = E^j_{i\lambda} y^i \frac{\partial f}{\partial y^j} \quad (\lambda = n+1, \cdots, n+s),$$

and D's are structural constants for $\mathfrak{G}$.

From the relations

$$\left(\tilde{\xi}^j_a \frac{\partial \tilde{\eta}^i_\lambda}{\partial y^j} - \tilde{\eta}^j_\lambda \frac{\partial \tilde{\xi}^i_a}{\partial y^j} \right)_0 = D^k_{a\lambda} (\tilde{\xi}^i_k)_0$$

and from

$$(\tilde{\xi}^i_a)_0 = \delta^i_a, \quad (\tilde{\eta}^i_\lambda)_0 = 0 \quad \left(\frac{\partial \tilde{\eta}^i_\lambda}{\partial y^j} \right)_0 = E^i_{j\lambda}$$

we get

(4.4) $$D^i_{a\alpha} = E^i_{a\alpha} = C^i_{a\alpha}, \qquad D^i_{a\alpha'} = E^i_{a\alpha'}.$$

Making use of the Jacobi relations for G_{n+r}

$$C^j_{l\alpha} C^i_{j\beta} + C^\gamma_{\alpha\beta} C^i_{\gamma l} + C^j_{\beta l} C^i_{j\alpha} = 0,$$

we get

$$\tilde{\eta}^j_\alpha \frac{\partial \tilde{\eta}^i_\beta}{\partial y^j} - \tilde{\eta}^j_\beta \frac{\partial \tilde{\eta}^i_\alpha}{\partial y^j} = C^\gamma_{\alpha\beta} \tilde{\eta}^i_\gamma .$$

Hence we have

(4.5) $$D^\gamma_{\alpha\beta} = C^\gamma_{\alpha\beta}, \qquad D^{\gamma'}_{\alpha\beta} = 0.$$

From § 2, the components of the curvature tensor are given at 0 by

$$B^i_{jkl} = D^i_{\lambda j} D^\lambda_{kl} = C^i_{\alpha j} D^\alpha_{kl} + E^i_{\alpha' j} D^{\alpha'}_{kl}.$$

On the other hand, B^i_{jkl} are given by (4.1), therefore we must have

$$C^i_{\alpha j} D^\alpha_{kl} + E^i_{\alpha' j} D^{\alpha'}_{kl} = C^i_{\alpha j} C^\alpha_{kl}.$$

Since matrix $||E^i_{\lambda j}||$ where λ denotes rows and i and j columns is of rank s, we obtain

(4.6) $$D^\alpha_{kl} = C^\alpha_{kl}, \qquad D^{\alpha'}_{kl} = 0.$$

From (4.3), (4.4), (4.5) and (4.6) we see that $\mathfrak{G}$ contains the subgroup generated by $\tilde{X}_i$, $\tilde{Y}_\alpha$ which is isomorphic to G_{n+r}.

REMARK. In the case where G_{n+r} is not effective, let g_{r-t} be the maximal invariant subgroup of G_{n+r} which is contained in H_r. We consider the factor group $G_{n+t}=G_{n+r}/g_{r-t}$. This group G_{n+t} has the structural equations similar to (3.1), and is effective. It is easily seen that the symmetric A_n which is defined from G_{n+r} in the same manner as in the proof of Theorem 1 is equivalent to the symmetric affine space which is defined from G_{n+t} in the same manner as in the proof of Theorem 1. According to Theorem 1, the complete group of affine collineations of this space A_n contains the subgroup isomorphic to G_{n+t} and consequently homomorphic to G_{n+r}.

From Theorem 1 and the above Remark, if G_{n+r} is given, then we obtain a symmetric A_n whose complete group of affine collineations contains the subgroup isomorphic or homomorphic to G_{n+r} according as G_{n+r} is effective or not. We shall call this symmetric A_n *symmetric A_n determined by G_{n+r}*.

§ 5. Decomposable symmetric affine space

If an affinely connected space without torsion is decomposable, then the curvature tensor is product tensor in any code. Hence we have the following

THEOREM 2. *A decomposable affinely connected space without torsion is symmetric if and only if each composition space is symmetric.*

Let $J=\{1,\cdots,n\}$ be the index set for the n-dimensional space A_n. We decompose J into two subsets $J^1=\{1,\cdots,p\}$ and $J^2=\{p+1,\cdots,n\}$, and we fix this decomposition. If the values of the components of a tensor with respect to a coordinate system vanish at one point when its indices are of different kind, then we shall say that they are *breakable* (with respect to the decomposition $J=J^1+J^2$).

THEOREM 3. *A symmetric affine space A_n is decomposable if and only if there exists a coordinate system such that the components of the curvature tensor evaluated at any point in this coordinate system are breakable.*

PROOF. If a symmetric A_n is decomposable, then the curvature tensor is product tensor in a code. Therefore the components B^i_{jkl} of the curvature tensor evaluated at a point in this code are breakable.

Conversely, suppose that there exists a coordinate system (x^i) such that the components in this coordinate system of the curvature tensor are breakable at a point 0. We introduce in A_n the normal coordinate system (y^i) at 0 corresponding to the coordinate system (x^i). Let

$N^i_{jkl_1\cdots l_u}$ be the u-th normal tensor of A_n. Since our space is symmetric, we have [3]

$$N^i_{jkl_1\cdots l_{2s}}=0 \qquad (s=1,2,\cdots).$$

Next, we shall prove that $N^i_{jkl_1\cdots l_{2s+1}}(0)$ are breakable.

Given two tensors $S^i_{j_1\cdots j_a}$ and $T^i_{k_1\cdots k_b}$, we can define new tensors $S^l_{j_1\cdots j_a}T^i_{k_1\cdots k_{t-1}lk_{t+1}\cdots k_b}$ $(T^l_{k_1\cdots k_b}S^i_{j_1\cdots j_{t-1}lj_{t+1}\cdots j_a})$. We shall call this multiplication the C_S (C_T)-process. If two tensors $S^i_{j_1\cdots j_a}$ and $T^i_{k_1\cdots k_b}$ are both breakable, then $C_S(T^i_{k_1\cdots k_b})$ and $C_T(S^i_{j_1\cdots j_a})$ are breakable.

Now, let Π^i_{jk} be the components of the connection with respect to the normal coordinate system (y^i), then we have

$$\begin{aligned} R^{*i}_{jkl;\,l_1;\cdots;\,l_t} &= R^{*i}_{jkl;\,l_1;\cdots;\,l_{t-1},l_t}+\Pi_{l_t}(R^{*i}_{jkl;\,l_1;\cdots;\,l_{t-1}}) \\ &= R^{*i}_{jkl,l_1,\cdots,l_t}+\sum_{u=0}^{t-1}(\Pi_{l_{u+1}}(R^{*i}_{jkl;\,l_1;\cdots;\,l_u}),_{\,l_{u+2},\cdots,l_t}, \end{aligned}$$

where

$$\begin{aligned} \Pi_h(R^{*a}_{bcd;\,d_1;\cdots;\,d_u}) &= \Pi^a_{lh}R^{*l}_{bcd;\,d_1;\cdots;\,d_u}-\Pi^l_{bh}R^{*a}_{lcd;\,d_1;\cdots;\,d_u} \\ &\quad -\Pi^l_{ch}R^{*a}_{bld;\,d_1;\,d_2;\cdots;\,d_u}-\Pi^l_{dh}R^{*a}_{bcl;\,d_1;\cdots;\,d_u}-\sum\Pi^l_{d_th}R^{*a}_{bcd;\,d_1;\cdots;\,d_{t-1};\,l;\,d_{t+1};\cdots;d_u}. \end{aligned}$$

(R^{*i}_{jkl}; the components of the curvature tensor with respect to (y^i)).

On the other hand, we have

$$P(N^i_{jkl_1\cdots l_u})=0$$

where P denotes the sum of the $(u+1)(u+2)/2$ terms obtained by the permutations of the lower indices which do not yield equivalent terms. [cf. L. P. Eisenhart, Non-Riemannian Geometry]. Hence from the above relations and from the fact that our space is symmetric, we can derive, by a direct calculation, the following expression:

$$N^i_{jkl_1\cdots l_{2s+1}}=\varphi(N),$$

where $\varphi(N)$ is the polynomial of the normal tensors $N_{(2s-1)}$ of order $\leqq 2s-1$ and its each term is obtained by effecting the C_N-processes on $N_{(2s-1)}$. From the relation

$$N^i_{jkl}(0)=\frac{1}{3}(B^i_{jkl}+B^i_{kjl})$$

and from the breakability of B^i_{jkl}, we can conclude that $N^i_{jkl_1\cdots l_{2s+1}}(0)$ are breakable.

Now, we have

$$(5.1)\qquad \prod\nolimits^i_{jk} = N^i_{jkl_1}(0)\, y^{l_1} + \frac{1}{3!}\, N^i_{jkl_1l_2l_3}(0)\, y^{l_1}y^{l_2}y^{l_3} + \cdots$$

$$\cdots + \frac{1}{(2s+1)!}\, N_{jkl_1\cdots l_{2s+1}}(0)\, y^{l_1}\cdots y^{l_{2s+1}} + \cdots .$$

Since every $N^i_{jkl_1\cdots l_{2s+1}}(0)$ in (5.1) are breakable, we see that $\prod^i_{jk}$ vanishes if its indices are of different kinds and $\prod^{i^1}_{j^1k^1}$ and $\prod^{i^2}_{j^2k^2}$ depend only on y^{i^1} and y^{i^2} respectively. Hence our symmetric A_n is decomposable.

THEOREM 4. *The symmetric A_n determined by an effective group G_{n+r} is a product space of A_p and A_{n-p} of dimensions p and $n-p$ respectively if and only if the n-dimensional vector space V spanned by the transvections of G_{n+r} is a direct sum of p-dimensional subspace V_1 and $(n-p)$-dimensional subspace V_2 satisfying the following conditions;*

1° $[X_{i^1}, X_{i^2}]=0$

2° $[[X_{i^\lambda}, X_{j^\lambda}], X_{k^\lambda}]$ $(\lambda=1,2)$ *are linear combinations of X_{l^λ} only, where X_{i^λ} are the bases of V_λ $(\lambda=1,2)$ and $i^1, j^1, k^1, l^1=1,\cdots,p$ and $i^2, j^2, k^2, l^2=p+1,\cdots,n$.*

PROOF. Suppose that the symmetric A_n determined by G_{n+r} is decomposable. We write the structural equations for G_{n+r} in the form

$$(5.2)\qquad \begin{aligned}[][X_i, X_j] &= C^\alpha_{ij} Y_\alpha\\ [X_i, Y_\alpha] &= C^j_{i\alpha} X_j\\ [Y_\alpha, Y_\beta] &= C^\gamma_{\alpha\beta} Y_\gamma .\end{aligned}$$

In the same manner as in § 4, we obtain the symmetric A_n with normal coordinates (y^i). In this normal coordinate system, the components of the curvature tensor evaluated at the origin 0 are given by

$$(5.3)\qquad B^i_{jkl} = C^i_{\alpha j} C^\alpha_{kl} .$$

In general, we can not state that B^i_{jkl} are breakable. But by the assumption of decomposability of our space, we can introduce in A_n the normal code (y'^{i^1}, y'^{i^2}) at the point 0 such that the curvature tensor R'^i_{jkl} in this code is a product tensor. Hence, if we evaluate the transformation law for the curvature tensor at 0, we have

$$a^i_a(R'^a_{bcd})_0 = a^j_b a^k_c a^l_d B^i_{jkl}\,,$$

where a^i_j are constants and det $|a^i_j| \neq 0$.

Consequently, by effecting the change of base of the transvections

$$X'_i = a^j_i X_j\,,$$

we have

$$(5.4)\qquad \begin{aligned} &(a)\quad [[X'_{i^1}, X'_{j^1}], X'_{k^1}] = (R'^{l^1}_{k^1 i^1 j^1})_0 X'_{l^1} \\ &(b)\quad [[X'_{i^1}, X'_{j^1}], X'_{k^2}] = 0 \\ &(c)\quad [[X'_{i^1}, X'_{j^2}], X'_{k^1}] = 0 \\ &(d)\quad [[X'_{i^1}, X'_{j^2}], X'_{k^2}] = 0 \\ &(e)\quad [[X'_{i^2}, X'_{j^2}], X'_{k^1}] = (R'^{l^2}_{k^2 i^2 j^2})_0 X'_{l^2} \\ &(f)\quad [[X'_{i^2}, X'_{j^1}], X'_{k^2}] = 0 \end{aligned}$$

because $(R'^a_{bcd})_0$ are breakable.

Let V_1 be the p-dimensional subspace spanned by X'_{i^1} and let V_2 be the $(n-p)$-dimensional subspace spanned by X'_{i^2}. Then we have

$$(5.5)\qquad V = V_1 + V_2\,.$$

From (5.4 c, d) we get

$$[[X'_{i^1}, X'_{j^2}], X'_k] = 0\,.$$

Since G_{n+r} is effective, we get

$$(5.6)\qquad [X'_{i^1}, X'_{j^2}] = 0\,.$$

From (5.4 a, e), (5.5) and (5.6) we have proved the first part of Theorem 4.

Conversely, let us assume that G_{n+r} satisfies the conditions of Theorem 4. We consider the symmetric A_n determined by such G_{n+r}. The components of the curvature tensor at the origin are given by (5.3). From the condition 1°, we get

$$C^\alpha_{k^1 l^2} = 0\,.$$

Hence we get

$$(5.7)\qquad B^i_{j k^1 l^2} = 0\,.$$

Furthermore in the relations

$$B^i_{jk^1l^2}+B^i_{k^1l^2j}+B^i_{l^2jk^1}=0\,,$$

by putting $j=j^1$ and $j=j^2$, we get

$$B^i_{l^2j^1k^1}=0 \tag{5.8}$$

and

$$B^i_{k^1l^2j^2}=0 \tag{5.9}$$

Finally, by the condition 2°

$$[[X_{i^1},X_{j^1}],X_{k^1}]=C^{\alpha}_{i^1j^1}C^{l}_{\alpha k^1}X_l$$

are linear combinations of X_{l^1} only. Hence we get

$$B^{i^2}_{j^1k^1l^1}=0\,. \tag{5.10}$$

Similarly, we get

$$B^{i^1}_{j^2k^2l^2}=0\,. \tag{5.11}$$

It follows from, (5.7), (5.8), (5.9), (5.10) and (5.11) that B^i_{jkl} are breakable. Thus our space is decomposable by Theorem 3.

It is easily seen that each composition space is equivalent to the space determined by the subgroup generated by X_{i^λ}, $[X_{i^\lambda},X_{j^\lambda}]$ $(\lambda=1,2)$ and consequently composition spaces are of p and $n-p$ dimensional respectively.

THEOREM 5. *If the symmetric A_n determined by G_{n+r} is decomposable where G_{n+r} is semi-simple and effective, then G_{n+r} is decomposed into a direct product of two groups by which composition spaces are determined.*

PROOF. According to Theorem 4, the vector space V spanned by the transvections of G_{n+r} is direct sum of two subspaces V_1 and V_2 such that the conditions 1° and 2° of Theorem 4 are satisfied. Let X_{i^1} $(i^1=1,\cdots,p)$ and X_{i^2} $(i^2=p+1,\cdots,n)$ be the bases of V_1 and V_2 respectively.

Since G_{n+r} is semi-simple and effective, the subgroup of H_r generated by $[X_i,X_j]$ coincides with H_r ([2]). From this fact and from the condition 1° of Theorem 4 we can write each base Y_α of H_r in the form

$$Y_\alpha=a^{i^1j^1}_\alpha[X_{i^1},X_{j^1}]+b^{i^2j^2}_\alpha[X_{i^2},X_{j^2}]$$

with constants a's and b's.

From this we can easily see that X_{i^λ}; $[X_{i^\lambda},X_{j^\lambda}]$ $(\lambda=1,2)$ generate the invariant subgroups g_λ $(\lambda=1,2)$ of G_{n+r}.

On the other hand, if we denote by h_λ $(\lambda=1, 2)$ the invariant subgroups of H_r generated by $[X_{i^\lambda}, X_{j^\lambda}]$ $(\lambda=1, 2)$, then H_r is direct product of h_1 and h_2 because of effectiveness. Consequently G_{n+r} is the direct product of g_1 and g_2. It is clear that composition spaces are determined by g_1 and g_2.

COROLLARY 1. *If G_{n+r} is simple (and semi-simple) and effective, then symmetric A_n determined by G_{n+r} is non decomposable.*

PROOF. Suppose, on the contrary, that A_n is decomposable. Then the group G_{n+r} contains invariant subgroup by Theorem 5. This contradicts the hypothesis.

Let $\tilde{H}_r$ be the linear adjoint group corresponding to H_r and acting on the transvections of G_{n+r}. Then we have the following

COROLLARY 2. *If G_{n+r} is effective and $\tilde{H}_r$ is irreducible, then the symmetric A_n determined by G_{n+r} is either flat or non-decomposable.*

PROOF. Since $\tilde{H}_r$ is irreducible, either G_{n+r} is semi-simple or $[X_i, X_j]=0$ ([2]). In the case $[X_i, X_j]=0$, our space is flat.

Now we consider the case where G_{n+r} is semi-simple. Suppose that A_n is decomposable. According to Theorem 4, the vector space V spanned by the transvections is direct sum of two subspaces V_1 and V_2. These two subspaces are invariant under $\tilde{H}_r$. In fact, let X_λ $(\lambda=1, 2)$ be any generator of V_λ and let Y be any generator of H_r. Since H_r is the direct product of h_1 and h_2, we can write Y in the form

$$Y=Y_1+Y_2,$$

where Y_1 and Y_2 are generators of the invariant subgroups h_1 and h_2 of H_r respectively. Then we have

$$[Y, X_\lambda]=[Y_1+Y_2, X_\lambda]=[Y_\lambda, X_\lambda] \quad (\lambda=1, 2).$$

Therefore $[Y, X_\lambda]$ $(\lambda=1, 2)$ are generators of V_λ, that is, V_1 and V_2 are invariant under $\tilde{H}_r$. This contradicts the hypothesis that $\tilde{H}_r$ is irreducible.

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Kyoto University.

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