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A note on unramified abelian covering surfaces of a closed Riemann surface.

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1. Introduction.

Let F be a closed Riemann surface of genus p, and $\tilde{F}$ be an unramified and unbounded covering surface of F. If, above any closed curve on F, there never lie two curves on $\tilde{F}$, one of which is closed and the other open, $\tilde{F}$ is said to be *regular*. As is well known, a regular covering surface $\tilde{F}$ admits *covering transformations* onto itself, which are one-to-one and continuous and carry each point $\tilde{P}$ on $\tilde{F}$ into a point $\tilde{P}'$ with the same projection as $\tilde{P}$. The totality of these transformations forms the *covering transformation group* $\Gamma(\tilde{F})$, which characterizes $\tilde{F}$.

DEFINITION. *A regular covering surface $\tilde{F}$ of F is called an unramified abelian covering surface, if its covering transformation group $\Gamma(\tilde{F})$ is abelian.*[1]

In the present note, we shall investigate the structure of unramified abelian covering surfaces in some detail (§§ 2–4), and prove some function-theoretic properties of these surfaces (§§ 5–6).

An example of such surfaces is given by the Riemann surface $\tilde{F}_w$ of an *abelian integral* w,[2] where dw is an analytic differential of the first or the second kind defined on F. $\tilde{F}_w$ is an unramified and unbounded covering surface of F characterized by the following property: a curve $\tilde{\gamma}$ on $\tilde{F}_w$ is closed if and only if its projection γ on F is closed

1) An "unramified abelian covering surface" of a closed Riemann surface corresponds to an "unramified abelian extension" of an algebraic function field. L. Sario [6], [7] used the term "Abelsche Überlagerungsfläche" or "abelian covering surface" for another sort of covering surfaces (one of which is called "die Überlagerungsfläche der Kommutatoren" in [1]), whose covering transformation groups are not abelian except for some simple cases.

2) The Riemann surface of a *multiplicative function* gives a more general example.

and $\int_\gamma dw=0$. Its covering transformation group $\Gamma(\tilde{F}_w)$ is isomorphic to the additive group of complex numbers generated by the $2p$ periods of dw, which are taken along $2p$ closed curves forming a homology base on F (cf. § 2). If there exist, between these $2p$ periods, no linear equations with not all vanishing integral coefficients, $\Gamma(\tilde{F}_w)$ is a free abelian group with $2p$ generators, and $\tilde{F}_w$ coincides with the *covering surface of integral functions*[3] (" die Überlagerungsfläche der Integralfunktionen " in [9] or " die Überlagerungsfläche der Homologien " in [1]) of F.

2. The group $\Gamma(\tilde{F})$; constructive definition of $\tilde{F}$.

Let $C_{2i-1}, C_{2i}, i=1,\cdots,p$, be $2p$ (oriented) piecewise analytic simple closed curves on F, such that

1) the system $C_j, j=1,\cdots,2p$ does not divide F into two or more parts;

2) any two of C_j, except the p pairs C_{2i-1}, C_{2i}, $i=1,\cdots,p$, have no points in common; and

3) for each i, C_{2i-1} and C_{2i} have precisely one point in common.

Let $\tilde{P}$ be an arbitrarily fixed point on $\tilde{F}$. For each $j=1,\cdots,2p$, we denote by $C_j(\tilde{P})$ the end point of a curve on $\tilde{F}$ starting from $\tilde{P}$, whose projection on F is closed and homotopic to the curve C_j. Some of these points may, of course, be mutually identical, or identical with $\tilde{P}$. Since $\tilde{F}$ is regular by the assumption, there exist unique covering transformations of $\tilde{F}$, which carry $\tilde{P}$ into $C_j(\tilde{P})$, $j=1,\cdots,2p$, respectively. These transformations form a *system of generators* of the group $\Gamma(\tilde{F})$, and shall be denoted by the same letters C_j.[4] As to these matters, cf. e. g. [1] or [9].

Now, $\Gamma(\tilde{F})$ being an abelian group generated by C_j, $j=1,\cdots,2p$, there exists a number of *defining relations* between these elements:

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3) I. e. the " minimal " regular covering surface of F, on which every integral function defined on F becomes single-valued.

4) In general, these transformations depend also on the choice of the point $\tilde{P}$. In the case of abelian $\Gamma(\tilde{F})$, however, they are uniquely determined by the curves C_j.

$$(1) \qquad \sum_{j=1}^{2p} \gamma_{kj} C_j = 0 \qquad \text{(the identical transformation),}$$

$$k=1,\cdots,q \quad (0 \leqq q \leqq 2p),$$

with integral coefficients γ_{kj}, whose $q \times 2p$ matrix

$$(2) \qquad (\gamma_{kj})_{k=1,\cdots,q\,;\, j=1,\cdots,2p}$$

is of rank q. The number $2p-q=r$ is the *rank* of the abelian group $\Gamma(\tilde{F})$.

We shall represent the elements $\sum_1^{2p} m_j C_j$ of $\Gamma(\tilde{F})$, where m_j are integers, by the lattice points $(m_1,\cdots,m_{2p})$ of a $2p$-dimensional euclidean space E^{2p}. Let $\mathfrak{T}$ be the group of transformations of E^{2p} generated by the $2p$ translations carrying the origin into $(1,0,\cdots,0),\cdots,(0,\cdots,0,1)$ respectively, and $\mathfrak{T}(\tilde{F})$ be its subgroup generated by the q translations carrying the origin into $(\gamma_{k1},\cdots,\gamma_{k\,2p})$, $k=1,\cdots,q$, respectively. Obviously, two lattice points of E^{2p} represent one and the same element of $\Gamma(\tilde{F})$ if and only if they are equivalent with respect to $\mathfrak{T}(\tilde{F})$, and $\Gamma(\tilde{F})$ is isomorphic to the factor group $\mathfrak{T}/\mathfrak{T}(\tilde{F})$.

Now, suppose that $\Gamma(\tilde{F})$ is given by the defining relations (1). We shall construct the covering surface $\tilde{F}$ from F as follows. Let the two shores of each of the curves C_{2i-1}, C_{2i}, $i=1,\cdots,p$, be denoted by C_{2i-1}^+, C_{2i-1}^-, C_{2i}^+, C_{2i}^- respectively, in such a manner that the oriented curve C_{2i} intersects C_{2i-1} from the shore C_{2i-1}^+ to the other shore C_{2i-1}^-, and that C_{2i-1} intersects C_{2i} from C_{2i}^+ to C_{2i}^-. We cut F along the $2p$ curves C_j to obtain a surface Φ of planar character having p boundary curves, each of which consists of four sides $C_{2i-1}^+, C_{2i}^+, C_{2i-1}^-, C_{2i}^-$. To each residue class $(m_1,\cdots,m_{2q})$ mod $\mathfrak{T}(\tilde{F})$, we associate a replica $\Phi(m_1,\cdots,m_{2p})$ of Φ. Next, we identify the side C_{2i}^+ of each $\Phi(m_1,\cdots,m_{2i-1},m_{2i},\cdots,m_{2p})$ with the side C_{2i}^- of $\Phi(m_1,\cdots,m_{2i-1}+1,m_{2i},\cdots,m_{2p})$, and C_{2i-1}^+ of $\Phi(m_1,\cdots,m_{2i-1},m_{2i},\cdots,m_{2p})$ with C_{2i-1}^- of $\Phi(m_1,\cdots,m_{2i-1},m_{2i}+1,\cdots,m_{2p})$, where each point on C_j^+ must be identified with the corresponding point on C_j^- $(j=1,\cdots,2p)$.

By these procedures, each side of each Φ is identified with some unique side of some other (or the same) Φ, and, at each vertex of each Φ, there meet four Φ's: $\Phi(m_1,\cdots,m_{2i-1},m_{2i},\cdots,m_{2p})$, $\Phi(m_1,\cdots,m_{2i-1}+1,m_{2i},\cdots,m_{2p})$, $\Phi(m_1,\cdots,m_{2i-1},m_{2i}+1,\cdots,m_{2p})$, and $\Phi(m_1,\cdots,m_{2i-1}+1,m_{2i}+1,\cdots,m_{2p})$ (some of these four may be mutually identical). Thus, an unramified and unbounded covering surface of F is constructed. It is evident that this covering surface has $\Gamma(\tilde{F})$ as its covering transformation group.

REMARK. Obviously, $\tilde{F}$ is closed if and only if the group $\Gamma(\tilde{F})$ is finite, i.e. $r=0$. Further, it is easily proved that, if $r=1$, the ideal boundary of $\tilde{F}$ consists of two components (Randstücke in [1]), and, if $r\geqq 2$, it consists of a single one (cf. § 3). As for the genus $\tilde{p}$ of $\tilde{F}$, we have the followings: if the order g of $\Gamma(\tilde{F})$ is finite, then $\tilde{p}=g(p-1)+1$; if $g=\infty$ and $p=1$, then $\tilde{p}=0$; and, if $g=\infty$ and $\tilde{p}\geqq 2$, then $\tilde{p}=\infty$.

3. An exhaustion $\{\tilde{F}_n\}$ of $\tilde{F}$.

Suppose that $r>0$, so that $\tilde{F}$ is open. Then, from the $2p$ basis vectors $(1,0,\cdots,0),\cdots,(0,\cdots,0,1)$ of E^{2p}, we can choose r ones: $\mathfrak{y}_1,\cdots,\mathfrak{y}_r$, such that the $q+r=2p$ vectors $\mathfrak{x}_k=(\gamma_{k1},\cdots,\gamma_{k\,2p})$, $k=1,\cdots,q$, and $\mathfrak{y}_l$, $l=1,\cdots,r$, are linearly independent. In order to construct a convenient *exhaustion* of the surface $\tilde{F}$, we shall introduce in E^{2p} a new coordinate system $(x_1,\cdots,x_q\,;\,y_1,\cdots,y_r)$ with the same origin as before and with $\mathfrak{x}_1,\cdots,\mathfrak{x}_q\,;\,\mathfrak{y}_1,\cdots,\mathfrak{y}_r$ as basis vectors:

$$(m_1,\cdots,m_{2p})=(x_1,\cdots,x_q\,;\,y_1,\cdots,y_r)T\,, \tag{3}$$

where T is a $2p\times 2p$ matrix obtained from (2) by attaching below r rows of the form $(0,\cdots,0,1,0,\cdots,0)$, We put

$$T^{-1}=(\alpha_{jk},\beta_{jl})\,, \tag{4}$$

where $j=1,\cdots,2p$; $k=1,\cdots,q$; $l=1,\cdots,r$.

Two lattice points $(m_1,\cdots,m_{2p})$ and $(m'_1,\cdots,m'_{2p})$ are equivalent with respect to $\mathfrak{T}(\tilde{F})$ if and only if their difference has integral x-coordinates and vanishing y-coordinates:

$$\begin{aligned}&\sum_{j=1}^{2p}\alpha_{jk}(m'_j-m_j)\equiv 0 \qquad \mathrm{mod}\ 1, \quad k=1,\cdots,q\,,\\ &\sum_{j=1}^{2p}\beta_{jl}(m'_j-m_j)=0\,, \qquad\qquad l=1,\cdots,r\,.\end{aligned} \tag{5}$$

The number Q of lattice points $(m_1,\cdots,m_{2p})$ contained in the $2p$-dimensional parallelepiped: $0\leqq x_k<1$, $0\leqq y_l<1$, $k=1,\cdots,q$, $l=1,\cdots,r$, is equal to the absolute value of the determinant of T (cf. Minkowski [2]).

Further we remark: if $(x_1,\cdots,x_q;y_1,\cdots,y_l,\cdots,y_r)$ represents a lattice point in E^{2p}, i.e. if the components of the vector $(x_1,\cdots,x_q;\ y_1,\cdots,y_r)T$ are all integers, $(x_1,\cdots,x_q;\ y_1,\cdots,y_l\pm 1,\cdots,y_r)$ also represent lattice points, and these are neighbours of the former one. In fact, as is seen from (3), $(0,\cdots,0;\ 0,\cdots,\ \pm 1,\cdots,0)T$ have the form $(0,\cdots,\ \pm 1,\cdots,0)$.

Consider two Φ's of $\tilde{F}$ adjacent to each other along some side or having some vertex in common: $\Phi(m_1,\cdots,m_{2i-1},m_{2i},\cdots,m_{2p})=\Phi[x_1,\cdots,x_q;\ y_1,\cdots,y_r]$[5] and $\Phi(m_1,\cdots,m_{2i-1}+\varepsilon_1,m_{2i}+\varepsilon_2,\cdots,m_{2p})=\Phi[x_1',\cdots,x_q';\ y_1',\cdots,y_r']$, where $\varepsilon_1,\varepsilon_2=0,\ \pm 1$, not both zero. By (3) and (4), we have $y_l'-y_l=\varepsilon_1\beta_{2i-1,l}+\varepsilon_2\beta_{2i,l}$, $l=1,\cdots,r$. Hence, if we put

$$M=[\operatorname*{Max}_{i,l}(|\beta_{2i-1,l}|+|\beta_{2i,l}|)]+1,$$

where [ ] denotes the integral part of the number lying in, we have

$$|y_l'-y_l|<M, \qquad l=1,\cdots,r. \tag{6}$$

Now, for any integer $n\geqq 0$, let $\tilde{F}_n$ be the part of $\tilde{F}$, which consists of all $\Phi(m_1,\cdots,m_{2p})=\Phi[x_1,\cdots,x_q;\ y_1,\cdots,y_r]$ satisfying $-Mn\leqq y_l<M(n+1)$, $l=1,\cdots,r$. The number of such Φ's is equal to $M^r(2n+1)^rQ$; and the sequence $\tilde{F}_n$, $n=0,1,\cdots$, exhausts $\tilde{F}$. Further, as is seen from (6), any Φ of $\tilde{F}$ having points in common with some Φ of $\tilde{F}_n$ belong to $\tilde{F}_{n+1}$. Hence, $\tilde{F}_n$, together with its boundary, is contained in the interior of $\tilde{F}_{n+1}$.

Finally we shall prove that $\tilde{F}_n$ is connected if n is sufficiently large: $n\geqq n_0$.[6] First, we connect each of M^rQ Φ's of $\tilde{F}_0$ to $\Phi[0,\cdots,0;\ 0,\cdots,0]$ by a chain of Φ's on $\tilde{F}$, and take n_0 so large that $\tilde{F}_{n_0}$ contains these M^rQ chains. Suppose that $\Phi[x_1,\cdots,x_q;\ y_1,\cdots,y_r]\subset\tilde{F}_n$. Then, while reducing the absolute values of y's one by one, we construct a chain of Φ's on $\tilde{F}_n$, which connects $\Phi[x_1,\cdots,x_q;\ y_1,\cdots,y_r]$ to one of the M^rQ Φ's of $\tilde{F}_0$. Thus, $\Phi[x_1,\cdots,x_q;\ y_1,\cdots,y_r]$ can be connected to $\Phi[0,\cdots,0;\ 0,\cdots,0]$ in $\tilde{F}_n$ if $n\geqq n_0$.

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5) In the notation of replicas of Φ, we shall use square brackets instead of round ones, if the corresponding lattice point is represented in terms of the (x,y)-coordinates.

6) E.g. it is sufficient to take $n_0=\Sigma_{k,j}|\gamma_{kj}|+rM$.

4. Estimation of boundary lengths of $\tilde{F}_n$.

Suppose that a curve on $\tilde{F}$ consists of a finite number of the arcs C_j^+, C_j^-, $j=1,\cdots,2p$, of some Φ's on $\tilde{F}$. For simplicity, we shall call the number of these arcs the " length " of that curve.

1) We shall first evaluate the *total length* L_n of the boundary Γ_n of $\tilde{F}_n$.

For any r integers $t_1,\cdots,t_r$, let $Z(t_1,\cdots,t_r)$ be the part of $\tilde{F}$ consisting of M^rQ $\Phi[x_1,\cdots,x_q; y_1,\cdots,y_r]$ satisfying $Mt_l \leqq y_l < M(t_l+1)$, $l=1,\cdots,r$. As is seen from (6), any Φ of $\tilde{F}$ adjacent to some Φ of $Z(t_1,\cdots,t_r)$ belongs to one of $Z(t_1+\delta_1,\cdots,t_r+\delta_r)$, where $\delta_l=0, \pm 1$, $l=1,\cdots,r$.

For not all vanishing δ's, let $\gamma(t_1,\cdots,t_r; \delta_1,\cdots,\delta_r)$ be the part of the boundary of $Z(t_1,\cdots,t_r)$, along which it adjoins to $Z(t_1+\delta_1,\cdots,t_r+\delta_r)$, and $L(\delta_1,\cdots,\delta_r)$ be its length. Since any $Z(t_1,\cdots,t_r)$ is congruent to $Z(0,\cdots,0)$ $=\tilde{F}_0$, $L(\delta_1,\cdots,\delta_r)$ does not depend on $t_1,\cdots,t_r$, The sum of $L(\delta_1,\cdots,\delta_r)$ for the 3^r-1 admissible value combinations of δ's is equal to the boundary length L_0 of $\tilde{F}_0$.

Suppose that $Z(t_1,\cdots,t_r)\subset\tilde{F}_n$, i.e. $-n\leqq t_l\leqq n$, $l=1,\cdots,r$. The curve $\gamma(t_1,\cdots,t_r; \delta_1,\cdots,\delta_r)$ belongs to the boundary of $\tilde{F}_n$ if and only if $Z(t_1+\delta_1,\cdots,t_r+\delta_r)\not\subset\tilde{F}_n$. As is easily seen, the number of such value combinations of t's for fixed δ's is equal to $(2n+1)^r-(2n)^{r'}(2n+1)^{r-r'}$, where r' is the number of non-vanishing δ's. Hence, the total boundary length of $\tilde{F}_n$ is found to be: $L_n=\sum\{(2n+1)^r-(2n)^{r'}(2n+1)^{r-r'}\}\times L(\delta_1,\cdots,\delta_r)\leqq\{(2n+1)^r-(2n)^r\}\sum L(\delta_1,\cdots,\delta_r)=\{(2n+1)^r-(2n)^r\}L_0$, where the summation ranges over the 3^r-1 value combinations of δ's.

Thus, we have

$$L_n=O(n^{r-1}) . \tag{7}$$

2) Next, we shall estimate from above the *length of each connected component* of Γ_n.

Suppose that a boundary arc α, one of $C_1^+, C_1^-, C_2^+, C_2^-$ say, of a $\Phi(m_1,\cdots,m_{2p})$ of $\tilde{F}_n$ belongs to Γ_n. At each end point of α, there meet four Φ's of $\tilde{F}$ (not all belonging to $\tilde{F}_n$): $\Phi(m_1, m_2, m_3,\cdots,m_{2p})$, $\Phi(m_1+\varepsilon_1, m_2, m_3,\cdots,m_{2p})$, $\Phi(m_1, m_2+\varepsilon_2, m_3,\cdots,m_{2p})$, and $\Phi(m_1+\varepsilon_1, m_2+\varepsilon_2, m_3,\cdots,m_{2p})$, where $\varepsilon_1, \varepsilon_2=\pm 1$. Hence, the arc of Γ_n, which adjoins to α at this point, must be one of $C_1^+, C_1^-, C_2^+, C_2^-$ of one of above four Φ's. Hence we see: the connected component of Γ_n containing α consists only of the arcs $C_1^+, C_1^-, C_2^+, C_2^-$ of Φ's with constant $m_3,\cdots, m_{2p}$ (i. e. Φ's, which allow one such representation respectively).

Now, let $\Phi(m_1^*, m_2^*, m_3^*,\cdots, m_{2p}^*)$ be a Φ belonging to $\tilde{F}_n$. We denote by $\Delta_n^{12}=\Delta_n^{12}(m_3^*,\cdots, m_{2p}^*)$ the part of $\tilde{F}_n$ consisting of Φ's of the form $\Phi(m_1, m_2, m_3^*,\cdots, m_{2p}^*)$, and by $\gamma_n^{12}=\gamma_n^{12}(m_3^*,\cdots, m_{2p}^*)$ the part of the boundary of Δ_n^{12} consisting of the arcs $C_1^+, C_1^-, C_2^+, C_2^-$. Then, in order to estimate from above the length of a component of Γ'_n consisting of $C_1^+, C_1^-, C_2^+, C_2^-$, it suffices, by the above remark, to estimate the length $L_n^{12}=L_n^{12}(m_3^*,\cdots, m_{2p}^*)$ of γ_n^{12}.

By the definition of $\tilde{F}_n$, $\Phi(m_1, m_2, m_3^*,\cdots, m_{2p}^*)$ belongs to $\tilde{F}_n$ if and only if

$$(8) \qquad -Mn \leqq \beta_{1l}\, m_1+\beta_{2l}\, m_2+\sum_{j=3}^{2p}\beta_{jl}\, m_j^* < M(n+1)\,, \qquad l=1,\cdots, r\,.$$

According to the rank of the $2\times r$ matrix

$$(9) \qquad \begin{pmatrix} \beta_{11},\cdots, \beta_{1r} \\ \beta_{21},\cdots, \beta_{2r} \end{pmatrix},$$

we distinguish three cases from each others.

(a): The matrix (9) has the rank 2.

Then, the inequalities (8) define a bounded convex region δ_n^{12} on the (m_1, m_2)-plane through $(m_1^*, m_2^*, m_3^*,\cdots, m_{2p}^*)$, i. e. the plane $m_3=m_3^*$, $\cdots, m_{2p}=m_{2p}^*$ in E^{2p}. Further, there exists on this plane no pair of mutually equivalent lattice points. For, if (m_1, m_2), (m_1', m_2') are such a pair, there hold by (5)

$$\beta_{1l}(m_1'-m_1)+\beta_{2l}(m_2'-m_2)=0\,, \qquad l=1,\cdots, r\,,$$

so that $m_1'=m_1$, $m_2'=m_2$.

To each lattice point (m_1, m_2) on this plane, we associate a square $S(m_1, m_2)$, whose centre is at (m_1, m_2) and whose sides are parallel to the coordinate axes and have the length unity respectively. $S(m_1, m_2)$ may be considered as a model of the replica $\Phi(m_1, m_2, m_3^*,\cdots, m_{2p}^*)$; the sides parallel to the m_1-axis are the models of the boundary arcs C_1^+ and C_1^-, and the sides parallel to the m_2-axis are those of C_2^+ and C_2^-. The part Δ_n^{12} is represented by the sum D_n^{12} of all $S(m_1, m_2)$ with $(m_1, m_2)\in\delta_n^{12}$, and L_n^{12} is equal to the length of the boundary of D_n^{12}.

Suppose, for instance, that $\beta_{11}\,\beta_{22}-\beta_{12}\,\beta_{21}\neq 0$. Then, the first two inequalities of (8) define a parallelogram π_n^{12} containing δ_n^{12}. Let P_n^{12} be the sum of all $S(m_1, m_2)$ with $(m_1, m_2) \in \pi_n^{12}$. Since δ_n^{12} is convex and contained in π_n^{12}, it is easily seen that the boundary length L_n^{12} of D_n^{12} does not exceed that of P_n^{12}.

The directions of the sides of π_n^{12} are independent of n, and their lengths are respectively proportional to n. Hence we see that the boundary length of P_n^{12} of is $\sim$const. n. This constant depends only on $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$, and not on $m_3^*, \cdots, m_{2p}^*$, since, as is seen from (8), a change in these latter quantities only causes a translation of π_n^{12}.

Hence, in the case (a), we have

$$L_n^{12}(m_3^*, \cdots, m_{2p}^*) \leqq \text{const.}\, n \qquad (n \geqq 1),$$

where the constant on the right-hand side depends only on the matrix T.

(b): The matrix (9) has the rank 1.

Suppose, for instance, that $|\beta_{11}| + |\beta_{21}| \neq 0$. In this case, the inequalities (8) define a parallel strip region δ_n^{12} on the (m_1, m_2)-plane, which is parallel to the straight-line $\beta_{11} m_1 + \beta_{21} m_2 = 0$.

A lattice point (m_1, m_2) on the (m_1, m_2)-plane is, by (5), an equivalent of the lattice point (0, 0), if and only if

$$(10) \qquad \begin{aligned} &\alpha_{1k} m_1 + \alpha_{2k} m_2 \equiv 0 \quad \text{mod } 1, \quad && k = 1, \cdots, q, \\ &\beta_{1l} m_1 + \beta_{2l} m_2 = 0, && l = 1, \cdots, r. \end{aligned}$$

Hence, any equivalent of (0, 0) on this plane lies on the straight-line $\beta_{11} m_1 + \beta_{21} m_2 = 0$. Further, since the elements of the matrix QT^{-1} are integers, the point $(-Q^2\beta_{21}, Q^2\beta_{11})$ is actually a lattice point $\neq (0, 0)$ satisfying (10).

Let $(\mu_1, \mu_2) \neq (0, 0)$ be one of the equivalents of $(0, 0)$ nearest to (0, 0). Then, any equivalent of (0, 0) is represented in the form $(\nu\mu_1, \nu\mu_2)$, $\nu = 0, \pm 1, \cdots$. Hence, two lattice points (m_1, m_2), (m_1', m_2') are mutually equivalent if and only if $m_1' = m_1 + \nu\mu_1$, $m_2' = m_2 + \nu\mu_2$ for some integer ν.

As in (a), we may consider the square $S(m_1, m_2)$ as a model of $\Phi(m_1, m_2, m_3^*, \cdots, m_{2p}^*)$. Then, a model of Δ_n^{12} is constructed from the sum of all $S(m_1, m_2)$ with $(m_1, m_2) \in \delta_n^{12}$ by identifying the squares $S(m_1 + \nu\mu_1, m_2 + \nu\mu_2)$, $\nu = 0, \pm 1, \cdots$, mutually. As is easily seen, the resulting cylinder-shaped figure has two boundary components, each of which has the length $|\mu_1|+|\mu_2|$. Hence, we have $L_n^{12}=2(|\mu_1|+|\mu_2|) \leqq 2Q^2(|\beta_{11}|+|\beta_{21}|)$.

Thus, in the case (b), we have

$$L_n^{12}(m_3^*,\cdots,m_{2p}^*)\leqq \text{const.},$$

where the constant on the right-hand side depends only on the matrix T.

(c): The matrix (9) has the rank 0.

In this case, any point on the (m_1, m_2)-plane satisfies (8); and the lattice points $(Q, 0)$ and $(0, Q)$ both satisfy (10), i. e. they are equivalents of $(0, 0)$. Then, by the well known Minkowski's procedure [2], we can find two lattice points (μ_1, μ_2) and (μ_1', μ_2') equivalent to $(0, 0)$, such that any equivalent of $(0, 0)$ is uniquely expressed in the form $(\nu\mu_1+\nu'\mu_1', \nu\mu_2+\nu'\mu_2')$ with integral coefficients ν, ν'.

As before, we take the square $S(m_1, m_2)$ as a model of $\Phi(m_1, m_2, m_3^*,\cdots, m_{2p}^*)$ and construct the model of Δ_n^{12} from the whole (m_1, m_2)-plane by identifying the mutually equivalent squares $S(m_1+\nu\mu_1+\nu'\mu_1', m_2+\nu\mu_2+\nu'\mu_2')$, $\nu, \nu'=0, \pm 1,\cdots$. The resulting torus-shaped figure has no boundary arcs, i. e. γ_n^{12} is empty.

Thus, in the case (c), we have

$$L_n^{12}(m_3^*,\cdots,m_{2p}^*)=0.$$

5. Main theorem and lemmas.

As usual, let O_G denote the class of Riemann surfaces with null boundary, and O_{AB}, O_{AD} the classes of those not tolerating non-constant analytic functions which are bounded or have finite Dirichlet integrals respectively. Now, we shall formulate our theorem in the following form.

THEOREM. *Let $\tilde{F}$ be an unramified abelian covering surface of a closed Riemann surface F, and $\Gamma(\tilde{F})$ be its covering transformation group.*

i) *Let r be the rank of the abelian group $\Gamma(\tilde{F})$. Then, $\tilde{F}\in O_G$ if and only if $r\leqq 2$.*

ii) *$\tilde{F}\in O_{AD}$.*

iii) *Let $C_{2i-1}, C_{2i}, i=1,\cdots,p$, be the system of generators of $\Gamma(\tilde{F})$ mentioned in § 2. If there exists, for each $i=1,\cdots,p$, a relation of the form*

(11) $$\gamma_{2i-1} C_{2i-1} + \gamma_{2i} C_{2i} = 0$$

with not both vanishing integral coefficients γ_{2i-1} *and* γ_{2i}[7], *then* $\hat{F} \in O_{AB}$.

Let Ψ be a surface of planar character obtained from F by cutting along p disjoint non-dividing loop cuts. According to Royden [5], we shall call an unramified and unbounded covering surface of F a *covering surface of type S*, if it consists of a (finite or infinite) number of replicas of Ψ. Then,

COROLLARY OF iii). *An abelian covering surface of type S of a closed Riemann surface belongs to the class* O_{AB}.

In fact, taking the p loop cuts as C_{2i-1}, $i=1,\cdots,p$, we can easily construct p curves $C_{2i}, i=1,\cdots,p$, such that the system $C_j, j=1,\cdots,2p$, satisfies the conditions mentioned in § 2. Then, (11) is satisfied with $\gamma_{2i-1}=1$, $\gamma_{2i}=0$.

Our proof of the theorem is based on the following existence criteria due to Royden, Nevanlinna, Sario, and Pfluger.

Let $\hat{F}$ be an unramified and unbounded covering surface of F, which is composed of an infinite number of replicas of Φ.[8] We may then represent the structure of $\hat{F}$ by the well known *Speiser linear graph* (Streckenkomplex), where each Φ of $\hat{F}$ is represented by a *knot* t^0_μ $(\mu=0,1,\cdots)$, and two knots representing two Φ's adjacent to each other are connected by a *segment* t^1_ν $(\nu=0,1,\cdots)$. If two Φ's of $\hat{F}$ adjoin to each other along two or more sides, their representative knots are connected by the same number of segments; and, if a Φ of $\hat{F}$ adjoins to itself along a number of sides, its representative knot has the same number of segments starting from it and returning to itself. Since Φ has $4p$ sides, there meet $4p$ segments at each knot (returning segments being counted twice). We fix an orientation for each segment t^1_ν once for all, and distinguish one knot t^0_0 from others.

LEMMA 1. (Royden [5])[9]. *If there exists, on the linear graph of* $\hat{F}$, *a one-dimensional chain* $\sum_0^\infty a_\nu t^1_\nu$ *with real coefficients* a_ν, *such that*

(12) $$\left(\sum_0^\infty a_\nu t^1_\nu\right)^{\cdot} = t^0_0 \quad \text{and} \quad \sum_0^\infty |a_\nu|^2 < +\infty,$$

then $\hat{F} \notin O_G$.

7) Clearly, the effectiveness of this condition depends on the choice of the curves C_j, $j=1,\cdots,2p$.

8) For the brevity of statements, we put this rather restrictive assumption.

9) A simple proof of this theorem was given by M. Tsuji [8].

Let W be an open Riemann surface, and A_n^κ, $\kappa=1,\cdots,k(n)<+\infty$, $n=0,1,\cdots$, be a collection of doubly connected subregions of W satisfying the following conditions:

1) each annulus A_n^κ is bounded by two piecewise analytic curves γ_n^κ and $\bar{\gamma}_n^\kappa$;

2) any two of the annuli A_n^κ have no points in common;

3) the complementary set of $\bigcup_{\kappa=1}^{k(n)} A_n^\kappa$ with respect to W has precisely one compact component B_n; and

4) B_n is bounded by the $k(n)$ curves γ_n^κ and contains the annuli $A_{n'}^\kappa$ with $n'<n$.

Let u_n^κ be the harmonic measure of $\bar{\gamma}_n^\kappa$ with respect to A_n^κ, i. e. the function harmonic in A_n^κ, continuous on the closure of A_n^κ, $=0$ on γ_n^κ, and $=1$ on $\bar{\gamma}_n^\kappa$. We denote by v_n^κ the conjugate harmonic function of u_n^κ, and put

$$\mu_n^\kappa=2\pi\Big/\int_{\gamma_n^\kappa} dv_n^\kappa, \qquad \sigma_n=1\Big/\sum_{\kappa=1}^{k(n)}\frac{1}{\mu_n^\kappa}.$$

These quantities are called the *harmonic moduli* of A_n^κ and $\bigcup_{\kappa=1}^{k(n)} A_n^\kappa$ respectively. We put $\mu_n=\mathrm{Min}_\kappa\,\mu_n^\kappa$, and $K(N)=\mathrm{Max}_{n\leq N}\,k(n)$.

LEMMA 2. (Nevanlinna [3]). *If*

$$\sum_n^\infty \sigma_n=+\infty,$$

then $W\in O_G$.

LEMMA 3. (Sario [6], [7]). *If*

$$\sum_n^\infty \mu_n=+\infty,$$

then $W\in O_{AD}$.

LEMMA 4. (Pfluger [4]).[10] *If*

$$\overline{\lim_{N\to\infty}}\left\{\sum_n^N \mu_n-\frac{1}{2}\log K(N)\right\}=+\infty,$$

then $W\in O_{AB}$.

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10) Pfluger states this theorem in terms of a conformal metric defined on W. Lemma 4 is proved by a slight modification of his proof.

6. Proof of the theorem.

Since the theorem is trivial for closed covering surfaces $\tilde{F}$, we assume that $\tilde{F}$ is open.

1) PROOF OF i): NECESSITY OF $r \leqq 2$.

Suppose that $r \geqq 3$. Since $\tilde{F}$ consists of an infinite number of Φ's, it can be represented by the Speiser linear graph mentioned in § 5. We shall construct on this graph a one-dimensional chain $\sum_0^\infty a_\nu t_\nu^1$ satisfying (12).

Let E^{2p} be the space mentioned in § 2. By segments of unit length, we connect every two neighbouring lattice points in E^{2p}: $(m_1, \cdots, m_j, \cdots, m_{2p})$ and $(m_1, \cdots, m_j+1, \cdots, m_{2p})$, so that a net of segments is constructed in E^{2p}. If we identify the lattice points and the segments with each others, which are mutually equivalent with respect to the group $\mathfrak{T}(\tilde{F})$, we obtain from this net a linear graph G^{2p} representing the structure of $\tilde{F}$.

Since $r \geqq 3$, we can choose, from the $2p$ generators C_j of $\Gamma(\tilde{F})$, three ones, C_1, C_2, and C_3 say, which are mutually free. Then, the three-dimensional subspace E^3: $m_4=0, \cdots, m_{2p}=0$, of E^{2p} contains no pair of points equivalent to each other with respect to $\mathfrak{T}(\tilde{F})$. Let G^3 be the subgraph of G^{2p} consisting of the knots (lattice points) and the segments lying in E^3.

To the segments t_ν^1 of G^{2p} not appearing in G^3, we attribute the value $a_\nu=0$. For the segments t_ν^1 in G^3, we determine a_ν as follows. Let M_ν be the middle point of t_ν^1, and O be the origin $(0, 0, 0)$ of E^3. Through M_ν we draw a plane in E^3 parpedicular to t_ν^1, and let S_ν be the square on this plane, whose centre is at M_ν, and whose sides are parallel to the coordinate axes of E^3 and have the length unity respectively. Let ω_ν be the solid angle spanned by S_ν at O. We assume that t_ν^1 is so oriented that its positive direction and $\overrightarrow{OM_\nu}$ make an angle $<\pi/2$, and put $a_\nu = -\omega_\nu/(4\pi)$.

Now, consider the boundary of the chain $\sum_0^\infty a_\nu t_\nu^1$. Obviously, a knot t_μ^0 of G^{2p} not appearing in G^3 has the coefficient zero in $(\sum_0^\infty a_\nu t_\nu^1)^\cdot = \sum_0^\infty a_\nu \dot{t}_\nu^1$. Suppose that t_μ^0 $(\mu=1, 2, \cdots)$ is a knot of G^3 other than the origin denoted by t_0^0. At t_μ^0 there meet six segments t_ν^1 of G^3, and the corresponding six squares S_ν form the surface of a unit cube with centre at t_μ^0. By the mentioned orientation of the segments t_ν^1 and by the fact that the origin lies outside that cube surface, we see that the coefficient of t_ν^0 in $\sum_0^\infty a_\nu \dot{t}_\nu^1$ vanishes. Similarly, the coefficient of t_0^0 in $\sum_0^\infty a_\nu \dot{t}_\nu^1$ is found to be $=1$. Hence, we have

$$(\textstyle\sum_0^\infty a_\nu t_\nu^1)^{\cdot}=t_0^0.$$

Since $\omega_\nu=o(1/\overline{OM}_\nu^2)$, we have $\sum_0^\infty |a_\nu|^2 \leqq \text{const.} \sum_0^\infty (1/\overline{OM}_\nu^4) \leqq \text{const.}$ $\sum'\{1/(m_1^2+m_2^2+m_3^2)^2\}<+\infty$, where the summation $\sum'$ ranges over all lattice points (m_1, m_2, m_3) in E^3 except $(0, 0, 0)$.

Hence, by Lemma 1, we have $\hat{F} \notin O_G$.

2) CONSTRUCTION OF ANNULI A_n^κ.

In order to prove the remaining parts of the theorem, we shall construct on $\hat{F}$ a collection of annuli A_n^κ satisfying the four conditions mentioned in § 5.

For each $j=1,\cdots,2p$, we construct on F a doubly connected strip region D_j containing the curve C_j in its interior, such that any two of D_j, except the p pairs D_{2i-1}, D_{2i}, $i=1,\cdots,p$ have no points in common, and that, for each i, $D_{2i-1} \cap D_{2i}$ is a simply connected region. Further, we assume that, by the cutting of F into Φ, D_j is transformed into two simply connected strip regions D_j^+ and D_j^- on Φ, respectively adjacent to the boundary arcs C_j^+ and C_j^- of Φ. Then, $D_{2i-1}^+ \cap D_{2i}^+$, $D_{2i-1}^+ \cap D_{2i}^-$, $D_{2i-1}^- \cap D_{2i}^+$, $D_{2i-1}^- \cap D_{2i}^-$ are simply connected subregions of Φ respectively having a vertex of Φ on the boundary.

Let $\{\hat{F}_n\}$ be the exhaustion of $\hat{F}$ defined in § 3, and Γ_n^κ $(\kappa=1,\cdots, k(n))$ be a connected component of the boundary Γ_n of $\hat{F}_n$. Let A_n^κ be the sum of the replicas of D_j^+, D_j^-, $D_{2i-1}^+ \cap D_{2i}^+$, $D_{2i-1}^+ \cap D_{2i}^-$, $D_{2i-1}^- \cap D_{2i}^+$, and $D_{2i-1}^- \cap D_{2i}^-$ on $\hat{F}_n$, which adjoin to the arcs of Γ_n^κ or have some vertex of Γ_n^κ on the boundary. A_n^κ is a doubly connected subregion of $\hat{F}_n$. Since $\Gamma_n=\bigcup_{\kappa=1}^{k(n)}\Gamma_n^\kappa$ is the whole boundary of $\hat{F}_n$, and since $\hat{F}_n \cup \Gamma_n$ is contained in the interior of $\hat{F}_{n+1}$, the annuli A_n^κ, $\kappa=1,\cdots,k(n)$, $n=n_0$, $n_0+1,\cdots$, satisfy the mentioned conditions.

Now, A_n^κ is composed of L_n^κ replicas of D_j^+ and D_j^-, and at most the same number of replicas of $D_{2i-1}^+ \cap D_{2i}^+$, $D_{2i-1}^+ \cap D_{2i}^-$, $D_{2i-1}^- \cap D_{2i}^+$, or $D_{2i-1}^- \cap D_{2i}^-$, where L_n^κ is the "length" of Γ_n^κ. Hence, it is easily seen that the harmonic modulus μ_n^κ of A_n^κ satisfies an inequality

$$\mu_n^\kappa \geqq c/L_n^\kappa \tag{13}$$

with a positive constant c independent of κ and n. Hence, we have for the harmonic modulus σ_n of $\bigcup_{\kappa=1}^{k(n)} A_n^{\kappa}$,

(14) $$\sigma_n \geqq c/L_n\,,$$

where $L_n=\sum_{\kappa=1}^{k(n)} L_n^{\kappa}$ is the total "length" of Γ_n.

3) PROOF OF i): SUFFICIENCY OF $r\leqq 2$.

If $r\leqq 2$, we have, by (7) in 1) of § 4 and by (14), $\sigma_n \geqq \text{const.}(1/n) > 0$. Hence

$$\sum_n^{\infty} \sigma_n = +\infty\,,$$

so that, by Lemma 2, $\tilde{F}\in O_G$.

4) PROOF OF ii).

By the results of 2) of § 4, we have $\mathrm{Max}_{\kappa} L_n^{\kappa}=O(n)$. Hence, by (13), $\mu_n=\mathrm{Min}_{\kappa}\, \mu_n^{\kappa} \geqq \text{const.}(1/n) > 0$. Consequently

$$\sum_n^{\infty} \mu_n = +\infty\,,$$

so that, by Lemma 3, $\tilde{F}\in O_{AD}$.

5) PROOF OF iii).

Suppose that (11) holds for each $i=1,\cdots,p$. Then, the p lattice points $(\gamma_1, \gamma_2, 0,\cdots,0),\cdots, (0,\cdots,0,\gamma_{2i-1},\gamma_{2i},0,\cdots,0),\cdots, (0,\cdots,0,\gamma_{2p-1},\gamma_{2p})$ in E^{2p} are equivalents of $(0,\cdots,0)$ with respect to the group $\mathfrak{T}(\tilde{F})$, so that, on any (m_{2i-1}, m_{2i})-plane through any lattice point in E^{2p}, there exist pairs of lattice points equivalent to each other. Hence, among the three cases distinguished in 2) of § 4, the case (a) cannot occur. Consequently, we have $L_n^{\kappa} \leqq \text{const.}$ for any n and κ, whence, by (13), $\mu_n=\mathrm{Min}_{\kappa}\, \mu_n^{\kappa} \geqq \text{const.} > 0$. On the other hand, since $L_n=O(n^{r-1})$, we have $k(n)=O(n^{r-1})$, so that $K(N)=O(N^{r-1})$. Hence,

$$\sum_n^{N} \mu_n - \frac{1}{2}\log K(N) \geqq \text{const.}\, N - O(\log N).$$

The right-hand side tends to $+\infty$ with N, whence, by Lemma 4, $\tilde{F}\in O_{AB}$ follows.

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