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SEIDEL ELEMENTS AND POTENTIAL FUNCTIONS OF HOLOMORPHIC DISC COUNTING

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Abstract. Let M be a symplectic manifold equipped with a Hamiltonian circle action and let L be an invariant Lagrangian submanifold of M. We study the problem of counting holomorphic *disc sections* of the trivial M-bundle over a disc with boundary in L through degeneration. We obtain a conjectural relationship between the potential function of L and the Seidel element associated to the circle action. When applied to a Lagrangian torus fibre of a semi-positive toric manifold, this degeneration argument reproduces a conjecture (now a theorem) of Chan-Lau-Leung-Tseng [8, 9] relating certain correction terms appearing in the Seidel elements with the potential function.

1. Introduction. Let M be a symplectic manifold with a Hamiltonian circle action. Seidel [27] constructed an invertible element of the quantum cohomology of M by counting pseudo-holomorphic sections of the associated M-bundle E over S^2:

$$E = (M \times S^3)/S^1$$

where S^1 acts by the diagonal action and $S^3 \to S^2$ is the Hopf fibration. Seidel elements have been used to detect essential loops in the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. McDuff-Tolman [26] used them to verify Batyrev's presentation of quantum cohomology rings for toric varieties.

In a previous paper [21], we computed Seidel elements of semi-positive toric manifolds and found that they are closely related to Givental's mirror transformation [19]. Chan-Lau-Leung-Tseng [8] conjectured that certain correction terms appearing in our computation of Seidel elements determine the potential function of a Lagrantian torus fibre. The potential function here is given by counting holomorphic discs with boundary in a Lagrangian torus fibre and is thought of as a mirror of the toric variety. The conjecture was proved by themselves [9] in a recent preprint. In this paper, we propose an alternative approach which relates Seidel elements and potential functions via *degeneration*. Our method should apply to a general symplectic manifold M with a Hamiltonian S^1-action and an invariant Lagrangian.

We assume that M is a smooth projective variety, equipped with a $\mathbb{C}^\times$-action and an S^1-invariant Kähler form ω. Let L be an S^1-invariant Lagrangian submanifold of M. Let $\mathcal{M}_1(\beta)$ denote the moduli space of genus-zero bordered stable holomorphic maps from $(\Sigma, \partial\Sigma)$ to

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(M, L) with one boundary marking and representing $\beta \in H_2(M, L)$. By the fundamental work of Fukaya-Oh-Ohta-Ono [13, 16], $\mathcal{M}_1(\beta)$ is compact and carries a Kuranishi structure with boundary and corner. Let β be a class of Maslov index two. Under certain assumptions (see §2.1), the virtual fundamental *chain* of $\mathcal{M}_1(\beta)$ is a *cycle* of dimension $\dim_{\mathbb{R}} L$ and one can define the *open Gromov-Witten invariant* $n_\beta \in \mathbb{Q}$ by

$$\mathrm{ev}_*[\mathcal{M}_1(\beta)]^{\mathrm{vir}} = n_\beta [L]$$

where $\mathrm{ev}\colon \mathcal{M}_1(\beta) \to L$ is the evaluation map. The potential function W is

$$W = \sum_{\beta \in H_2(M,L):\mu(\beta)=2} n_\beta z^\beta .$$

The idea of degeneration is that instead of counting discs in (M, L), we consider the problem of counting *disc sections* of the trivial bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ to the union $E \cup_M (M \times \mathbb{D})$. From this geometry we expect the following *degeneration formula* (see §3.3 for details):

$$\varphi_* \mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}} \tag{1}$$

if both-hand sides carry virtual fundamental *cycles*, instead of *chains*. Here $\hat{\beta} \in H_2(M \times \mathbb{D}, L \times S^1)$ denotes a disc section class corresponding to $\beta \in H_2(M, L)$ and $\mathcal{M}_1(\hat{\beta})$ is the corresponding moduli space of disc sections. The summation in the right-hand side is taken over all possible decompositions $\sigma + \hat{\alpha}$ of the class $\hat{\beta}$ into a section class σ of E and another disc section class $\hat{\alpha}$ under the degeneration. Also $\mathcal{M}_{\mathrm{S}}(\sigma)$ is a moduli space of holomorphic sections of E in the class σ, which is relevant to the Seidel element. This formula relates disc counts of different boundary types; the boundary classes $\partial\alpha$ and $\partial\beta$ from the both-hand sides differ exactly by the S^1-action.

The degeneration formula predicts a relationship between the Seidel element of the S^1-action and the potential function W. We need the following conditions in order to extract meaningful information from the formula (1):

(i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.
(ii) The maximal fixed component $F_{\max} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is -1.
(iii) $c_1(M)$ is semi-positive.
(iv) $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma))$ is disjoint from L for all $\sigma \in H_2^{\mathrm{sec}}(E)$ such that $\langle c_1^{\mathrm{vert}}(E), \sigma \rangle = -1$.

THEOREM 1.1 (Corollary 3.21). *Assume that M is simply-connected and L is connected. Assume that the degeneration formula* (1) *holds* (*see Conjecture* 3.17 *for a precise formulation*) *and that the above conditions* (i)–(iv) *are satisfied. Then*

$$z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW \rangle + \widetilde{S}^{(0)}$$

holds in a certain "open" Novikov ring Λ^{op} (*see* §2.1), *where*

- $\alpha_0 \in H_2(M, L)$ *is the maximal disc class defined by rotating a path connecting* L *and* $F_{\max}$ *by the* S^1*-action* (*see* §3.2);
- $dW = \sum_{\mu(\beta)=2} \beta \otimes n_\beta z^\beta$ *is the logarithmic derivative of* W;
- $\widetilde{S} = \widetilde{S}^{(0)} + \widetilde{S}^{(2)}$ *is the Seidel element associated to the* S^1*-action and* $\widetilde{S}^{(i)} \in H^i(M) \otimes \Lambda$ (Λ *is the "closed" Novikov ring in Remark* 2.9);
- $\widehat{S}^{(2)} \in H^2(M, L) \otimes \Lambda$ *is a lift of* $\widetilde{S}^{(2)}$ (*see Definition* 3.19).

In particular,

$$\mathrm{KS}(\widetilde{S}) = [z^{\alpha_0}]$$

holds in a certain Jacobi algebra of W*, where* KS *denotes the Kodaira-Spencer mapping* (*see the end of* §3.3.3).

In the second half of the paper, we apply these to a semi-positive toric manifold X and calculate the potential function of a Lagrangian torus fibre $L \subset X$. In toric case, the potential function can be regarded as a function on the moduli space $\mathfrak{M}_{\mathrm{opcl}}$ of Lagrangian torus fibres L together with complexfied Kähler classes $-\omega + iB$ and lifts $h \in H^2(X, L; U(1))$ of $\exp(iB)$ (see §4.2.1, h defines a $U(1)$-local system on L when $B = 0$). The potential function is of the form:

$$W = w_1 + \cdots + w_m$$

with $w_i = f_i(q) z_i$, where $f_i(q) \in \Lambda$ is the *correction term* defined by

$$f_i(q) = \sum_{d \in H_2(X;\mathbb{Z}):\langle c_1(X), d\rangle = 0} n_{\beta_i + d} q^d .$$

Each term w_i corresponds to a prime toric divisor $D_i \subset X$ and arises from disc counting of fixed boundary type $b_i \in H_1(L)$. Applying the degeneration formula, we get:

THEOREM 1.2 (Theorem 4.13). *Assume that the degeneration formula* (1) (*Conjecture* 3.17) *holds for* (X, L) *equipped with the* $\mathbb{C}^\times$*-action* ρ_j *rotating around the prime toric divisor* D_j (*see* §4.3). *Let* $\widetilde{S}_j \in H^2(X) \otimes \Lambda$ *be the Seidel element* ρ_j *and let* $\widehat{S}_j \in H^2(X, L) \otimes \Lambda$ *be its lift. Then we have*

$$\langle \widehat{S}_j, dw_k \rangle = \delta_{jk} z_j .$$

In particular we have $\mathrm{KS}(\widetilde{S}_j) = [z_j]$.

We observe in Theorem 4.14 that the degeneration formula reproduces the following conjecture (now a theorem) of Chan-Lau-Leung-Tseng [8, 9].

THEOREM 1.3 ([8, Conjecture 4.12], [9, Theorem 1.1]). *Let* $g_0^{(j)}(y)$, $j = 1, \ldots, m$ *be explicit hypergeometric functions in variables* $y_1, \ldots, y_r$ ($r = \dim H^2(X)$) *given in equation* (38). *Then we have*

$$f_j(q) = \exp\left(g_0^{(j)}(y)\right)$$

under an explicit change of variables (*mirror transformation*) *of the form* $\log q_i = \log y_i + g_i(y)$, $i = 1, \ldots, r$ *with* $g_i(y) \in \mathbb{Q}[[y_1, \ldots, y_r]]$ *and* $g_i(0) = 0$.

In [21], we introduced *Batyrev elements* $\widetilde{D}_j$ as mirror analogues of the divisor classes D_j. They satisfy the relations of Batyrev's quantum ring [5] for toric varieties. The hypergeometric functions $g_0^{(j)}(y)$ originally appeared in our computation [21] as the difference between the Seidel and the Batyrev elements:

$$\widetilde{D}_j = \exp\left(g_0^{(j)}(y)\right)\widetilde{S}_j\,.$$

Hence by Theorem 1.2, $\widetilde{S}_j$ and $\widetilde{D}_j$ correspond respectively to $[z_j]$ and $[w_j]$ under the Kodaira-Spencer mapping (see also [9, Theorem 1.5]).

Finally we discuss briefly the method of Chan-Lau-Leung-Tseng [9]. Their approach is different from ours but is closely related. They observed that a holomorphic disc in (X, L) whose boundary class is $b_j \in H_1(L)$ can be completed to a holomorphic *sphere* in the M-bundle E'_j associated to the inverse $\mathbb{C}^\times$-action ρ_j^{-1}. Using this, they identified open Gromov-Witten invariants of (X, L) with certain closed invariants of E'_j. The associated bundle E' of the inverse action also appears in our story as the central fibre $E \cup_M E'$ of the degeneration of the closed manifold $M \times \mathbb{P}^1$ (instead of $M \times \mathbb{D}$) in §3.1.

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2. Preliminaries. In this section, we review a potential function of a Lagrangian submanifold and a Seidel element associated to a Hamiltonian circle action.

2.1. Potential function of a Lagrangian submanifold. The potential of a Lagrangian submanifold arises as the 0-th operation $\mathfrak{m}_0$ of the corresponding A_∞-algebra in Lagrangian Floer theory of Fukaya-Oh-Ohta-Ono [13]. In this paper, we do not use the full generality of A_∞-formalism developed in [13]; instead we consider potential functions under certain restrictive assumptions.

Let (M, ω) be a closed symplectic manifold and L be a Lagrangian submanifold. For simplicity, we restrict ourselves to the case where M is a smooth projective variety. We assume that L is oriented, relatively-spin and fix a relative spin structure [13, Definition 8.1.2] of L so that the moduli space of bordered stable maps to (M, L) has an oriented Kuranishi structure. Let $\mu\colon H_2(M, L) \to \mathbb{Z}$ denote the Maslov index. It takes values in $2\mathbb{Z}$ since L is oriented.

Let $\mathcal{M}_1(\beta)$ denote the moduli space of stable holomorphic maps from a genus-zero bordered Riemann surface $(\Sigma, \partial\Sigma)$ to (M, L) with one boundary marked point and in the class $\beta \in H_2(M, L)$. This was denoted by $\mathcal{M}_1^{\mathrm{main}}(\beta)$ in [13]. By [13, Proposition 7.1.1] (see also [16, Theorem 15.3]), $\mathcal{M}_1(\beta)$ is compact and equipped with an oriented Kuranishi structure (with boundary and corner) and has virtual dimension $n + \mu(\beta) - 2$, where $n = \dim_{\mathbb{R}} L$. Let $\mathrm{ev}\colon \mathcal{M}_1(\beta) \to L$ denote the evaluation map. Define an open version of Novikov ring Λ^{op} to be the space of all formal power series

$$\sum_{\beta \in H_2(M,L)} c_\beta z^\beta$$

with $c_\beta \in \mathbb{Q}$ such that

$$\sharp \left\{ \beta \,:\, c_\beta \neq 0, \ \int_\beta \omega < E \right\} < \infty$$

holds for all $E \in \mathbb{R}$.

DEFINITION 2.1. Assume that $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \le 0$. Then $\mathcal{M}_1(\beta)$ with $\mu(\beta) = 2$ has no boundary and carries a virtual fundamental *cycle* of dimension $n = \dim_{\mathbb{R}} L$ [13, Lemma A.1.32]. We define *open Gromov-Witten invariants* $n_\beta \in \mathbb{Q}$ by

$$\mathrm{ev}_*[\mathcal{M}_1(\beta)]^{\mathrm{vir}} = n_\beta [L]$$

for β with $\mu(\beta) = 2$, where $[L] \in H_n(L)$ is the fundamental class of L. The *potential function* of L is defined to be the formal sum:

$$W = \sum_{\beta \in H_2(M,L): \mu(\beta)=2} n_\beta z^\beta \,.$$

This is an element of Λ^{op}.

We can decompose W according to boundary classes of discs.

DEFINITION 2.2. Under the same assumption as in Definition 2.1, we write

$$W = \sum_{\gamma \in H_1(L)} W_\gamma$$

with $W_\gamma \in \Lambda^{\mathrm{op}}$ given by

$$W_\gamma := \sum_{\beta \in H_2(M,L):\, \mu(\beta)=2, \partial\beta=\gamma} n_\beta z^\beta \,.$$

REMARK 2.3. The potential function does depend on the choice of a complex structure on M and this is a reason why we restricted to a smooth projective variety M. For example, the Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$ together with their Lagrangian torus fibres are symplectomorphic to each other, but the potential functions are different. See Auroux [3] for wall-crossing of disc counting.

2.2. Seidel elements. Seidel element is an invertible element of quantum cohomology associated to a loop in the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of a symplectic manifold (M, ω). In this paper we restrict to the case where M is a smooth projective variety equipped with an algebraic $\mathbb{C}^\times$-action. In this case, the associated S^1-action is Hamiltonian and yields a loop in $\mathrm{Ham}(M, \omega)$. We refer the reader to [27, 24, 25] for the original definitions and to [26, 20] for applications in symplectic topology.

Let M be a smooth projective variety, equipped with a $\mathbb{C}^\times$-action.

DEFINITION 2.4. The *associated bundle* of the $\mathbb{C}^\times$-action on M is the M-bundle over $\mathbb{P}^1$

$$E := M \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \to \mathbb{P}^1 ,$$

where $\mathbb{C}^\times$ acts with the diagonal action $\lambda \cdot (x, (z_1, z_2)) = (\lambda x, (\lambda z_1, \lambda z_2))$.

REMARK 2.5. In symplectic geometric terms, the associated bundle is in fact a clutched bundle obtained by gluing two trivial M-bundles over the unit disc, along the boundary, using the action. More precisely,

$$E = (M \times \mathbb{D}_0) \cup_g (M \times \mathbb{D}_\infty)$$

where $\mathbb{D}_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$ and the gluing map $g\colon M \times \partial\mathbb{D}_0 \to M \times \partial\mathbb{D}_\infty$ is given by

$$g(x, e^{i\theta}) = (e^{-i\theta} \cdot x, e^{i\theta}) .$$

This construction can be generalized to a loop in the group of Hamiltonian diffeomorphims and yields a Hamiltonian bundle $E \to \mathbb{P}^1$ in general. One can equip a symplectic form ω_E on the total space E of the Hamiltonian bundle such that ω_E restricts to the symplectic form ω_M on each fibre [27].

By Atiyah's theorem [1], there exists a unique $\mathbb{C}^\times$-fixed component $F_{\max} \subset M^{\mathbb{C}^\times}$ such that the normal bundle of $F_{\max}$ has only negative $\mathbb{C}^\times$-weights. For a Hamiltonian function H generating the S^1-action, $F_{\max}$ is the locus where H takes the maximum value. Each fixed point $x \in M^{\mathbb{C}^\times}$ defines a section σ_x of E. We denote by σ_0 the section associated to a fixed point in $F_{\max}$. We call it a *maximal section*. This defines a splitting[1]

$$H_2(E; \mathbb{Z}) \cong \mathbb{Z}[\sigma_0] \oplus H_2(M; \mathbb{Z}) . \tag{2}$$

Let $\mathrm{NE}(M) \subset H_2(M, \mathbb{R})$ denote the Mori cone, that is the cone generated by effective curves and set $\mathrm{NE}(M)_\mathbb{Z} := \{d \in H_2(M; \mathbb{Z}) : d \in \mathrm{NE}(M)\}$. We introduce $\mathrm{NE}(E)$ and $\mathrm{NE}(E)_\mathbb{Z}$ similarly.

LEMMA 2.6 ([21, Lemma 2.2]). $\mathrm{NE}(E)_\mathbb{Z} = \mathbb{Z}_{\geq 0}[\sigma_0] + \mathrm{NE}(M)_\mathbb{Z}$.

Let $H_2^{\sec}(E; \mathbb{Z})$ denote the affine subspace of $H_2(E; \mathbb{Z})$ which consists of section classes, i.e. the classes that project to the positive generator of $H_2(\mathbb{P}^1; \mathbb{Z})$. We set $\mathrm{NE}(E)_\mathbb{Z}^{\sec} := \mathrm{NE}(E)_\mathbb{Z} \cap H_2^{\sec}(E; \mathbb{Z})$. The above lemma shows that

$$\mathrm{NE}(E)_\mathbb{Z}^{\sec} = [\sigma_0] + \mathrm{NE}(M)_\mathbb{Z} . \tag{3}$$

For $d \in \mathrm{NE}(X)_\mathbb{Z}$ and $\sigma \in \mathrm{NE}(E)_\mathbb{Z}$, we denote by q^d and q^σ the corresponding elements in the group ring $\mathbb{Q}[\mathrm{NE}(X)_\mathbb{Z}]$ and $\mathbb{Q}[\mathrm{NE}(E)_\mathbb{Z}]$ respectively. We write:

$$q^\sigma = q_0^k q^d \quad \text{when} \quad \sigma = k[\sigma_0] + d$$

[1] The section σ_0 gives a splitting of the Serre spectral sequence. In general one has a non-canonical splitting $H^*(E; \mathbb{Q}) \cong H^*(M; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q})$ for any Hamiltonian bundle $E \to \mathbb{P}^1$ [25].

where $q_0 = q^{\sigma_0}$ is the variable corresponding to the maximal section σ_0. For $\sigma \in \mathrm{NE}(E)^{\mathrm{sec}}_{\mathbb{Z}}$, let $\mathcal{M}_{\mathrm{S}}(\sigma)$ denote the moduli space of stable maps from genus-zero closed nodal Riemann surfaces to E in the class σ with one marked point whose image lies in a fixed fibre $\iota\colon M \hookrightarrow E$. We can write

$$\mathcal{M}_{\mathrm{S}}(\sigma) = \mathcal{M}_1(\sigma) \times_E M$$

using the usual moduli space $\mathcal{M}_1(\sigma)$ of genus-zero one-pointed stable maps to E in the class σ. Since $\mathcal{M}_1(\sigma)$ has a Kuranishi structure (without boundary) of virtual real dimension $2n + 2\langle c_1(E), \sigma\rangle - 2$ (with $n := \dim_{\mathbb{C}} M$) and we may assume that the evaluation map $\mathcal{M}_1(\sigma) \to E$ is weakly submersive (see [18, Theorem 7.11]), the fibre product $\mathcal{M}_{\mathrm{S}}(\sigma)$ is equipped with the induced Kuranishi structure of virtual dimension:

$$\text{vir.dim}_{\mathbb{R}} \mathcal{M}_{\mathrm{S}}(\sigma) = 2n + 2\left\langle c_1^{\mathrm{vert}}(E), \sigma\right\rangle. \tag{4}$$

Here $c_1^{\mathrm{vert}}(E)$ denotes the first Chern class of the vertical tangent bundle $T_{\mathrm{vert}}E$,

$$T_{\mathrm{vert}}E := \mathrm{Ker}(d\pi\colon TE \to \pi^* T\mathbb{P}^1)$$

with $\pi\colon E \to \mathbb{P}^1$ the natural projection. (Note that $\langle c_1(E), \sigma\rangle = \left\langle c_1^{\mathrm{vert}}(E), \sigma\right\rangle + 2$.) Let $\mathrm{ev}\colon \mathcal{M}_{\mathrm{S}}(\sigma) \to M$ be the evaluation map and let $[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}$ be the virtual fundamental cycle of $\mathcal{M}_{\mathrm{S}}(\sigma)$.

DEFINITION 2.7. The *Seidel element* associated to the $\mathbb{C}^\times$-action on M is the class

$$S := \sum_{\sigma \in \mathrm{NE}(E)^{\mathrm{sec}}_{\mathbb{Z}}} \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_S(\sigma)]^{\mathrm{vir}}\right) q^\sigma \tag{5}$$

in $H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(E)_{\mathbb{Z}}]\!]$. Here PD stands for the Poincaré duality isomorphism. By (3), we can factorize S as $S = q_0\widetilde{S}$ with $\widetilde{S}$ in the small quantum cohomology ring

$$QH(M) := H(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$$

and $q_0 := q^{\sigma_0}$ as above. Then $\widetilde{S}$ is an invertible element of $QH(M)[q^{-d} : d \in \mathrm{NE}(M)_{\mathbb{Z}}]$: the Seidel element $\widetilde{S}'$ associated with the reverse $\mathbb{C}^\times$-action satisfies $\widetilde{S}' \star \widetilde{S} = q^{d_0}$ for some $d_0 \in H_2(M,\mathbb{Z})$ [27, 24, 25].

REMARK 2.8. Using genus zero one-point Gromov-Witten invariants for E, we can write

$$S = \sum_{\sigma \in \mathrm{NE}(E)^{\mathrm{sec}}_{\mathbb{Z}}} \sum_i \langle \iota_*\phi_i\rangle^E_{0,1,\sigma}\, \phi^i q^\sigma$$

where $\{\phi_i\}$ is a basis of $H^*(M;\mathbb{Q})$, $\{\phi^i\}$ is the dual basis with respect to the Poincaré pairing and $\iota\colon M \to E$ is the inclusion of a fibre. (We followed the standard notation of Gromov-Witten invariants as in [11].)

REMARK 2.9. For a general symplectic manifold M, we use the Novikov ring Λ

$$\Lambda := \left\{ \sum_{d \in H_2(M;\mathbb{Z})} c_d q^d \; : \; c_d \in \mathbb{Q}, \; \sharp\{d \; : \; c_d \neq 0, \; \langle \omega, d \rangle \leq E\} < \infty \text{ for all } E \in \mathbb{R} \right\}$$

instead of $\mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$. The Seidel elements associated to loops in $\mathrm{Ham}(M,\omega)$ define a group homomorphism [27, 24, 25]:

$$\pi_1(\mathrm{Ham}(M,\omega)) \to QH(M)_\Lambda^\times \big/ \left\{ q^d \; : \; d \in H_2(M;\mathbb{Z}) \right\}$$

which is called the *Seidel representation*, where $QH(M)_\Lambda = H^*(M;\mathbb{Q}) \otimes \Lambda$ denotes the quantum cohomology ring over Λ.

3. Degeneration Formula. Let M be a smooth projective variety equipped with a $\mathbb{C}^\times$-action. We take an S^1-invariant Kähler form ω on M. Let L be a Lagrangian submanifold of M which is preserved by $S^1 \subset \mathbb{C}^\times$, i.e. $\lambda L \subset L$ for $\lambda \in S^1$. Instead of counting holomorphic discs in (M, L), we shall consider the problem of counting holomorphic *disc sections* of the bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ into the union of the associated bundle E and $M \times \mathbb{D}$. From this we expect a certain relationship between Seidel elements and disc counting invariants. We assume that M is a smooth projective variety with a $\mathbb{C}^\times$-action for simplicity, but the degeneration formula in this section makes sense for a symplectic manifold with a Hamiltonian circle action (or a loop in the group of Hamiltonian diffeomorphisms) in general.

3.1. Degeneration of $M \times \mathbb{D}$. Let $\mathbb{D}$ denote the unit disc $\{z \in \mathbb{C} \; : \; |z| \leq 1\}$. A degeneration of the disc $\mathbb{D}$ into the union $\mathbb{D} \cup \mathbb{P}^1$ is given by the blowup $\mathrm{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C})$ of $\mathbb{D} \times \mathbb{C}$ at the origin. The projection $\pi \colon \mathrm{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C}) \to \mathbb{C}$ satisfies $\pi^{-1}(t) \cong \mathbb{D}$ for $t \neq 0$ and $\pi^{-1}(0) \cong \mathbb{D} \cup \mathbb{P}^1$. Explicitly:

$$\mathrm{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C}) = \left\{ (z, t, [\alpha, \beta]) \in \mathbb{D} \times \mathbb{C} \times \mathbb{P}^1 \; : \; z\beta - t\alpha = 0 \right\}.$$

An M-bundle $\mathcal{E}$ over $\mathrm{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C})$ is defined as follows.

$$\mathcal{E} := \left\{ (x, z, t, (\alpha, \beta)) \in M \times \mathbb{D} \times \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\}) \; : \; z\beta - t\alpha = 0 \right\} \big/ \mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts as $(x, z, t, (\alpha, \beta)) \mapsto (\lambda x, z, t, (\lambda\alpha, \lambda\beta))$. We have a natural projection $\pi : \mathcal{E} \to \mathbb{C}$. One can see that

$$\mathcal{E}_t = \pi^{-1}(t) = \begin{cases} M \times \mathbb{D} & \text{if } t \neq 0; \\ E \cup_M (M \times \mathbb{D}) & \text{if } t = 0 \end{cases} \tag{6}$$

where E is the associated bundle (Definition 2.4) of the $\mathbb{C}^\times$-action on M. One can also construct $\mathcal{E}$ as a symplectic quotient:

$$\mathcal{E} = \left\{ (x, z, t, (\alpha, \beta)) \; : \; z\beta - t\alpha = 0, \; H(x) + |\alpha|^2 + |\beta|^2 = c \right\} \big/ S^1$$

where $H \colon M \to \mathbb{R}$ is the moment map of the S^1-action and $c > \max_{x \in M} H(x)$ is a real number. We can equip $\mathcal{E}$ with a symplectic structure. The boundary $\partial\mathcal{E}_t$ can be identified with

$M \times S^1$ via the map:

$$M \times S^1 \ni (x, z) \mapsto [x, z, t, (z, t)] \in \partial \mathcal{E}_t \, . \tag{7}$$

Via this identification, $\mathcal{E}_t$ contains a Lagrangian submanifold $\widehat{L_t} := L \times S^1$ in the boundary $M \times S^1 \cong \partial \mathcal{E}_t$.

We can close $\mathcal{E}_t$ by attaching $M \times \mathbb{D}$ to the boundary for each t and get a degenerating family $\overline{\mathcal{E}}$ of closed manifolds. More explicitly, we define:

$$\overline{\mathcal{E}} = \left\{(x, (z, w), t, (\alpha, \beta)) \in M \times (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\}) \;:\; t\alpha w = z\beta\right\} \big/ \mathbb{C}^\times \times \mathbb{C}^\times$$

where $\mathbb{C}^\times \times \mathbb{C}^\times$ acts as

$$(x, (z, w), t, (\alpha, \beta)) \mapsto (\lambda_1^{-1}\lambda_2 x, (\lambda_1 z, \lambda_1 w), t, (\lambda_2 \alpha, \lambda_2 \beta)) \, .$$

This is an M-bundle over

$$\mathrm{Bl}_{(0,0)}(\mathbb{P}^1 \times \mathbb{C}) = \left\{([z, w], t, [\alpha, \beta]) \in \mathbb{P}^1 \times \mathbb{C} \times \mathbb{P}^1 \;:\; t\alpha w = z\beta\right\} .$$

With respect to the projection $\pi : \overline{\mathcal{E}} \to \mathbb{C}$ to the t-plane, we have

$$\overline{\mathcal{E}}_t = \pi^{-1}(t) = \begin{cases} M \times \mathbb{P}^1 & \text{if } t \neq 0 \, ; \\ E \cup_M E' & \text{if } t = 0 \end{cases}$$

where E' is the associated bundle of the $\mathbb{C}^\times$-action on M *inverse* to the original one. Note that $\mathcal{E}$ is contained in $\overline{\mathcal{E}}$ as the locus $\{w = 1, \; |z| \leq 1\}$ and $\overline{\mathcal{E}} = \mathcal{E} \cup_{M \times S^1 \times \mathbb{C}} (M \times \mathbb{D}^2 \times \mathbb{C})$. We can also equip $\overline{\mathcal{E}}$ with a symplectic structure by describing it as a symplectic quotient in a similar manner.

A topological description is given as follows. We start from a trivial M-bundle $M \times \mathbb{P}^1$ over $\mathbb{P}^1$. We cut $\mathbb{P}^1$ into 3 pieces: $\mathbb{P}^1 = \mathbb{D}_0 \cup A \cup \mathbb{D}_\infty$, where $\mathbb{D}_0 = \{|z| \leq 1/2\}$, $A = \{1/2 \leq |z| \leq 2\}$ and $\mathbb{D}_\infty = \{|z| \geq 2\} \cup \{\infty\}$. One can twist the clutching function along $\partial \mathbb{D}_0$ and $\partial \mathbb{D}_\infty$ by the given S^1-action on M; namely

$$M \times \mathbb{P}^1 = (M \times \mathbb{D}_0) \cup_{g_1} (M \times A) \cup_{g_2} (M \times \mathbb{D}_\infty) \tag{8}$$

where the clutching functions g_1, g_2 are given respectively by

$$\begin{aligned} g_1 &: M \times \partial \mathbb{D}_0 \ni (x, \tfrac{1}{2} e^{i\theta}) \longmapsto (e^{-i\theta} x, \tfrac{1}{2} e^{i\theta}) \in M \times \partial_0 A \\ g_2 &: M \times \partial_\infty A \ni (x, 2e^{i\theta}) \longmapsto (e^{i\theta} x, 2e^{i\theta}) \in M \times \partial \mathbb{D}_\infty \end{aligned}$$

where we set $\partial A = \partial_0 A \cup \partial_\infty A$. Collapsing $M \times S^1 \subset M \times A$ down to M, we get the singular central fibre $E \cup_M E'$. In fact, for $|t| < 1$, one can decompose $\overline{\mathcal{E}}_t$ as

$$\begin{aligned} \overline{\mathcal{E}}_t = &\left\{[x, (tz, 1), t, (z, 1)] \;:\; |z| \leq 1\right\} \\ &\cup \left\{[x, (z, 1), t, (1, \beta)] \;:\; t = \beta z, \; |\beta| \leq 1, \; |z| \leq 1\right\} \\ &\cup \left\{[x, (1, w), t, (1, wt)] \;:\; |w| \leq 1\right\} . \end{aligned}$$

This corresponds to the decomposition (8) of $M \times \mathbb{P}^1$ above.

REMARK 3.1. We shall consider stable holomorphic discs in $(\overline{\mathcal{E}}_t, \widehat{L}_t)$ which project onto the holomorphic disc $(\mathbb{D}^2, S^1) \subset (\mathbb{P}^1, S^1)$. Such stable holomorphic discs are entirely contained in the half-space $\mathcal{E}_t$ of $\overline{\mathcal{E}}_t$, so the choice of "closing" of $\mathcal{E}_t$ is not relevant.

REMARK 3.2. We can perform a similar construction for a general symplectic manifold (M, ω) equipped with a Lagrangian submanifold L and a loop $\{\phi_\theta\}_{\theta\in[0,2\pi]}$ in the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms such that $\phi_\theta(L) = L$ for all θ. We can twist the clutching function of the trivial M-bundle $M \times \mathbb{P}^1$ as in (8) where g_1, g_2 there are replaced with

$$g_1(x, \tfrac{1}{2}e^{i\theta}) = (\phi_{-\theta}(x), \tfrac{1}{2}e^{i\theta}), \quad g_2(x, 2e^{i\theta}) = (\phi_\theta(x), 2e^{i\theta}).$$

Then we can degenerate the annulus A into the union of two discs (in a one-parameter family) in the middle part $M \times A$. In the degeneration family, we have a family of Lagrangian submanifolds $L \times S^1$ lying in the boundary of $M \times \mathbb{D}_0 \cup_{g_1} M \times A$.

3.2. Relative homology classes of degenerating discs. We write $\mathcal{L} = \bigcup_{t\in\mathbb{C}} \widehat{L}_t$. The total space $(\mathcal{E}, \mathcal{L})$ of the family has a deformation retraction to the central fibre $(\mathcal{E}_0, \widehat{L}_0)$. This gives a retraction map for $t \neq 0$:

$$r\colon H_2(\mathcal{E}_t, \widehat{L}_t) \longrightarrow H_2(\mathcal{E}, \mathcal{L}) \cong H_2(\mathcal{E}_0, \widehat{L}_0).$$

Let $\pi : \mathcal{E} \to \mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})$ denote the natural projection. We have the following commutative diagram:

$$\begin{array}{ccc} H_2(\mathcal{E}_t, \widehat{L}_t) & \xrightarrow{\ r\ } & H_2(\mathcal{E}_0, \widehat{L}_0) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ H_2(\mathbb{D}, S^1) & \xrightarrow{\ r\ } & H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1). \end{array}$$

Under the natural identifications $H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1; \mathbb{Z}) \cong H_2(\mathbb{P}^1; \mathbb{Z}) \oplus H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, the bottom arrow is given by $n \mapsto (n, n)$. We are interested in *section classes* lying in the following groups:

$$H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) = \pi_*^{-1}(1), \quad \text{for } t \neq 0, \quad \text{and} \quad H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) = \pi_*^{-1}(1, 1).$$

There is an induced retraction map $r\colon H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ for $t \neq 0$.

LEMMA 3.3. *Assume that M is simply connected and L is connected. Then we have*

$$\begin{aligned} H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) &\cong H_2(M, L) \quad \textit{for } t \neq 0 \\ H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) &\cong H_2^{\mathrm{sec}}(E) \times_{H_2(M)} H_2(M, L). \end{aligned} \tag{9}$$

PROOF. Recall that $(\mathcal{E}_t, \widehat{L}_t) \cong (M \times \mathbb{D}, L \times S^1)$ for $t \neq 0$. We show the isomorphism:

$$(p_{1*}, p_{2*})\colon H_2(M \times \mathbb{D}, L \times S^1) \cong H_2(M, L) \times H_2(\mathbb{D}, S^1)$$

where p_1, p_2 are natural projections. Because we have sections $i_1, i_2\colon (M, L) \to (M \times \mathbb{D}, L \times S^1)$ such that $p_1 \circ i_1 = \mathrm{id}$, $p_2 \circ i_2 = \mathrm{id}$, $p_2 \circ i_1 = \mathrm{const}$ and $p_1 \circ i_2 = \mathrm{const}$, the map

(p_{1*}, p_{2*}) is surjective. To show that it is injective, we use the commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H_2(\mathbb{D}, S^1) & \longrightarrow & H_1(S^1) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle p_{2*}} & & \uparrow \\
H_2(L \times S^1) & \longrightarrow & H_2(M \times \mathbb{D}) & \longrightarrow & H_2(M \times \mathbb{D}, L \times S^1) & \longrightarrow & H_1(L \times S^1) \\
{\scriptstyle \mathrm{epi}}\downarrow & & {\scriptstyle \cong}\downarrow & & {\scriptstyle p_{1*}}\downarrow & & \downarrow \\
H_2(L) & \longrightarrow & H_2(M) & \longrightarrow & H_2(M, L) & \longrightarrow & H_1(L).
\end{array}$$

Here all the horizontal sequences are exact. The injectivity of (p_{1*}, p_{2*}) follows from the diagram chasing and $H_1(L \times S^1) \cong H_1(L) \oplus H_1(S^1)$ (here we use the condition that L is connected). Then $H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \cong H_2(M, L)$ for $t \neq 0$ follows.

The Mayer-Vietoris exact sequence for $\mathcal{E}_0 = E \cup_M (M \times \mathbb{D})$ gives

$$H_2(M) \longrightarrow H_2(E) \oplus H_2(M \times \mathbb{D}, L \times S^1) \longrightarrow H_2(\mathcal{E}_0, \widehat{L}_0) \longrightarrow 0\,.$$

Here we used $H_1(M) = 0$. The formula for $H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ follows. □

Henceforth we assume that L *is connected and* M *is simply-connected.*

REMARK 3.4. The natural map $H_2(\mathcal{E}_t, \widehat{L}_t) \to H_2(\overline{\mathcal{E}}_t, \widehat{L}_t)$ is injective because the composition:

$$H_2(M \times \mathbb{D}, L \times S^1) \to H_2(M \times \mathbb{P}^1, L \times S^1) \xrightarrow{(p_{1*}, p_{2*})} H_2(M, L) \oplus H_2(\mathbb{P}^1, S^1)$$

is injective.

NOTATION 3.5. We denote by

$$\begin{aligned}
\hat{\beta} &\in H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \cong H_2^{\mathrm{sec}}(M \times \mathbb{D}, L \times S^1) \quad (t \neq 0) \\
\sigma + \hat{\beta} &\in H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)
\end{aligned}$$

the homology classes corresponding to $\beta \in H_2(M, L)$ and to $[\sigma, \beta] \in H_2^{\mathrm{sec}}(E) \times_{H_2(M)} H_2(M, L)$ respectively, under the isomorphism (9) in Lemma 3.3.

Let $u \colon \mathbb{D} \to M$ be a map such that $u(e^{i\theta}) = e^{i\theta} \cdot u(1)$, namely, u is a disc contracting an S^1-orbit in M. This defines a section $\sigma(u)$ of the associated bundle $E \to \mathbb{P}^1$:

$$\begin{aligned}
\sigma(u)|_{\mathbb{D}_0} &\colon \mathbb{D}_0 \to E|_{\mathbb{D}_0} \cong M \times \mathbb{D}_0\,, \quad z \mapsto (u(1), z) \\
\sigma(u)|_{\mathbb{D}_\infty} &\colon \mathbb{D}_\infty \to E|_{\mathbb{D}_\infty} \cong M \times \mathbb{D}_\infty\,, \quad z \mapsto (u(z^{-1}), z)
\end{aligned}$$

where $\mathbb{D}_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$; here we used the gluing construction of E in Remark 2.5.

Recall the maximal section class σ_0 of E in §2.2. We introduce a similar *maximal disc class* $\alpha_0 \in H_2(M, L)$ as follows. Take a path $\gamma \colon [0, 1] \to M$ such that $\gamma(0) \in F_{\max}$ and $\gamma(1) \in L$, where $F_{\max}$ is the maximal fixed component. We define α_0 to be the class represented by the disc $\mathbb{D} \ni re^{i\theta} \mapsto e^{-i\theta} \cdot \gamma(r) \in M$. The homotopy class here is independent of the choice of a path γ because M is simply-connected and L is connected. The boundary of α_0 is an inverse S^1-orbit on L.

PROPOSITION 3.6. *The retraction map* $r\colon H_2^{\rm sec}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\rm sec}(\mathcal{E}_0, \widehat{L}_0)$ (*for* $t \neq 0$) *of section classes is an isomorphism. It is given by* (*under Notation* 3.5)

$$r(\hat{\beta}) = \sigma(u) - \hat{u} + \hat{\beta} = \sigma_0 + \hat{\alpha}_0 + \hat{\beta} \quad \textit{for} \ \beta \in H_2(M, L)$$

where $u\colon \mathbb{D} \to M$ *is an arbitrary disc whose boundary is an* S^1*-orbit in* L*,* σ_0 *is the maximal section and* α_0 *is the maximal disc. In particular we have the commutative diagram*

$$\begin{array}{ccc} H_2^{\rm sec}(\mathcal{E}_t, \widehat{L}_t) & \xrightarrow[r]{\cong} & H_2^{\rm sec}(\mathcal{E}_0, \widehat{L}_0) \\ \partial \big\downarrow & & \partial \big\downarrow \\ H_1(L) & \xrightarrow[-\lambda]{\cong} & H_1(L) \end{array}$$

where the bottom map is the subtraction of the class $\lambda = [\partial u]$ *of an* S^1*-orbit on* L*.*

PROOF. Consider a constant section $s_{\rm triv}(z) = (x, z)$ of $M \times \mathbb{D} \cong \mathcal{E}_t$ with $x \in L$. By the topological description of the degeneration given in §3.1, we see that $s_{\rm triv}$ can degenerate to the union:

$$\sigma(u) \cup \tilde{u}\colon \mathbb{P}^1 \cup \mathbb{D} \to E \cup_M (M \times \mathbb{D})$$

where $u\colon \mathbb{D} \to M$ is a disc contracting the S^1-orbit $e^{i\theta}x$ on L and $\tilde{u}\colon \mathbb{D} \to M \times \mathbb{D}$ is given by $z \mapsto (u(\overline{z}), z)$. This shows that $r([s_{\rm triv}]) = \sigma(u) - \hat{u}$. Since the retraction map is a homomorphism of $H_2(M, L)$-modules, we have $r(\hat{\beta}) = \sigma(u) - \hat{u} + \hat{\beta}$ in general. When u is a disc of the form: $\mathbb{D} \ni re^{i\theta} \mapsto e^{i\theta} \cdot \gamma(r) \in M$, where $\gamma\colon [0, 1] \to M$ is a path such that $\gamma(0) \in F_{\max}$ and $\gamma(1) \in L$, $\sigma(u)$ is homotopic to the maximal section σ_0 and $[u] = -\alpha_0$. This shows the formula $r(\hat{\beta}) = \sigma_0 + \hat{\alpha}_0 + \hat{\beta}$. It is easy to check that r is an isomorphism between section classes. □

REMARK 3.7. The latter statement in Proposition 3.6 is a consequence of the difference of trivializations of $\partial\mathcal{E}_t$ ($t \neq 0$) and $\partial\mathcal{E}_0$. Recall that we have a trivialization $\partial\mathcal{E}_t \cong M \times S^1$ in (7) depending smoothly on $t \in \mathbb{C}$. For $t \neq 0$, this trivialization is induced from the isomorphism $\mathcal{E}_t \cong M \times \mathbb{D}$ in (6); however for $t = 0$, this trivialization differs by the S^1-action from the one induced by the isomorphism $\mathcal{E}_0 \cong E \cup_M (M \times \mathbb{D})$ in (6).

LEMMA 3.8 (Maslov index and vertical Chern number). *Let* $u\colon \mathbb{D} \to M$ *be a disc with boundary an* S^1*-orbit on* L*, i.e.* $u(e^{i\theta}) = e^{i\theta} \cdot u(1)$ *and* $u(1) \in L$*. Then* u *defines a class in* $\pi_2(M, L)$ *and we have* $\mu(u) = 2\left\langle c_1^{\rm vert}(E), [\sigma(u)]\right\rangle$*.*

PROOF. We recall the definition of Maslov index of a disc $u\colon (\mathbb{D}, S^1) \to (M, L)$. We set $\gamma = u|_{\partial\mathbb{D}}$. Note that $u^*TM|_{S^1}$ is a complexfication of the subbundle γ^*TL. Thus $\det(u^*TM)|_{S^1}$ is a complexification of the real line bundle $\det_{\mathbb{R}}(\gamma^*TL)$. On the other hand $\det_{\mathbb{R}}(\gamma^*TL)^{\otimes 2}$ has a canonical orientation. Take a positive (nowhere vanishing) section s_0 of

$\det_{\mathbb{R}}(\gamma^* TL)^{\otimes 2}$. The Maslov index of u is the signed count of zeros of a transverse section s of $\det(u^*TM)^{\otimes 2}$ such that $s|_{\partial\mathbb{D}} = s_0$.

When $u|_{\partial\mathbb{D}}$ is an S^1-orbit of L, we can take s_0 above to be S^1-equivariant. A transverse section s of $\det(u^*TM)^{\otimes 2}$ with $s|_{\partial\mathbb{D}} = s_0$ defines a section $t \in \det(\sigma(u)^* T_{\mathrm{vert}} E)^{\otimes 2}$ by

$$t|_{\mathbb{D}_0}(z) = s_0(1)\,, \quad t|_{\mathbb{D}_\infty}(z) = s(z^{-1})\,.$$

Then the numbers of zeros of t and s coincide. The lemma follows. □

Proposition 3.6 and Lemma 3.8 show the following corollary:

COROLLARY 3.9. *Let $r\colon H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ be the retraction map for $t \neq 0$. Suppose that $r(\hat\beta) = \sigma + \hat\alpha$ with $\alpha, \beta \in H_2(M, L)$. Then $\mu(\beta) = 2\langle c_1^{\mathrm{vert}}(E), \sigma\rangle + \mu(\alpha)$.*

REMARK 3.10. We have $\mu(\hat\beta) = \mu(\beta) + 2$ for $\beta \in H_2(M, L)$.

3.2.1. Example. We give an example of degenerating holomorphic discs. Consider a family of (constant) holomorphic disc sections $u_t\colon (\mathbb{D}, S^1) \to (\mathcal{E}_t, \widehat{L}_t)$ given by

$$u_t(z) = [x_0, z, t, (z, t)]$$

for some $x_0 \in L$. For a fixed non-zero $z \in \mathbb{D}$, we have

$$\varphi(z) := \lim_{t\to 0} u_t(z) = [x_0, z, 0, (z, 0)]\,.$$

This can be completed to a holomorphic disc section $\varphi\colon \mathbb{D} \to M \times \mathbb{D} \subset \mathcal{E}_0$. Note that the limit

$$\lim_{z\to 0} \varphi(z) = [x_1, 0, 0, (1, 0)] \quad \text{where} \quad x_1 := \lim_{z\to 0} z^{-1} x_0 \in M\,,$$

exists by the completeness of M, and it is fixed by the $\mathbb{C}^\times$ action. On the other hand, we can see a bubbling off holomorphic sphere at $z = 0$ by the usual rescaling:

$$\psi(z) := \lim_{t\to 0} u_t(tz) = \lim_{t\to 0}[t^{-1}x_0, tz, t, (z, 1)] = [x_1, 0, 0, (z, 1)]\,.$$

This defines a holomorphic section $\psi\colon \mathbb{P}^1 \to E \subset \mathcal{E}_0$ associated to the $\mathbb{C}^\times$-fixed point $x_1 \in M$. Note that $\psi(\infty) = \varphi(0)$ and $\partial\varphi$ is an inverse S^1-orbit on L.

3.3. Degeneration formula. In what follows, we propose a conjectural degeneration formula and discuss its consequences. As before, M denotes a smooth projective variety equipped with a $\mathbb{C}^\times$-action and an S^1-invariant Kähler form ω; L is a Lagrangian submanifold which is preserved by the S^1-action. We assume that M is simply-connected and L is connected. Moreover we assume that L is oriented and relatively spin and we fix a relative spin structure [13, Definition 8.1.2].

Take $\beta \in H_2(M, L)$. We consider the moduli space $\mathcal{M}_1(\hat\beta)$ of stable holomorphic maps from genus zero bordered Riemann surface $(\Sigma, \partial\Sigma)$ to $(\overline{\mathcal{E}}_t, \widehat{L}_t) \cong (M \times \mathbb{P}^1, L \times S^1)$ with one boundary marked point and in the class $\hat\beta \in H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t)$ (where $t \neq 0$; see Notation 3.5). Such stable maps project onto the disc $(\mathbb{D}, S^1) \subset (\mathbb{P}^1, S^1)$ on the base and so are contained in $\mathcal{E}_t$ (see Remarks 3.1 and 3.4). The virtual dimension of $\mathcal{M}_1(\hat\beta)$ is $n + 1 + \mu(\hat\beta) - 2 =$

$n+1+\mu(\beta)$ with $n := \dim_{\mathbb{C}} M$. The corresponding moduli space at $t=0$ should be described as the fibre product:

$$\bigcup_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$$

where $\mathcal{M}_{\mathrm{S}}(\sigma)$ is the moduli space of holomorphic sections of E appearing in §2.2 and $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ is the moduli space of stable holomorphic maps from genus zero bordered Riemann surfaces to $(M \times \mathbb{P}^1, L \times S^1)$ in the class $\hat{\alpha} \in H_2^{\mathrm{sec}}(M \times \mathbb{D}, L \times S^1)$ with one boundary marked point and one interior marked point such that the image of the interior marked point lies in $M \times \{0\}$. The superscript "rel" (which means "relative") signifies the last condition. The fibre product above is taken with respect to the interior evaluation maps. One can write:

$$\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) = \mathcal{M}_{1,1}(\hat{\alpha}) \times_{M \times \mathbb{P}^1} (M \times \{0\}) \tag{10}$$

using the moduli space $\mathcal{M}_{1,1}(\hat{\alpha})$ of bordered stable maps to $(M \times \mathbb{P}^1, L \times S^1)$ of class $\hat{\alpha}$ with one boundary marking and one interior marking. Then a Kuranishi structure on $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ is induced from the Kuranishi structure on $\mathcal{M}_{1,1}(\hat{\alpha})$ (as defined in [13, §7.1]) via this presentation. The virtual dimension is

$$\mathrm{vir.dim}\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) = n + 1 + \mu(\alpha)\,.$$

We write $\mathrm{ev}^{(\mathrm{i})} \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to M$ for the interior evaluation map and $\mathrm{ev}^{(\mathrm{b})} \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to L \times S^1$ for the boundary evaluation map.

When the virtual fundamental *chains* on the moduli spaces $\mathcal{M}_1(\hat{\beta})$ and $\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ happen to be *cycles*, we expect the following degeneration formula:

$$\varphi_* \mathrm{ev}_* [\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathrm{ev}_* [\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}} \tag{11}$$

in $H_*(L \times S^1)$. Here ev on the both-hand sides denotes the evaluation map at the boundary markings taking values in $\widehat{L}_t \cong L \times S^1$ and $\varphi \colon L \times S^1 \to L \times S^1$ is the map $(x, e^{i\theta}) \mapsto (e^{-i\theta} \cdot x, e^{i\theta})$ which corresponds to the difference of boundary trivializations (see Remark 3.7). We will study below when the both-hand sides of (11) make sense as cycles; then will calculate them in terms of Seidel elements and open Gromov-Witten invariants.

3.3.1. The left-hand side of (11). When $\beta = 0$, $\mathcal{M}_1(\hat{\beta})$ consists of constant disc sections and $\mathrm{ev} \colon \mathcal{M}_1(\hat{\beta}) \to L \times S^1$ is a homeomorphism. All constant disc sections are Fredholm regular. When $\beta \neq 0$, we have a natural map

$$\mathcal{M}_1(\hat{\beta}) \to \mathcal{M}_1(\beta)$$

induced by the projection $\mathcal{E}_t \to M$, where $\mathcal{M}_1(\beta)$ is the moduli space of one-pointed bordered stable maps to (M, L) in the class β. By taking the graph of a disc component, we can see that this map is surjective. Therefore, for $\beta \neq 0$, $\mathcal{M}_1(\hat{\beta})$ is non-empty if and only if $\mathcal{M}_1(\beta)$ is non-empty. Moreover, if $\mathcal{M}_1(\beta)$ is non-empty, $\mathcal{M}_1(\hat{\beta})$ has boundary (see [13, §7.1.1] for the boundary description) since a bordered stable map of class $\hat{\beta}$ can be constructed as the union of a constant disc section and a disc of class β (which is constant in the $\mathbb{D}$-direction). Therefore, we have

LEMMA 3.11. *The virtual cycle* $\mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}}$ *is well-defined if* $\mathcal{M}_1(\beta) = \emptyset$. *We have*

$$\varphi_*\mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \begin{cases} [L \times S^1] & \text{if } \beta = 0\,; \\ 0 & \text{if } \beta \neq 0 \text{ and } \mathcal{M}_1(\beta) = \emptyset\,. \end{cases}$$

3.3.2. The right-hand side of (11). Take $(\sigma, \alpha) \in H_2^{\mathrm{sec}}(E) \times H_2(M, L)$ such that $\sigma + \hat{\alpha} = r(\hat{\beta})$. By Corollary 3.9 and Proposition 3.6, we have

$$\mu(\beta) = 2\left\langle c_1^{\mathrm{vert}}(E), \sigma \right\rangle + \mu(\alpha) \tag{12}$$

$$\partial\beta = \partial\alpha + \lambda \tag{13}$$

where $\lambda \in H_1(L)$ is the class of an S^1-orbit.

Suppose $\alpha = 0$. This can happen only when $\partial\beta = \lambda$ by (13). Since $\alpha = 0$, $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ consists of constant disc sections and $\mathrm{ev}^{(\mathrm{b})}\colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \cong L \times S^1$. The interior evaluation $\mathrm{ev}^{(\mathrm{i})}\colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to M$ is given by the projection $L \times S^1 \to L \subset M$. Thus

$$\begin{aligned} \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}} &= \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M (L \times S^1)]^{\mathrm{vir}} \\ &= (\mathcal{S}_\sigma \cap [L]) \times [S^1] \end{aligned} \tag{14}$$

where

$$\mathcal{S}_\sigma := \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}\right) \in H^{-\mu(\beta)}(M)\,. \tag{15}$$

Here we used the virtual dimension formula (4) and (12).

Suppose $\alpha \neq 0$. By the same argument as in §3.3.1, $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ is non-empty if and only if $\mathcal{M}_1(\alpha)$ is non-empty; also $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ has boundary if $\mathcal{M}_1(\alpha)$ is non-empty. Assume that $\mathcal{M}_1(\alpha)$ has no boundary. This means that every stable map in $\mathcal{M}_1(\alpha)$ has only one disc component[2] (but possibly with sphere bubbles). Let us study the moduli space $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ and its boundary. Since $\alpha \neq 0$, we have a map

$$\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)\colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to \mathcal{M}_1(\alpha) \times S^1\,. \tag{16}$$

The first factor $\mathfrak{f}_1$ is given by projecting bordered stable maps to M, forgetting the interior marking and collapsing unstable components; the second factor $\mathfrak{f}_2$ is the boundary evaluation $\mathrm{ev}^{(\mathrm{b})}\colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to L \times S^1$ followed by the projection $L \times S^1 \to S^1$. The map $\mathfrak{f}$ can be viewed as a tautological family of stable discs over $\mathcal{M}_1(\alpha) \times S^1$. In fact we have the following result.

LEMMA 3.12. *Let* $u\colon (\Sigma, \partial\Sigma) \to (M, L)$ *be a one-pointed bordered stable map of class* α *and* $x \in \partial\Sigma$ *be the boundary marking. Suppose that* Σ *has only one disc component. Then the fibre* $\mathfrak{f}^{-1}([u, \Sigma, x], z)$ *at* $([u, \Sigma, x], z) \in \mathcal{M}_1(\alpha) \times S^1$ *can be identified with the oriented real blow-up* $\widehat{\Sigma}$ *of* Σ *at* x *(see the proof below for the definition of* $\widehat{\Sigma}$*) and the interior evaluation* $\mathrm{ev}^{(\mathrm{i})}$ *on* $\mathfrak{f}^{-1}([u, \Sigma, x], z)$ *can be identified with the map* $\widehat{\Sigma} \to \Sigma \xrightarrow{u} M$.

[2] See [13, §7.1.1] for the boundary description of the moduli spaces.

PROOF. The assumption that Σ has only one disc component was made for simplicity's sake (and this is the case we are interested in). In general, the fibre of $\mathfrak{f}$ can be identified with a smoothing of $\widehat{\Sigma}$ at the boundary singularities. See [13, Lemma 7.1.45] for a similar statement.

We identify a neighbourhood of $x \in \Sigma$ with the upper-half disc $\mathbb{D}_+ = \{w \in \mathbb{D} : \mathrm{Im}(w) \geq 0\}$ where x corresponds to $0 \in \mathbb{D}_+$. The oriented real blow-up $\widehat{\Sigma}$ is defined by replacing this neighbourhood with $[0, \pi] \times [0, 1]$:

$$\widehat{\Sigma} = (\Sigma \setminus \{x\}) \cup_{\mathbb{D}_+ \setminus \{0\}} ([0, \pi] \times [0, 1])$$

where $\mathbb{D}_+ \setminus \{0\}$ is identified with $[0, \pi] \times (0, 1]$ by the map $w \mapsto (\arg(w), |w|)$. Note that $\widehat{\Sigma}$ is a real analytic manifold (with boundary and corner) equipped with a natural projection $\widehat{\Sigma} \to \Sigma$.

For a point $p \in \widehat{\Sigma}$, we shall construct a bordered stable map in the fibre $\mathfrak{f}^{-1}((u, \Sigma, x), z)$. Suppose $p \in \widehat{\Sigma} \setminus \partial\widehat{\Sigma} \cong \Sigma \setminus \partial\Sigma$. Note that Σ is a union of one disc component Σ_0 and trees of sphere bubbles. If p is in a tree of spheres bubbles, let q be the intersection point of the tree (on which p lies) and the disc Σ_0. If p is in the interior of Σ_0, set $q := p$. Take a unique holomorphic map $v : \Sigma_0 \to \mathbb{D}$ which sends q to $0 \in \mathbb{D}$ and $x \in \partial\Sigma_0$ to $z \in S^1$. Extend v to the whole Σ so that it is constant on each sphere component. Then we obtain a bordered stable map $\hat{u} = (u, v) : \Sigma \to M \times \mathbb{D}$ of class $\hat{\alpha}$ with p a new interior marked point. (If p is a node, we insert at the node a trivial sphere with an interior marking.)

Next consider the case $p \in \partial\widehat{\Sigma}$. In this case, the corresponding bordered stable map is in the boundary of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. See Figure 1 below. If p is not in the exceptional locus $[0, \pi]$ of $\widehat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to Σ by identifying $1 \in \partial\mathbb{D}$ with $p \in \partial\Sigma$ and define a map $\hat{u} : \mathbb{D} {}_1\cup_p \Sigma \to M \times \mathbb{D}$ by

$$\hat{u}|_{\mathbb{D}}(w) = (u(p), zw), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

A new interior marking is taken to be $0 \in \mathbb{D}$. If p corresponds to an interior point $\theta \in (0, \pi)$ of the exceptional locus $[0, \pi]$ of $\widehat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to Σ by identifying $1 \in \partial\mathbb{D}$ with $x \in \partial\Sigma$ and define a map $\hat{u} : \mathbb{D} {}_1\cup_x \Sigma \to M \times \mathbb{D}$ by

$$\hat{u}|_{\mathbb{D}}(w) = (u(x), e^{-2i\theta} zw), \quad \hat{u}|_{\Sigma}(y) = (u(y), e^{-2i\theta} z).$$

We put a new boundary marking at $e^{2i\theta} \in \mathbb{D}$ and a new interior marking at $0 \in \mathbb{D}$. If p is a boundary point of the exceptional locus $[0, \pi]$, say, $0 \in [0, \pi]$, we consider the domain $\mathbb{D}^{(1)} {}_1\cup_{-1} \mathbb{D}^{(2)} {}_1\cup_x \Sigma$ (subscripts signify how to identify boundary points) with a boundary marking $i \in \mathbb{D}^{(2)}$ and an interior marking $0 \in \mathbb{D}^{(1)}$ and define a map $\hat{u} : \mathbb{D}^{(1)} \cup \mathbb{D}^{(2)} \cup \Sigma \to M \times \mathbb{D}$ by:

$$\hat{u}|_{\mathbb{D}^{(1)}}(w) = (u(x), zw), \quad \hat{u}|_{\mathbb{D}^{(2)}}(w) = (u(x), z), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

When p corresponds to $\pi \in [0, \pi]$, we take $-i \in \mathbb{D}^{(2)}$ in place of $i \in \mathbb{D}^{(2)}$ as a boundary marking. One can see that the above construction defines a homeomorphism $\widehat{\Sigma} \cong \mathfrak{f}^{-1}([u, \Sigma, x], z)$. The last statement is obvious. □

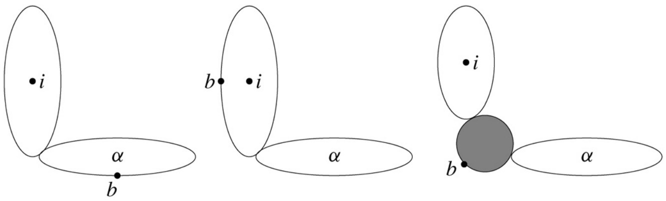

FIGURE 1. Three types of boundary points of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. The horizontal direction is M and the vertical direction is $\mathbb{D}$. The boundary/interior markings are denoted by b and i respectively. The shaded disc is a component where the map is constant.

From the previous lemma and its proof, we have:

COROLLARY 3.13. *Suppose* $\partial\mathcal{M}_1(\alpha) = \emptyset$. *The boundary* $\partial\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ *maps to L under the interior evaluation map* $\mathrm{ev}^{(\mathrm{i})}\colon \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \to M$.

COROLLARY 3.14. *Suppose* $\partial\mathcal{M}_1(\alpha) = \emptyset$ *and* $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$. *Then the fibre product* $\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ *has no boundary. In particular, the virtual fundamental cycle* $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ *is well-defined (see* [13, Lemma A.1.32]*).*

We proceed to calculate the cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ under the assumption of Corollary 3.14. By Corollary 3.13, taking a sufficiently small perturbation, we get a virtual fundamental *chain*

$$\mathcal{P}_\alpha := (\mathrm{ev}^{(\mathrm{i})} \times \mathrm{ev}^{(\mathrm{b})})_*[\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$$

whose boundary lies in $\nu(L) \times L \times S^1$, where $\nu(L) \subset M$ is an arbitrarily small tubular neighbourhood of L. In other words, $\mathcal{P}_\alpha$ defines a *relative* homology class of the pair $(M \times L \times S^1, \nu(L) \times L \times S^1)$ whose dimension is $n + 1 + \mu(\alpha)$ (where $n = \dim_{\mathbb{C}} M$). On the other hand, since $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$, taking a sufficiently small perturbation again, we obtain a virtual cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}$ in $M \setminus \nu(L)$. By Poincaré-Lefschetz duality this defines a class

$$\widehat{\mathcal{S}_\sigma} := \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}\right) \in H^{\mu(\alpha)-\mu(\beta)}(M, L)\,. \tag{17}$$

Here we used the virtual dimension formula (4) and (12). (Note that we put "hat" to distinguish $\widehat{\mathcal{S}_\sigma} \in H^*(M, L)$ from the element $\mathcal{S}_\sigma \in H^*(M)$ appearing in (15).) The virtual cycle of the fibre product can be evaluated as the pairing of the two classes:

$$\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}} = \langle \widehat{\mathcal{S}_\sigma}, \mathcal{P}_\alpha \rangle \tag{18}$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between relative cohomology and homology (with Künneth decomposition):

$$\begin{aligned} &H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\alpha)}(M \times L \times S^1, \nu(L) \times L \times S^1) \\ &\quad\longrightarrow H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\beta)}(L \times S^1) \\ &\quad\longrightarrow H_{n+1+\mu(\beta)}(L \times S^1)\,. \end{aligned}$$

We now compute the class $\mathcal{P}_\alpha$ in terms of the class $[\mathcal{M}_1(\alpha)^{\mathrm{vir}}]$. For this, consider the diagram

$$\begin{array}{ccccc} \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) & \xrightarrow{\mathfrak{f}} & \mathcal{M}_1(\alpha) \times S^1 & \xrightarrow{\mathrm{ev}\times\mathrm{id}} & L \times S^1 \\ {\scriptstyle \mathrm{ev}^{(\mathrm{i})}}\big\downarrow & & & & \\ M & & & & \end{array} \tag{19}$$

where $\mathfrak{f}$ is given in (16). The composition of the horizontal arrows is the boundary evaluation map $\mathrm{ev}^{(\mathrm{b})}$. As we saw in Lemma 3.12, $(\mathfrak{f}, \mathrm{ev}^{(\mathrm{i})})$ can be viewed as a universal family of bordered stable maps of class α.

LEMMA 3.15. *Suppose* $\partial\mathcal{M}_1(\alpha) = \emptyset$ *and* $\alpha \neq 0$. *Then the relative homology class* $\mathcal{P}_\alpha$ *is given by*

$$\mathcal{P}_\alpha = \alpha \otimes \left(\mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times [S^1]\right) \tag{20}$$

in $H_*(M \times L \times S^1, \nu(L) \times L \times S^1) \cong H_*(M, L) \otimes H_*(L \times S^1)$.

Notice that if one ignores the technical details on the construction of virtual chains, as well as the expected functoriality relating the respective chains on these spaces, the lemma follows directly from Diagram (19).

PROOF. We briefly overview the steps towards the proof of this result. We first recall the definition of Kuranishi structure and the construction of Kuranishi neighbourhoods for $\mathcal{M}_1(\alpha)$. Roughly speaking, a Kuranishi neighbourhood V of $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$ consists of smooth maps from stable discs to (M, L) which are sufficiently "close" to $\mathfrak{r}_0$. It is equipped with an obstruction bundle $E \to V$ satisfying a transversality condition with respect to the Cauchy-Riemann operator, together with a section s of E such that a neighbourhood of $\mathfrak{r}_0$ in $\mathcal{M}_1(\alpha)$ is homeomorphic to $s^{-1}(0) \subset V$. As a next step, we describe how a Kuranishi neighbourhood V for $\mathcal{M}_1(\alpha)$ induces a Kuranishi neighbourhood $\widehat{V}$ for $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. Following exactly the method of Lemma 3.12, for a given smooth map $u\colon \Sigma \to M$ in the Kuranishi neighbourhood V *and* a point of $\widehat{\Sigma} \times S^1$, we can canonically construct a smooth map $\hat{u}\colon \widetilde{\Sigma} \to M \times \mathbb{P}^1$ which is holomorphic in the $\mathbb{P}^1$-direction, where $\widetilde{\Sigma}$ is obtained from Σ by possibly adding disc or sphere bubbles. This constructs a Kuranishi neighbourhood $\widehat{V}$ for $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ which is a fibration over $V \times S^1$ with fibres stable discs. The key point is that we can take an obstruction bundle over $\widehat{V}$ to be the *pull-back* of the obstruction bundle E over V. This allows us to choose the virtual chain of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ to be "fiber bundle" over a virtual cycle of $\mathcal{M}_1(\alpha) \times S^1$ with fibre the corresponding stable discs, and the conclusion of the lemma follows.

Kuranishi structure on $\mathcal{M}_1(\alpha)$. We refer the reader to [13, §7.1], [16, Part 3, 4] for a detailed description. Recall [13, Definition A1.1] that a *Kuranishi neighbourhood* of a point $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$ is a tuple (V, E, Γ, ψ, s) where

- V is a finite dimensional manifold (possibly with boundary and corner);
- E is a finite dimensional real vector space;
- Γ is a finite group; it acts on V smoothly and effectively and on E linearly;
- s is a smooth Γ-equivariant map $V \to E$;

- ψ is a homeomorphism between $s^{-1}(0)/\Gamma$ and an open neighbourhood of $\mathfrak{r}_0$ in $\mathcal{M}_1(\alpha)$.

Every point of $\mathcal{M}_1(\alpha)$ is equipped with a certain Kuranishi neighbourhood, and these Kuranishi neighbourhoods are related by certain co-ordinate changes [13, Definition A.1.3] and $\mathcal{M}_1(\alpha)$ becomes a Kuranishi space (a space endowed with a Kuranishi structure) [13, Definition A.1.5, Proposition 7.1.1]. The Kuranishi neighbourhoods are cut-off from maps satisfying the Cauchy-Riemann equation modulus a finite dimensional obstruction. Such construction depends on several parameters, as we now describe in the case at hand.

Let $(u_0\colon \Sigma_0 \to M, x_0 \in \partial\Sigma_0)$ be a marked bordered stable map representing $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$. The finite group Γ is given by the set of holomorphic automorphisms $\varphi\colon \Sigma_0 \to \Sigma_0$ such that $u_0 \circ \varphi = u_0$ and $\varphi(x_0) = x_0$. Since (Σ_0, x_0) has only one marking, it is unstable if we forget the map u_0. We add interior markings $w_{0,1}, \dots, w_{0,l}$ in Σ_0 so that $(\Sigma_0, x_0, \{w_{0,1}, \dots, w_{0,l}\})$ is now stable. We require that the set $\{w_{0,1}, \dots, w_{0,l}\}$ is preserved by the Γ-action [16, Definition 17.5], so that Γ permutes the indices. We can therefore regard Γ as a subgroup of the symmetric group $\mathfrak{S}_l$. We take real codimension 2 submanifolds $Q_1, \dots, Q_l$ of M such that u_0 intersects Q_i transversely at $w_{0,i}$ (so u_0 is necessarily an immersion at $w_{0,i}$); moreover we require that $Q_i = Q_{\sigma(i)}$ for every permutation $\sigma \in \Gamma \subset \mathfrak{S}_l$. Let $\mathcal{M}_{1,l}$ denote the moduli space of genus-zero stable bordered Riemann surfaces with one boundary and l interior markings and let $\mathfrak{a}_0 \in \mathcal{M}_{1,l}$ be the point represented by $(\Sigma_0, x_0, \{w_{0,1}, \dots, w_{0,l}\})$. The group Γ acts on $\mathcal{M}_{1,l}$ by permutation of the l interior markings and $\mathfrak{a}_0$ is fixed by Γ. Let $N \subset \mathcal{M}_{1,l}$ be a Γ-invariant small open neighbourhood of $\mathfrak{a}_0$. Fukaya-Oh-Ohta-Ono [16, Section 16] constructed N in two steps. First they considered a subset $\mathfrak{V} \subset \mathcal{M}_{1,l}$ consisting of deformations of $\mathfrak{a}_0$ having the same combinatorial type, i.e. the same associated dual graph as $\mathfrak{a}_0$. Then, they introduced smoothing (or gluing) near the nodes with parameters $T \in \mathbb{R}$ and $\theta \in S^1$. This constructs the neighbourhood N of $\mathfrak{V}$. This construction yields a Γ-equivariant *tautological* family $\mathcal{R} \to N$ [16, Lemma 16.9] of stable bordered Riemann surfaces, where the fibre at $\mathfrak{b} \in N$ corresponds to its underlying surface. Note that when $\gamma \cdot \mathfrak{b} = \mathfrak{b}'$ for $\mathfrak{b}, \mathfrak{b}' \in N$ and $\gamma \in \Gamma$, there is a canonical isomorphism between the underlying surfaces of $\mathfrak{b}$ and $\mathfrak{b}'$ which induces the permutation γ of the interior markings; this defines the Γ-action on $\mathcal{R}$. We take a Γ-invariant closed subset $\mathcal{K} \subset \mathcal{R}$ such that the fibre $K_0 = \Sigma_0 \cap \mathcal{K}$ at $\mathfrak{a}_0$ is the complement in Σ_0 of small neighbourhoods of the nodes in Σ_0, and that the family $\mathcal{K} \to N$ is C^∞-trivial. We choose a Γ-equivariant C^∞-trivialization $\mathcal{K} \cong K_0 \times N$ which preserves the markings, i.e. the section of the ith interior marking in $\mathcal{K}$ corresponds to $\{w_{0,i}\} \times N \subset K_0 \times N$. $\mathcal{K}$ is called the *core* and its complement is called the *neck region* (for further details see [16, Definitions 16.2, 16.4, 16.6, 16.7].)

For a bordered Riemann surface Σ appearing as a fibre of $\mathcal{R} \to N$, the core $K = \Sigma \cap \mathcal{K}$ is identified with K_0 by the given trivialization $\mathcal{K} \cong K_0 \times N$, and thus u_0 induces a map $u_0\colon K \to M$. We consider an infinite dimensional space $\mathcal{U}$ consisting of tuples $(u, \Sigma, x, \{w_1, \dots, w_l\})$, where $(\Sigma, x, \{w_1, \dots, w_l\})$ represents a point of N and $u\colon (\Sigma, \partial\Sigma) \to (M, L)$ is a smooth map of degree α which is "sufficiently close" to u_0 in the sense that (see [16, Definitions 17.12, 18.10]) for an $\varepsilon > 0$

- u is ε-close to u_0 in the C^{10}-topology on the core $K = \Sigma \cap \mathcal{K}$;
- u is holomorphic on the neck region $\Sigma \setminus K$;
- the diameter of the image of each connected component of the neck region under u is smaller than ε.

The group Γ acts on $\mathcal{U}$ by permutation of interior marked points. Next we choose an obstruction bundle $\mathbb{E}$ over $\mathcal{U}$ as follows (see [16, Definitions 17.7, 17.15]). Take a Γ-equivariant smooth family of finite dimensional subspaces $\mathbb{E}_{\mathfrak{a}}$

$$\mathbb{E}_{\mathfrak{a}} \subset C_c^{\infty}(\mathrm{Int}(K), u_0^* TM \otimes \Lambda^{0,1})$$

parametrized by $\mathfrak{a} = (\Sigma, x, \{w_1, \dots, w_l\}) \in N$, where $K = \Sigma \cap \mathcal{K}$ is the core of Σ and $\Lambda^{0,1}$ is the bundle of $(0,1)$-forms on Σ. Then we extend this family to the whole $\mathcal{U}$ via parallel transport, i.e. for each point $\mathfrak{r} = (u, \Sigma, x, \{w_1, \dots, w_l\}) \in \mathcal{U}$ over $\mathfrak{a} = (\Sigma, x, \{w_1, \dots, w_l\}) \in N$, we define

$$\mathbb{E}_{\mathfrak{r}} \subset C_c^{\infty}(\mathrm{Int}(K), u^* TM \otimes \Lambda^{0,1})$$

as the parallel transport of $\mathbb{E}_{\mathfrak{a}}$ along geodesics joining $u(y)$ and $u_0(y)$, for $y \in \mathrm{Int}(K)$. Here we use a connection on TM such that TL is preserved by parallel translation [16, §11]. By construction, the bundle $\mathbb{E} \to \mathcal{U}$ is Γ-equivariant. The *Kuranishi neighbourhood* $V \subset \mathcal{U}$ is cut out by the equations:

$$\begin{aligned} &\overline{\partial} u \equiv 0 \mod \mathbb{E}_{\mathfrak{r}} \\ &u(w_i) \in Q_i \quad i = 1, \dots, l \end{aligned} \tag{21}$$

for $\mathfrak{r} = (u, \Sigma, x, \{w_1, \dots, w_l\}) \in \mathcal{U}$. We need to choose $\mathbb{E}$ so that the equations (21) are transversal (see below). The Γ-action on $\mathcal{U}$ preserves V and thus the obstruction bundle restricts to a Γ-equivariant vector bundle $E = \mathbb{E}|_V$ over V. The Cauchy-Riemann operator $\overline{\partial}$ induces a section s of $E \to V$ and $s^{-1}(0)/\Gamma$ gives a neighbourhood of $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$.

The required transversality for (21) is stated as follows (see [16, Lemmata 18.16, 20.7]). For a smooth map $u \colon (\Sigma, \partial\Sigma) \to (M, L)$, let $L^2_{m,\delta}(\Sigma, \partial\Sigma; u^*TM, u^*TL)$ denote a weighted Sobolev space, for m sufficiently large and $\delta > 0$, and consisting of $L^2_{m,\mathrm{loc}}$-sections of u^*TM which take values in u^*TL along the boundary $\partial\Sigma$, see [16, Definitions 10.1, 19.8]. Let $L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$ denote a similar weighted Sobolev space of sections of $u^*TM \otimes \Lambda^{0,1}$ (see [16, Definition 19.9]). Let

$$D_{\mathfrak{r}}\overline{\partial} \colon L^2_{m+1,\delta}(\Sigma, \partial\Sigma; u^*TM, u^*TL) \to L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$$

denote the linearization of $\overline{\partial}$ at $\mathfrak{r} = (u, \Sigma, x, \{w_1, \dots, w_l\}) \in \mathcal{U}$. We require that $\mathrm{Im}(D_{\mathfrak{r}}\overline{\partial})$ and $\mathbb{E}_{\mathfrak{r}}$ span $L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$ for each $\mathfrak{r} \in \mathcal{U}$. (This is called "Fredholm regularity".) Let $\mathcal{M} \subset \mathcal{U}$ denote the subspace cut out only by the first equation of (21). Let $\mathrm{ev}_{\mathrm{ad}} \colon \mathcal{M} \to M^l$ be the evaluation map at the l additional markings. We also require $\mathrm{ev}_{\mathrm{ad}}$ to be transversal to $\prod_{i=1}^l Q_i \subset M^l$. Then $V = \mathrm{ev}_{\mathrm{ad}}^{-1}(\prod_{i=1}^l Q_i)$ is the desired neighbourhood.

Induced Kuranishi structure on $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. Recall that $\mathfrak{f}$ is the forgetful morphism $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \to \mathcal{M}_1(\alpha) \times S^1$ as in (16). We now construct a Kuranishi neighbourhood of

$\mathfrak{f}^{-1}(\mathfrak{r}_0 \times S^1) \subset \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ from the Kuranishi neighbourhood (V, E, Γ, ψ, s) of $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$ above. We have $\mathfrak{f}^{-1}(\mathfrak{r}_0 \times S^1) \cong \widehat{\Sigma}_0 \times S^1$ by Lemma 3.12, where $\widehat{\Sigma}_0$ is the oriented real blow-up of Σ_0 at x_0. We perform this oriented real blow-up in families. The family $\mathcal{R} \to N$ is equipped with a section $x\colon N \to \mathcal{R}$ corresponding to the boundary marked point. Let $\widehat{\mathcal{R}}$ denote the oriented real-blow up along x. The proof of Lemma 3.12 shows that a point $p \in \widehat{\mathcal{R}}$ parametrizes a marked stable bordered Riemann surface[3] $(\widetilde{\Sigma}, x, p, \{w_1, \ldots, w_l\})$ with a new interior marking p (see also [13, Lemma 7.1.45]). More precisely, letting $p \in \widehat{\mathcal{R}}$ be on the blow-up of a fibre $\Sigma \subset \mathcal{R}$: if p is neither a node nor a boundary point, $\widetilde{\Sigma} = \Sigma$; if p is an interior node, $\widetilde{\Sigma}$ is obtained from Σ by adding a sphere bubble at the node; if p is a boundary point, $\widetilde{\Sigma}$ is obtained from Σ by adding at most two disc bubbles (see Figure 1). It is possible that a new interior marking p on $\widetilde{\Sigma}$ coincides with one of the w_i's. Let $\mathfrak{R} \to \widehat{\mathcal{R}}$ denote the corresponding family of marked stable bordered Riemann surfaces. The Γ-action on $\widehat{\mathcal{R}}$ naturally lifts to that on the tautological family $\mathfrak{R} \to \widehat{\mathcal{R}}$. Let $p \in \widehat{\mathcal{R}}$ be a point on the blow-up $\widehat{\Sigma}$ of a fiber $\Sigma \subset \mathcal{R}$ of $\mathcal{R} \to N$ and let $\widetilde{\Sigma}$ be the fibre of $\mathfrak{R} \to \widehat{\mathcal{R}}$ at $p \in \widehat{\mathcal{R}}$ as above. Then the core $K = \mathcal{K} \cap \Sigma$ of Σ induces a compact subset $\widetilde{K}$ of $\widetilde{\Sigma}$ which maps isomorphically onto K under the natural map $\widetilde{\Sigma} \to \Sigma$. (The set $\widetilde{K}$ is like the strict transform of K.) The union of these subsets $\widetilde{K}$ gives a Γ-invariant subset $\mathfrak{K} \subset \mathfrak{R}$ equipped with a Γ-equivariant C^∞-trivialization $\mathfrak{K} \cong K_0 \times \widehat{\mathcal{R}}$. Notice that $\mathrm{Int}(\mathfrak{K})$ is disjoint from the components contracted under $\widetilde{\Sigma} \to \Sigma$.

Let $\mathfrak{U}$ be the space of tuples $(\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \ldots, w_l\})$ where $(\widetilde{\Sigma}, x, p, \{w_1, \ldots, w_l\})$ is a marked bordered Riemann surface corresponding to a point of $\widehat{\mathcal{R}}$ (i.e. arises as a fibre of $\mathfrak{R} \to \widehat{\mathcal{R}}$) and $\hat{u}\colon (\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to (M \times \mathbb{P}^1, L \times S^1)$ is a smooth map of class $\hat{\alpha}$ which satisfies the following conditions:

- $\pi_M \circ \hat{u}$ is C^{10}-close to u_0 on $\widetilde{K} = \widetilde{\Sigma} \cap \mathfrak{K}$, where $\pi_M\colon M \times \mathbb{P}^1 \to M$ is the projection (since $\widetilde{K}$ is identified with K_0 via the given trivialization $\mathfrak{K} \cong K_0 \times \widehat{\mathcal{R}}$, u_0 defines a map $u_0\colon \widetilde{K} \to M$);
- $\hat{u}$ is holomorphic on $\widetilde{\Sigma} \setminus \widetilde{K}$;
- the diameter of the image of each connected component of $\widetilde{\Sigma} \setminus \widetilde{K}$ under $\pi_M \circ \hat{u}$ is small.

The obstruction bundle $\mathbb{E} \to \mathcal{U}$ induces an obstruction bundle $\mathbf{E} \to \mathfrak{U}$ as follows. Take an element $\mathfrak{s} = (\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \ldots, w_l\}) \in \mathfrak{U}$ and let $\mathfrak{a} = (\Sigma, x, \{w_1, \ldots, w_l\}) \in N$ denote the marked Riemann surface given by forgetting p and collapsing unstable components of the source. Define the obstruction space at $\mathfrak{s} \in \mathfrak{U}$

$$\mathbf{E}_{\mathfrak{s}} \subset C^\infty_c(\mathrm{Int}(\widetilde{K}), (\pi_M \circ \hat{u})^* TM \otimes \Lambda^{0,1}) \subset C^\infty_c(\mathrm{Int}(\widetilde{K}), \hat{u}^* T(M \times \mathbb{P}^1) \otimes \Lambda^{0,1})$$

(with $\widetilde{K} = \widetilde{\Sigma} \cap \mathfrak{K}$) to be the parallel transport of $\mathbb{E}_{\mathfrak{a}} \subset C^\infty_c(\mathrm{Int}(\widetilde{K}), u_0^* TM \otimes \Lambda^{0,1})$ along geodesics joining $u_0(y)$ and $(\pi_M \circ \hat{u})(y)$. Let $C \subset \widetilde{\Sigma}$ be the union of the contracted components of $\widetilde{\Sigma} \to \Sigma$. Because $\pi_M \circ \hat{u}|_{\widetilde{K}}$ is sufficiently close to u_0, $\pi_M \circ \hat{u}|_{\widetilde{\Sigma}\setminus\widetilde{K}}$ is holomorphic and $C \subset \widetilde{\Sigma} \setminus \mathrm{Int}(\widetilde{K})$, by choosing a smaller neck region from the beginning if necessary (see

[3]By abuse of notation, we denote by p a point of $\widehat{\mathcal{R}}$ and at the same time a new interior marking on $\widetilde{\Sigma}$.

"extending the core" [16, Definition 17.21]), we may assume that $\pi_M \circ \hat{u}$ is constant on C (since the symplectic area of $(\pi_M \circ \hat{u})(\widetilde{\Sigma} \setminus \widetilde{K})$ has to be small). Hence $\pi_M \circ \hat{u}$ induces a map $u\colon (\Sigma, \partial\Sigma) \to (M, L)$ belonging to $\mathcal{U}$. Therefore we have a projection $\mathfrak{U} \to \mathcal{U}$ and $\mathbf{E}$ is identified with the pull-back of $\mathbb{E}$. The group Γ acts on $\mathfrak{U}$ and $\mathbf{E}$ and $\mathbf{E} \to \mathfrak{U}$ is Γ-equivariant. The Kuranishi neighbourhood $\widehat{V}$ for $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ is cut out from $\mathfrak{U}$ by the equations:

$$
\begin{aligned}
&\overline{\partial}\hat{u} \equiv 0 \mod \mathbf{E}_{\mathfrak{s}} \\
&\hat{u}(p) \in M \times \{0\} \\
&\hat{u}(w_i) \in Q_i \times \mathbb{P}^1 \quad i = 1, \dots, l
\end{aligned}
\tag{22}
$$

for $\mathfrak{s} = (\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\}) \in \mathfrak{U}$. The second equation of (22) corresponds to the fibre product presentation (10) of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$.

Let $\widehat{\mathcal{M}} \subset \mathfrak{U}$ denote the subspace cut out by the first and the second equations of (22). Consider the map $\mathfrak{U} \to \mathcal{U} \times S^1$, where the first factor is the projection we discussed and the second factor is the evaluation map at the boundary marking x followed by the projection $L \times S^1 \to S^1$. We claim that $\widehat{\mathcal{M}}$ is a tautological family of (blown-up) Riemann surfaces over $\mathcal{M} \times S^1$ under the map $\widehat{\mathcal{M}} \subset \mathfrak{U} \to \mathcal{U} \times S^1$. (Recall that $\mathcal{M} \subset \mathcal{U}$ is cut out by the first equation of (21).) More precisely, it is identified with the restriction to $\mathcal{M} \times S^1$ of the family $\mathrm{pr}^*\widehat{\mathcal{R}} \to \mathcal{U} \times S^1$ where $\mathrm{pr}\colon \mathcal{U} \times S^1 \to N$ is the natural projection. By the choice of $\mathbf{E}$, each element $(\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\})$ of $\widehat{\mathcal{M}}$ is holomorphic in the $\mathbb{P}^1$-factor and its image $(u, \Sigma, x, \{w_1, \dots, w_l\})$ in $\mathcal{U}$ belongs to $\mathcal{M}$. By the same argument as in the proof of Lemma 3.12, it follows that $\hat{u}$ is uniquely reconstructed from $u\colon \Sigma \to M$, $p \in \widehat{\Sigma}$ and $(\pi_{\mathbb{P}^1} \circ \hat{u})(x) \in S^1$. This proves the claim. Cutting down the moduli space $\widehat{\mathcal{M}}$ by the third equation of (22), we obtain $\widehat{V}$ as a tautological family of (blown-up) Riemann surfaces over $V \times S^1$, with V the Kuranishi neighbourhood of $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$. The obstruction bundle $\widehat{E} = \mathbf{E}|_{\widehat{V}}$ and its section $\hat{s} := \overline{\partial}$ are the pull-backs of $E \to V$ and $s = \overline{\partial}$ respectively. These data $(\widehat{V}, \widehat{E}, \hat{s})$ are Γ-equivariant and give a Kuranishi neighbourhood $(\widehat{V}, \widehat{E}, \Gamma, \hat{\psi}, \hat{s})$ of $\mathfrak{f}^{-1}(\mathfrak{r}_0 \times S^1) \subset \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$.

We need to check the transversality of (22). We first show the transversality of the $\overline{\partial}$-equation. Let $\hat{u}\colon \widetilde{\Sigma} \to M$ be a map in $\mathfrak{U}$ satisfying the first equation of (22). Let $v := \pi_{\mathbb{P}^1} \circ \hat{u}\colon (\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to (\mathbb{P}^1, S^1)$ be the vertical component of $\hat{u}$, which is a holomorphic map of degree one. Let $\tilde{u} = \pi_M \circ \hat{u}\colon (\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to (M, L)$ be the horizontal component of $\hat{u}$. The image $u\colon (\Sigma, \partial\Sigma) \to (M, L)$ of $\hat{u}$ in $\mathcal{U}$ is obtained from $\tilde{u}$ by collapsing some components of $\widetilde{\Sigma}$ on which $\tilde{u}$ is constant. It suffices to show that

- v is Fredholm regular, i.e. $D_v\overline{\partial}$ is surjective; and
- $\tilde{u}$ is Fredholm regular for $\mathbb{E}_u$, i.e. $\mathrm{Im}(D_{\tilde{u}}\overline{\partial}) + \mathbb{E}_u = L^2_{m,\delta}(\widetilde{\Sigma}, \tilde{u}^*(TM) \otimes \Lambda^{0,1})$.

The Fredholm regularity for v can be rephrased as the vanishing of the sheaf cohomology (see [23, §3.4], [10, §6]):

$$H^1(\widetilde{\Sigma}, (v^*T\mathbb{P}^1, v^*TS^1)) = 0$$

where $(v^*T\mathbb{P}^1, v^*TS^1)$ denotes the sheaf of holomorphic sections of $v^*T\mathbb{P}^1$ which take values in v^*TS^1 on $\partial\widetilde{\Sigma}$. Let $\widetilde{\Sigma} = \bigcup_i \Sigma_i$ be the decomposition into irreducible components and write $v_i = v|_{\Sigma_i}$. Then we have the following standard normalization sequence for sheaves on $\widetilde{\Sigma}$:

$$0 \to (v^*T\mathbb{P}^1, v^*TS^1) \to \bigoplus_i (v_i^*T\mathbb{P}^1, v_i^*TS^1) \to \bigoplus_x T_{v(x)}\mathbb{P}^1 \oplus \bigoplus_y T_{v(y)}S^1 \to 0$$

where x ranges over interior nodes of $\widetilde{\Sigma}$ and y ranges over boundary nodes of $\widetilde{\Sigma}$. Since we have $H^1(v_i^*T\mathbb{P}^1, v_i^*TS^1) = 0$ for each component Σ_i by [10, Lemma 6.4], it suffices to show that the map $\bigoplus_i H^0(v_i^*T\mathbb{P}^1, v_i^*TS^1) \to \bigoplus_x T_{v(x)}\mathbb{P}^1 \oplus \bigoplus_y T_{v(x)}S^1$ is surjective: this follows easily by induction on the number of components (removing degree-zero tails one by one). The Fredholm regularity of $\tilde{u}$ with respect to $\mathbb{E}$ follows from the assumed regularity for u with respect to $\mathbb{E}$. For example, consider the case where $\widetilde{\Sigma} = \Sigma \cup \mathbb{D}$. The obstruction space $\mathbb{E}_u$ is supported on the core $K \subset \Sigma$ and $\tilde{u}$ is constant on $\mathbb{D}$. Write $x = \tilde{u}(\mathbb{D}) \in M$. Let (ξ_1, ξ_2) be an element of

$$L^2_{m,\delta}(\widetilde{\Sigma}, \tilde{u}^*TM \otimes \Lambda^{0,1}) = L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1}) \oplus L^2_{m,\delta}(\mathbb{D}, T_xM \otimes \Lambda^{0,1}) \,.$$

The assumed regularity implies that there exists $\varepsilon \in \mathbb{E}_u$ and $\nu_1 \in L^2_{m+1,\delta}(\Sigma, \partial\Sigma; u^*TM, u^*TL)$ such that $\xi_1 = (D_u\overline{\partial})\nu_1 + \varepsilon$. The vanishing $H^1(T_xM, T_xL) = 0$ of the sheaf cohomology on $\mathbb{D}$ implies that there exists $\nu_2 \in L^2_{m+1,\delta}(\mathbb{D}, \partial\mathbb{D}; T_xM, T_xL)$ such that $\overline{\partial}\nu_2 = \xi_2$. By adding a constant element in T_xL to ν_2, we may assume that ν_1 and ν_2 agree on the boundary node $\mathbb{D} \cap \Sigma$, and then we have $(\xi_1, \xi_2) = (D_{\tilde{u}}\overline{\partial})(\nu_1, \nu_2) + \varepsilon$. This shows the regularity of $\tilde{u}$. The other cases are similar.

Let $\mathcal{N} \subset \mathfrak{U}$ denote the moduli space cut out only by the first equation of (22). The holomorphic automorphism group Autt($\mathbb{D}$) acts on the target $(M \times \mathbb{P}^1, L \times S^1)$ and also on the moduli space $\mathcal{N}$. The transversality for the second equation of (22) follows from the fact that the Autt($\mathbb{D}$)-action on Int($\mathbb{D}$) is transitive. The first and the second equations of (22) define the modui space $\widehat{\mathcal{M}}$. The evaluation map $\mathrm{ev}_{\mathrm{ad}} \colon \widehat{\mathcal{M}} \to (M \times \mathbb{P}^1)^l$ at the markings $w_1, \ldots, w_l$ is transversal to $\prod_{i=1}^l (Q_i \times \mathbb{P}^1)$ by the transversality assumption for the second equation of (21). The transversality for (22) is now proved.

Comparison of virtual cycles. A virtual chain is defined by multi-valued perturbations (multisections) of s on Kuranishi neighbourhoods which are compatible under co-ordinate changes, and it is independent of the choice of the obstruction bundle (see [13, §A1.1], [16, Part 2]). By the above construction of Kuranishi neighbourhoods, and the hypothesis $\partial\mathcal{M}_1(\alpha) = \emptyset$, we can define a virtual cycle $[\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ by pulling back multisections used to define a virtual cycle $[\mathcal{M}_1(\alpha)]^{\mathrm{vir}}$, and this is independent of choices. Then $[\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ becomes a fibre bundle over $[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times S^1$ with fibre the corresponding stable bordered Riemann surfaces. Each fibre is of class α under the interior evaluation map. The lemma follows. □

Summarizing the discussion, we obtain (see (14), (18), (20)):

LEMMA 3.16. *The virtual cycle* $\mathrm{ev}_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ *is well-defined if one of the following holds:*

(a) $\mathcal{M}_S(\sigma) = \emptyset$ *or*;

(b) $\mathcal{M}_1(\alpha) = \emptyset$ *or*;

(c) $\partial\mathcal{M}_1(\alpha) = \emptyset$ *and* $\mathrm{ev}(\mathcal{M}_S(\sigma)) \cap L = \emptyset$.

When one of the above conditions holds, we have:

$$\mathrm{ev}_*[\mathcal{M}_S(\sigma)\times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}} = \begin{cases} (\mathcal{S}_\sigma \cap [L]) \times [S^1] & \text{if } \alpha = 0 \text{ (then (b) holds)};\\ \langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle \mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times [S^1] & \text{if (c) holds};\\ 0 & \text{if } \alpha \neq 0 \text{ and (a) or (b) holds.}\end{cases}$$

3.3.3. Conjecture and expected results. We now state our conjecture:

CONJECTURE 3.17 (Degeneration Formula). *Let* $\beta \in H_2(M, L)$ *be such that* $\mathcal{M}_1(\beta) = \emptyset$. *Assume that every pair* $(\sigma, \alpha) \in H_2^{\mathrm{sec}}(E) \times H_2(M, L)$ *with* $r(\hat{\beta}) = \sigma + \hat{\alpha}$ *satisfies one of the three conditions* (a), (b), (c) *in Lemma* 3.16. *Then the degeneration formula* (11)

$$\varphi_*\mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathrm{ev}_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$$

holds. This implies, by Lemmata 3.11 *and* 3.16, *that*

$$\delta_{\beta,0}[L] = \sum_{\substack{(\sigma,\alpha):\, r(\hat{\beta})=\sigma+\hat{\alpha},\ \alpha\neq 0\\ \text{satisfying (c) of Lemma 3.16}}} \langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle \mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} + \delta_{\partial\beta,\lambda} \sum_{\sigma: r(\hat{\beta})=\sigma+\hat{0}} \mathcal{S}_\sigma \cap [L] \tag{23}$$

holds in $H_{n+\mu(\beta)}(L; \mathbb{Q})$. *Here* $\mathcal{S}_\sigma$, $\widehat{\mathcal{S}}_\sigma$ *are defined in* (15), (17) *and* $\lambda \in H_1(L)$ *is the class of an* S^1*-orbit.*

Note that the second term in the right-hand side of (23) arises from the case $\alpha = 0$ (recall the discussion around (14)).

In practice it is not easy to make all the assumptions here to be satisfied and to obtain a non-trivial result from (23). Notice that the both-hand sides of (23) are zero unless $\mu(\beta) \leq 0$ for dimensional reason. Also the term $\langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle$ is zero unless $\widehat{\mathcal{S}}_\sigma \in H^2(M, L)$, i.e. $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$. Hence by (12), the first term of the right-hand side is the sum over classes α satisfying $\mu(\alpha) = \mu(\beta) + 2$. This motivates the following (rather restrictive) assumption:

ASSUMPTION 3.18. (i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.

(ii) The maximal fixed component $F_{\max} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is -1.

(iii) $c_1(M)$ is semi-positive.

(iv) $\mathrm{ev}(\mathcal{M}_S(\sigma))$ is disjoint from L for all $\sigma \in H_2^{\mathrm{sec}}(E)$ such that $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$.

We assume Assumption 3.18 in the rest of this section. Recall from Definition 2.1 that open Gromov-Witten invariants n_α are defined when $\mu(\alpha) = 2$ by the assumption (i) and so the potential function W of L is also defined. The role of the assumptions (ii) and (iii) is as follows. The assumption (ii) implies that $\langle c_1^{\mathrm{vert}}(E), \sigma_0\rangle = -1$ for a maximal section σ_0. Note that by (3) $\mathcal{M}_{\mathrm{S}}(\sigma)$ is empty unless $\sigma = \sigma_0 + d$ for some $d \in \mathrm{NE}(M)_{\mathbb{Z}}$. Therefore by (iii), $\mathcal{M}_{\mathrm{S}}(\sigma)$ is empty unless $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \geq -1$. This implies that the Seidel element S in Definition 2.7 is in $H^{\leq 2}(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(E)_{\mathbb{Z}}]\!]$.

DEFINITION 3.19. Under Assumption 3.18, we can decompose the Seidel element as

$$S = q_0 \widetilde{S} = q_0(\widetilde{S}^{(0)} + \widetilde{S}^{(2)})$$

with $\widetilde{S}^{(i)} \in H^i(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$ and $q_0 = q^{\sigma_0}$. Furthermore, we can define a lift $\widehat{S}^{(2)}$ of $\widetilde{S}^{(2)}$ as follows:

$$\widehat{S}^{(2)} := \sum_{\sigma:\, \langle c_1^{\mathrm{vert}}(E),\sigma\rangle = -1} \widehat{\mathcal{S}}_\sigma q^{\sigma-\sigma_0}$$

where $\widehat{\mathcal{S}}_\sigma \in H^2(M,L;\mathbb{Q})$ (see (17)) is well-defined by Assumption 3.18 (iv). The lift $\widehat{S}^{(2)}$ is an element of $H^2(M,L;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$ which maps to $\widetilde{S}^{(2)}$ under the natural map $H^2(M,L) \to H^2(M)$.

Under Assumption 3.18, the conditions in Conjecture 3.17 are satisfied for all β with $\mu(\beta) = 0$. In fact, if $r(\hat{\beta}) = \sigma + \hat{\alpha}$, then $\mu(\alpha) + 2\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = 0$ by (12), and thus

- if $\mu(\alpha) \leq 0$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \geq 0$, then $\mathcal{M}_1(\alpha) = \emptyset$ by the assumption (i);
- if $\mu(\alpha) \geq 4$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \leq -2$, then $\mathcal{M}_{\mathrm{S}}(\sigma) = \emptyset$ by the assumptions (ii), (iii);
- if $\mu(\alpha) = 2$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$, then $\mathcal{M}_1(\alpha)$ has no boundary by the assumption (i) and $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$ by the assumption (iv).

Fix a class $\gamma \in H_1(L)$. We now apply the formula (23) for β with $\mu(\beta) = 0$ and $\partial\beta = \gamma + \lambda$. In this case, (23) yields the following equality in $H_n(L;\mathbb{Q}) \cong \mathbb{Q}$:

$$\delta_{\beta,0} = \sum_{\substack{(\sigma,\alpha):\, r(\hat{\beta}) = \sigma + \hat{\alpha} \\ \mu(\alpha)=2,\ \langle c_1^{\mathrm{vert}}(E),\sigma\rangle = -1}} \langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle n_\alpha + \delta_{\partial\beta,\lambda} \sum_{\sigma:\, r(\hat{\beta}) = \sigma + \hat{0}} \mathcal{S}_\sigma \tag{24}$$

where n_α is the open Gromov-Witten invariant defined in Definition 2.1. Note that $\mathcal{S}_\sigma$ in the second term of the right-hand side lies in $H^0(L;\mathbb{Q}) \cong \mathbb{Q}$. We consider a generating function in the "open" Novikov ring Λ^{op} which was introduced before Definition 2.1. We have a (not necessarily injective) homomorphism from the "closed" Novikov ring Λ (see Remark 2.9) to the "open" Novikov ring Λ^{op}

$$\Lambda \to \Lambda^{\mathrm{op}}, \quad q^d \mapsto z^d .$$

Thus Λ^{op} is a Λ-algebra. Note that $r(\hat{\beta}) = \sigma + \hat{\alpha}$ means

$$z^{\alpha_0+\beta} = q^{\sigma-\sigma_0} z^\alpha \quad \text{in } \Lambda^{\mathrm{op}}$$

by Proposition 3.6 where σ_0, α_0 are maximal section/disc classes. We multiply the both-hand sides of (24) by $z^{\alpha_0+\beta} = q^{\sigma-\sigma_0} z^\alpha$ and sum over all β with $\mu(\beta) = 2$ and $\partial\beta = \gamma + \lambda$. About the first term of the right-hand side, this summation boils down to the sum over all (σ,α) with $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$, $\mu(\alpha) = 2$, $\partial\alpha = \gamma$ (see (13)); about the second term of the right-hand side (which occurs when and only when $\gamma = 0$), this boils down to the sum over all σ with $\langle c_1^{\mathrm{vert}}(E), \sigma \rangle = 0$. Therefore we have:

THEOREM 3.20. *Assume that the degeneration formula* (*Conjecture* 3.17) *and Assumption 3.18 hold for* (M, L). *For any* $\gamma \in H_1(L)$, *we have*

$$\delta_{\gamma+\lambda,0} z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW_\gamma \rangle + \delta_{\gamma,0} \widetilde{S}^{(0)} \tag{25}$$

in Λ^{op}, *where* $\widetilde{S}^{(0)}$ *and* $\widehat{S}^{(2)}$ *are in Definition* 3.19, W_γ *is in Definition* 2.2 *and* dW_γ *is its logarithmic derivative:*

$$dW_\gamma := \sum_{\alpha \in H_2(M,L):\, \mu(\alpha)=2,\ \partial\alpha=\gamma} \alpha \otimes n_\alpha z^\alpha \in H_2(M, L) \otimes \Lambda^{\mathrm{op}}.$$

Recall that α_0 *is the maximal disc class introduced before Proposition* 3.6 *and* $\lambda \in H_1(L)$ *is the class of an* S^1*-orbit on* L.

Summing over all $\gamma \in H_1(L)$ in (25), we obtain:

COROLLARY 3.21. *Assume that the degeneration formula* (*Conjecture* 3.17) *and Assumption 3.18 hold for* (M, L). *Then we have*

$$z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW \rangle + \widetilde{S}^{(0)} \quad \text{in } \Lambda^{\mathrm{op}}. \tag{26}$$

Via the natural map $H^1(L) \to H^2(M, L)$, an element of $H^1(L)$ can be regarded as a vector field tangent to the fibre of the map $\mathrm{Spec}\Lambda^{\mathrm{op}} \to \mathrm{Spec}\Lambda$. We define the (relative) *Jacobi algebra* of the potential W as

$$\mathrm{Jac}(W) := \Lambda^{\mathrm{op}} / \Lambda^{\mathrm{op}} \langle H^1(L), dW \rangle$$

where $\Lambda^{\mathrm{op}}\langle H^1(L), dW\rangle$ denotes the ideal of Λ^{op} generated by $\langle \varphi, dW \rangle$, $\varphi \in \mathrm{Im}(H^1(L) \to H^2(M, L))$. As a class in the Jacobi algebra, the right-hand side of (26) depends only on the Seidel element $\widetilde{S}$ itself, not on the lift $\widehat{S}^{(2)}$. We can interpret it as the derivative of the bulk-deformed potential $W + t^0$ with respect to $\widetilde{S}$, where t^0 is a co-ordinate on $H^0(M)$. The derivative of $W + t^0$ defines the so-called *Kodaira-Spencer mapping*:

$$\mathrm{KS}\colon H^{\le 2}(M) \otimes \Lambda \to \mathrm{Jac}(W).$$

Then the equation (26) implies

$$\mathrm{KS}(\widetilde{S}) = [z^{\alpha_0}] \qquad \text{in} \quad \mathrm{Jac}(W).$$

REMARK 3.22. Assumption 3.18 (i)–(iii) ensures that the conditions in Conjecture 3.17 hold for all β with $\mu(\beta) \le -2$. Using the formula (23) for β with $\mu(\beta) \le -2$ and $\partial\beta = \lambda$, we find:

$$\sum_{d \in i_* H_2(L)} \mathcal{S}_{\sigma+d} \cap [L] = 0 \quad \text{if} \quad \langle c_1^{\mathrm{vert}}(E), \sigma \rangle \le -1.$$

This supports the validity of Assumption 3.18 (iv).

REMARK 3.23. A more intuitive explanation for the formula (26) is as follows. One can think of the moduli space $\mathcal{M}_{1,1}(\beta)$ of stable holomorphic discs with boundaries in L and with one interior and one boundary marked points as giving a correspondence between M and the free loop space $\mathcal{L}L = \mathrm{Map}(S^1, L)$ of L. This correspondence should give rise to a map (bulk-boundary map)

$$C_*(M) \to C_*(\mathcal{L}L)$$

of chain complexes. One can view this as an analogue of the Kodaira-Spencer map. One can speculate that this map is an intertwiner between the Seidel homomorphism $S\colon C_*(M) \to C_*(M)$ and the map $\mathcal{L}L \to \mathcal{L}L$ induced by the S^1-action.

4. Potential function of a semi-positive toric manifold. Using the degeneration formula (Conjecture 3.17), we compute the potential function of a Lagrangian torus fibre of a semi-positive toric manifold X. This confirms a conjecture (now a theorem [9]) of Chan-Lau-Leung-Tseng [8].

4.1. Toric manifolds. We fix notation on toric geometry. For more details we refer the reader to [2, 11, 12]. For this paper a *toric manifold* X is a smooth projective toric variety, as constructed from the following data.

(a) An integral lattice $N \cong \mathbb{Z}^n$ and its dual $M = \mathrm{Hom}(N, \mathbb{Z})$. We denote by $\langle\cdot,\cdot\rangle$ the natural pairing between N and M.
(b) A fan Σ in $N_{\mathbb{R}} := N \otimes \mathbb{R}$ consisting of a collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$, which is closed under intersections and taking faces.

In order for X to be smooth and projective, we need to assume that Σ is complete, regular and admits a strongly convex piecewise-linear function. Let $\Sigma(1)$ denote the set of 1-cones (rays) in Σ, and we let $b_1, \dots, b_m$ denote integral primitive generators of the 1-cones. The *fan sequence* of X is the exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \longrightarrow N \longrightarrow 0, \tag{27}$$

where the third arrow takes the canonical basis to the primitive generators $b_1, \dots, b_m \in N$ and $\mathbb{L}$ is defined to be the kernel of the third arrow. The dual of the sequence (27) is the *divisor sequence*

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^m \xrightarrow{\ \kappa\ } \mathbb{L}^\vee \longrightarrow 0. \tag{28}$$

The second arrow takes $v \in M$ into the tuple $(\langle b_i, v\rangle)_{i=1}^m$. The third arrow is denoted by $\kappa\colon \mathbb{Z}^m \to \mathbb{L}^\vee$.

The fan sequence tensored with $\mathbb{C}^\times$ gives the exact sequence of tori:

$$1 \longrightarrow \mathbb{G} \longrightarrow (\mathbb{C}^\times)^m \longrightarrow \mathbb{T} \longrightarrow 1$$

with $\mathbb{G} := \mathbb{L} \otimes \mathbb{C}^\times$ and $\mathbb{T} := N \otimes \mathbb{C}^\times$. Let the torus $\mathbb{G}$ act on $\mathbb{C}^m$ by the second arrow $\mathbb{G} \to (\mathbb{C}^\times)^m$. The combinatorics of the fan defines a stability condition of this action as follows. Let $Z(\Sigma)$ denote the union

$$Z(\Sigma) := \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I, \quad \mathbb{C}^I = \{(x_1, \dots, x_m) : x_i = 0 \text{ for } i \notin I\}, \tag{29}$$

where $\mathcal{A}$ is the collection of anti-cones, that is the complements of the subsets of indices that yields a cone in the fan

$$\mathcal{A} := \left\{ I \ : \sum_{i \in \{1,\dots,m\} \setminus I} \mathbb{R}_{\geq 0} b_i \in \Sigma \right\}.$$

The toric variety X is defined as the quotient

$$X := \mathcal{U}_\Sigma / \mathbb{G}; \quad \mathcal{U}_\Sigma := \mathbb{C}^m \setminus Z(\Sigma).$$

The torus $\mathbb{T} = (\mathbb{C}^\times)^m/\mathbb{G}$ acts naturally on X. The toric manifold X contains $\mathbb{T}$ as an open free orbit; X is a compactification of $\mathbb{T}$ along the rays in $\Sigma(1)$.

Each character $\xi : \mathbb{G} \to \mathbb{C}^\times$ defines a line bundle

$$L_\xi := \mathbb{C} \times_{\xi, \mathbb{G}} \mathcal{U}_\Sigma \to X.$$

The correspondence $\xi \mapsto L_\xi$ yields an identification of the Picard group with the character group of $\mathbb{G}$. Thus, we have

$$\mathbb{L}^\vee = \mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times) \cong \mathrm{Pic}(X) \overset{c_1}{\cong} H^2(X; \mathbb{Z}).$$

The i-th toric divisor is given by

$$D_i := \{[x_1, \dots, x_m] \ : \ x_i = 0\} \subset X.$$

The Poincaré dual of D_i is the image $\kappa(e_i) \in \mathbb{L}^\vee \cong H^2(X; \mathbb{Z})$ of the standard basis $e_i \in \mathbb{Z}^m$ under the map κ in (28). By abuse of notation, D_i sometimes also denotes the corresponding cohomology class $\kappa(e_i)$ in $H^2(X; \mathbb{Z})$. We note that $\mathbb{L} = H_2(X; \mathbb{Z})$. The first Chern class $c_1(X)$ of X is given by $D_1 + \dots + D_m$.

The *Kähler cone* C_X of X, the cone consisting of Kähler classes, is given by

$$C_X := \bigcap_{I \in \mathcal{A}} \sum_{i \in I} \mathbb{R}_{>0} \kappa(e_i) \subset \mathbb{L}^\vee \otimes \mathbb{R} = H^2(X; \mathbb{R}).$$

The cone C_X is nonempty if and only if X is projective. Set $r := m - n$. We choose a nef integral basis $p_1, \dots, p_r$ of $H^2(X; \mathbb{Z})$, that is an integral basis such that $p_a \in \overline{C}_X$ for all $a = 1, \dots, r$. Then we write the toric divisor classes as

$$D_j = \kappa(e_j) = \sum_{a=1}^r m_{aj} p_a, \tag{30}$$

for some integer matrix (m_{aj}). The *Mori cone* $\mathrm{NE}(X) \subset H_2(X, \mathbb{R})$ is the dual of the cone $\overline{C}_X$. We set $\mathrm{NE}(X)_\mathbb{Z} := \mathrm{NE}(X) \cap H_2(X; \mathbb{Z})$.

The toric manifold X can be alternatively defined as a symplectic quotient. Let $\mathbb{G}_{\mathbb{R}} \cong (S^1)^r$ be the maximal compact subgroup in $\mathbb{G}$. The $\mathbb{G}_{\mathbb{R}}$-action on $\mathbb{C}^m$ is generated by the moment map

$$\phi\colon \mathbb{C}^m \to \mathfrak{g}_{\mathbb{R}}^{\vee}, \quad \phi(x_1, \ldots, x_m) = \kappa(|x_1|^2, \ldots, |x_m|^2)$$

where $\kappa\colon \mathbb{R}^m \to \mathbb{L}^{\vee} \otimes \mathbb{R}$ is the map in the divisor sequence (28) tensored with $\mathbb{R}$. For any Kähler class $\omega \in C_X$, we have a diffeomorphism ([2, 22])

$$\phi^{-1}(\omega)/\mathbb{G}_{\mathbb{R}} \cong X\,.$$

The left-hand side is a symplectic quotient and is equipped with a reduced symplectic form. The cohomology *class* of the reduced symplectic form coincides with ω; by abuse of notation we let ω also denote the reduced symplectic *form*.

Let $\mathbb{T}_{\mathbb{R}} \cong (S^1)^n$ be the maximal compact subgroup of $\mathbb{T}$. The $\mathbb{T}_{\mathbb{R}}$-action on the symplectic toric manifold (X, ω) admits a moment map:

$$\begin{aligned} &\Phi_\omega\colon X \longrightarrow \kappa^{-1}(\omega)\,, \\ &\Phi_\omega([x_1, \ldots, x_m]) = (|x_1|^2, \ldots, |x_m|^2) \quad \text{with} \quad (x_1, \ldots, x_m) \in \phi^{-1}(\omega)\,, \end{aligned}$$

where the affine subspace $\kappa^{-1}(\omega) \subset \mathbb{R}^m$ can be identified with $M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathfrak{t}_{\mathbb{R}}^{\vee}$ up to translation. The image of the moment map Φ_ω is the convex polytope:

$$\begin{aligned} P(\omega) &= \left\{(t_1, \ldots, t_m) \in \mathbb{R}^m \;:\; t_i \geq 0,\ \kappa(t_1, \ldots, t_m) = \omega\right\} \\ &\cong \{v \in M_{\mathbb{R}} \;:\; \langle b_i, v\rangle \leq -c_i,\ i = 1, \ldots, m\}\,. \end{aligned}$$

In the second line, we took a lift $(c_1, \ldots, c_m)$ of ω (such that $\omega = \kappa(c_1, \ldots, c_m)$) to identify $\kappa^{-1}(\omega)$ with $M_{\mathbb{R}}$. This is called *momentum polytope*. The facet $F_i \subset P$ of $P(\omega)$ normal to $b_i \in N$ corresponds to the toric divisor $D_i = \Phi_\omega^{-1}(F_i) \subset X$.

4.2. Potential function of a Lagrangian torus fibre. Cho-Oh [10] calculated potentials of Lagrangian torus fibres for Fano toric manifolds and matched them up with mirror Landau-Ginzburg potentials of Givental and Hori-Vafa. Fukaya-Oh-Ohta-Ono [15] studied potentials for general symplectic toric manifolds. Chan [6], Chan-Lau [7] and Chan-Lau-Leung-Tseng [8, 9] have studied the potential functions for semi-positive toric manifolds by establishing an equality between open and closed Gromov-Witten invariants.

Let X be a toric manifold in the previous section. Every free $\mathbb{T}_{\mathbb{R}}$-orbit in X is a fibre of the moment map $\Phi_\omega\colon X \to P(\omega)$ of an interior point in $P(\omega)$, and vice versa. We call it a *Lagrangian torus fibre* of X. For a Lagrangian torus fibre L, we have a homotopy exact sequence:

(31) $$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2(X, L) \xrightarrow{\ \partial\ } \pi_1(L) \longrightarrow 0\,.$$

Let $\beta_i \in \pi_2(X, L)$ denote the class represented by the holomorphic disc $u_i\colon \mathbb{D} \to X$:

(32) $$u_i(z) = [c_1, \ldots, c_{i-1}, c_i z, c_{i+1}, \ldots, c_m]\,, \quad |z| \leq 1$$

where $[c_1, \ldots, c_m] \in X$ is a point on the Lagrangian L (thus $c_i \neq 0$ for all i). The class β_i intersects with toric divisors as

$$\beta_i \cdot D_j = \delta_{ij} .$$

The relative homotopy group $\pi_2(X, L)$ is an abelian group freely generated by the classes $\beta_1, \ldots, \beta_m$ and the toric divisors $D_1, \ldots, D_m$ define a dual basis of $H^2(X, L)$. Under the identification:

$$\pi_2(X) \cong H_2(X; \mathbb{Z}) \cong \mathbb{L}, \quad \pi_2(X, L) \cong H_2(X, L; \mathbb{Z}) \cong \mathbb{Z}^m, \quad \pi_1(L) \cong N$$

the exact sequence (31) above can be identified with the fan sequence (27), i.e. $\partial\beta_i = b_i$. The *Maslov index*

$$\mu \colon \pi_2(X, L) \longrightarrow \mathbb{Z}$$

is given by the intersection with $2(D_1 + \cdots + D_m) \in H^2(X, L)$ [10, Theorem 5.1].

We consider the potential function (Definition 2.1) of a Lagrangian torus fibre $L \subset X$. As before, let $\mathcal{M}_1(\beta)$ denote the moduli space of bordered stable maps to (X, L) in the class $\beta \in \pi_2(X, L)$ with one boundary marked point.

PROPOSITION 4.1. *Suppose that $c_1(X)$ is semi-positive. Then $\mathcal{M}_1(\beta)$ is empty for all β with $\mu(\beta) \leq 0$. If $\mathcal{M}_1(\beta)$ is non-empty for β with $\mu(\beta) = 2$, then $\beta = \beta_i + d$ for some i and $d \in \mathrm{NE}(X)_{\mathbb{Z}}$ such that $\langle c_1(X), d\rangle = 0$.*

PROOF. Let β be a class of a bordered stable map to (X, L). By the classification of holomorphic discs by Cho-Oh [10], we find that β is of the form:

$$\beta = \sum_{i=1}^{m} k_i \beta_i + d \tag{33}$$

for some $k_i \geq 0$ and $d \in \mathrm{NE}(X)_{\mathbb{Z}}$. Here $\sum_{i=1}^m k_i\beta_i$ is the degree of disc components and d is the degree of sphere bubbles. Hence $\mu(\beta) = 2\sum_{i=1}^m k_i + 2\langle c_1(X), d\rangle \geq 0$. We claim that $(k_1, \ldots, k_m) = 0$ implies $\mu(\beta) \geq 4$. If $(k_1, \ldots, k_m) = 0$, a bordered stable map of class β is the union of a *constant* disc and sphere bubbles. In this case, at least one non-trivial sphere component has to touch L. Let d_1 be the degree of a non-trivial sphere component touching L and let d_2 be the degree of the remaining sphere bubbles. Then $d = d_1 + d_2$ with $d_1, d_2 \in \mathrm{NE}(X)_{\mathbb{Z}}$. Since D_i is disjoint from L, we have $\langle D_i, d_1\rangle \geq 0$. Since $d_1 \neq 0$, we have $\sum_{i=1}^m \langle D_i, d_1\rangle \geq 1$. Also it is impossible that $\sum_{i=1}^m \langle D_i, d_1\rangle = 1$ since d_1 gives the relation $\sum_{i=1}^m \langle D_i, d_1\rangle b_i = 0$ in N. Thus

$$\mu(\beta) = 2\langle c_1(X), d\rangle \geq 2\sum_{i=1}^{m} \langle D_i, d_1\rangle \geq 4 .$$

The claim follows. Consequently, $\mu(\beta) \leq 2$ implies $(k_1, \ldots, k_m) \neq 0$. The proposition follows easily. □

In particular, the potential function of a Lagrangian torus fibre (Definition 2.1) is well-defined for a semi-positive toric manifold.

REMARK 4.2. Fukaya-Oh-Ohta-Ono [15] defined the potential function of a Lagrangian torus fibre even without semi-positivity assumption. They defined virtual cycles and open Gromov-Witten invariants $n_\beta \in \mathbb{Q}$ for all β with $\mu(\beta) = 2$ using $\mathbb{T}_\mathbb{R}$-equivariant perturbations, see [15, Lemmata 11.2, 11.5, 11.6, 11.7]. In general, since every effective stable disc class β is of the form (33), the potential W lies in the completed group ring:

$$\mathbb{Q}[\![(\mathbb{Z}_{\geq 0})^m + \mathrm{NE}(X)_\mathbb{Z}]\!] \subset \Lambda^{\mathrm{op}}$$

where $\mathrm{NE}(X)_\mathbb{Z}$ is regarded as a subset of $\mathbb{Z}^m$ via the second arrow in the fan sequence (27). Notice that $(\mathbb{R}_{\geq 0})^m + \mathrm{NE}(X)$ is a strictly convex cone.

EXAMPLE 4.3 ([10]). When $\beta = \beta_i$, the moduli space $\mathcal{M}_1(\beta_i)$ consists of holomorphic discs of the form (32) and $\mathrm{ev}\colon \mathcal{M}_1(\beta_i) \cong L$; moreover all such discs are Fredholm regular [10, Theorem 6.1]. Therefore we have $n_{\beta_i} = 1$.

We write

$$z^\beta = z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m} \in \mathbb{Q}[H_2(X, L; \mathbb{Z})]$$

for $\beta = k_1\beta_1 + \cdots + k_m\beta_m$. Also we write

$$q^d = q_1^{\langle p_1, d\rangle} q_2^{\langle p_2, d\rangle} \cdots q_r^{\langle p_r, d\rangle} \in \mathbb{Q}[H_2(X; \mathbb{Z})]$$

for $d \in H_2(X; \mathbb{Z})$, where $p_1, \ldots, p_r$ is the nef integral basis of $H^2(X; \mathbb{Z}) \cong \mathbb{L}^\vee$ we chose in §4.1. Note that we have a natural inclusion of the group rings:

$$\mathbb{Q}[H_2(X; \mathbb{Z})] \hookrightarrow \mathbb{Q}[H_2(X, L; \mathbb{Z})].$$

By this we identify q^d with z^d; in co-ordinates:

$$q^d = z^d = z_1^{\langle D_1, d\rangle} z_2^{\langle D_1, d\rangle} \cdots z_m^{\langle D_m, d\rangle} \quad \text{or} \quad q_a = \prod_{i=1}^m z_i^{m_{ai}} \tag{34}$$

where (m_{ai}) is the divisor matrix in (30). Using these notations and Proposition 4.1, we can write the potential function of (X, L) in the following form when $c_1(X)$ is semi-positive.

DEFINITION 4.4. Let X be a semi-positive toric manifold. We present the potential function W of a Lagrangian torus fibre in the form:

$$W = w_1 + \cdots + w_m$$

where $w_i = f_i(q) z_i$ and

$$f_i(q) = \sum_{d \in \mathrm{NE}(X)_\mathbb{Z} \colon \langle c_1(X), d\rangle = 0} n_{\beta_i + d} q^d .$$

We call $f_i(q)$ the *correction term*. This decomposition of W is parallel to Definition 2.2.

Note that we have $f_i(q) = 1 + O(q)$ by Example 4.3. The correction term $f_i(q)$ was denoted by $1 + \delta_i(q)$ in [8]. When X is Fano, all the correction terms are 1 and

$$W = z_1 + \cdots + z_m \,.$$

This is the result of Cho-Oh [10]. By the *fan polytope*, we mean the convex hull of the ray vectors $b_1, \dots, b_m \in N_{\mathbb{R}}$. In the proof of [8, Corollary 4.12], Chan-Lau-Leung-Tseng showed the following:

PROPOSITION 4.5 (Chan-Lau-Leung-Tseng [8]). *Let $f_i(q)$ be the correction terms of the potential of a semi-positive toric manifold X. If the vector b_i is a vertex of the fan polytope of X, then $f_i(q) = 1$.*

4.2.1. Open-closed moduli space. We explain that the potential W of a Lagrangian torus fibre can be interpreted as a formal function on the *open-closed moduli space* introduced below.

The *closed moduli space* $\mathfrak{M}_{\mathrm{cl}}$ of X is defined to be:

$$\mathfrak{M}_{\mathrm{cl}} = \{\exp(-\omega + iB) \in \mathbb{L}^\vee \otimes \mathbb{C}^\times \ : \ \omega, B \in \mathbb{L}^\vee \otimes \mathbb{R},\ \omega \in C_X\}\,.$$

This is also called the *complexified Kähler moduli space*. The nef basis $p_1, \dots, p_r$ of $\mathbb{L}^\vee \cong H^2(X;\mathbb{Z})$ in §4.1 defines $\mathbb{C}^\times$-valued co-ordinates $(q_1, \dots, q_r)$ on $\mathfrak{M}_{\mathrm{cl}} \subset \mathbb{L}^\vee \otimes \mathbb{C}^\times$.

The *open-closed moduli space* $\mathfrak{M}_{\mathrm{opcl}}$ is defined to be the set of triples (q, L, h) such that

- a closed moduli $q = \exp(-\omega + iB) \in \mathfrak{M}_{\mathrm{cl}}$;
- a Lagrangian torus fibre $L = L_\eta = \Phi_\omega^{-1}(\eta)$ at $\eta \in P(\omega)^\circ$;
- a class $h \in H^2(X, L; U(1))$ which maps to $\exp(iB) \in H^2(X; U(1))$.

When the B-field vanishes $B = 0$, the class h defines a $U(1)$-local system on L via the exact sequence:

$$0 \longrightarrow H^1(L; U(1)) \longrightarrow H^2(X, L; U(1)) \longrightarrow H^2(X; U(1)) \longrightarrow 0\,.$$

Let $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m$ be the co-ordinates of η and write $h = (h_1, \dots, h_m)$ using the identification $H^2(X, L; U(1)) \cong (S^1)^m$; and set

$$z_i := \exp(-\eta_i) h_i \,. \tag{35}$$

The parameter $z = (z_1, \dots, z_m)$ here determines $\eta_i \in \mathbb{R}$, $h_i \in S^1$ by polar decomposition; then η determines ω by the condition $\eta \in P(\omega)^\circ$ (as $\omega = \kappa(\eta)$) and h determines $\exp(iB)$. Thus z determines a point of $\mathfrak{M}_{\mathrm{opcl}}$. We have:

$$\mathfrak{M}_{\mathrm{opcl}} \cong \left\{ z = (z_1, \dots, z_m) \in (\mathbb{C}^\times)^m \ : \ |z_i| < 1 \text{ for all } i,\ \kappa_{\mathbb{C}^\times}(z) \in \mathfrak{M}_{\mathrm{cl}} \right\}$$

where $\kappa_{\mathbb{C}^\times} : (\mathbb{C}^\times)^m \to \mathbb{L}^\vee \otimes \mathbb{C}^\times$ is the third arrow of the divisor sequence (28) tensored with $\mathbb{C}^\times$. A point $z = (z_1, \dots, z_m)$ of the right-hand side parametrizes

- a closed moduli $q = \exp(-\omega + iB) = \kappa_{\mathbb{C}^\times}(z)$;
- a Lagrangian torus fibre $L = L_\eta$ at $\eta = (-\log|z_1|, \dots, -\log|z_m|) \in P(\omega)^\circ$;
- a class $h = (z_1/|z_1|, \dots, z_m/|z_m|) \in H^2(X, L_\eta)$ which is a lift of $\exp(iB)$.

We regard W as a formal function on $\mathfrak{M}_{\mathrm{opcl}}$ via these co-ordinates $(z_1, \dots, z_m)$. The open-closed moduli is fibred over $\mathfrak{M}_{\mathrm{cl}}$:

$$\pi : \mathfrak{M}_{\mathrm{opcl}} \to \mathfrak{M}_{\mathrm{cl}}, \quad z \mapsto \kappa_{\mathbb{C}^\times}(z).$$

By pulling-back the co-ordinates $q_1, \dots, q_r$ by π, we obtain the same relation between z_i and q_a as in (34). The fibre $\mathfrak{M}_{\mathrm{opcl},q} = \pi^{-1}(q)$ has the structure of an $(M_{\mathbb{R}}/M) \cong (S^1)^n$-bundle over $P(\omega)^\circ$ via the map:

$$\mathfrak{M}_{\mathrm{opcl},q} \to P(\omega)^\circ, \quad (z_1, \dots, z_m) \mapsto \eta = (-\log|z_1|, \dots, -\log|z_m|).$$

This is a torus fibration dual to the moment map $\Phi_\omega \colon X \to P(\omega)$; we can view it as a mirror of (X, q).

PROPOSITION 4.6. *Via the co-ordinates $(z_1, \dots, z_m)$ on $\mathfrak{M}_{\mathrm{opcl}}$, the potential function of a Lagrangian torus fibre is identified with the following formal sum of functions on $\mathfrak{M}_{\mathrm{opcl}}$:*

$$W(q, L, h) = \sum_{\beta \in \pi_2(X,L) \,:\, \mu(\beta)=2} n_\beta h(\beta) e^{-\int_\beta \omega}$$

where $q = \exp(-\omega + iB)$.

PROOF. For $\beta = \beta_i$, we have (see [10, Theorem 8.1])

$$h(\beta_i) = \frac{z_i}{|z_i|}, \quad \int_{\beta_i} \omega = \eta_i = -\log|z_i|$$

and thus $h(\beta_i)e^{-\int_{\beta_i}\omega} = z_i$ (cf. (35)). Therefore $h(\beta)e^{-\int_\beta \omega} = z^\beta$ for every β. □

REMARK 4.7. When $B = 0$, the term $h(\beta)$ is the holonomy along the loop $\partial\beta \in \pi_1(L)$ of the $U(1)$-local system associated to h. This matches with the usual interpretation. In general, this term cannot be interpreted just as holonomy.

Fukaya-Oh-Ohta-Ono [14, Theorem 2.32] showed that the Jacobi algebra of the potential function restricted to the fibre $\mathfrak{M}_{\mathrm{opcl},q} = \pi^{-1}(q)$ is isomorphic to the quantum cohomology ring of (X, q) in a certain q-adic sense.

4.3. Seidel elements for toric varieties and Givental's mirror transformation. We review our previous computation [21] relating Seidel elements for toric varieties to Givental's mirror transformation [19]. Let X be a toric manifold from §4.1 with $c_1(X)$ semi-positive.

4.3.1. Seidel elements associated to the $\mathbb{C}^\times$-actions fixing toric divisors. For each toric divisor D_j of X, we can associate a $\mathbb{C}^\times$-action ρ_j on X rotating around D_j. It is given by:

$$\rho_j(\lambda) \colon [x_1, \dots, x_m] \longmapsto [x_1, \dots, \lambda^{-1}x_j, \dots, x_m], \quad \lambda \in \mathbb{C}^\times.$$

The toric divisor $D_j = \{x_j = 0\}$ is the maximal fixed component of this action. Let E_j denote the associated bundle of this $\mathbb{C}^\times$-action and let S_j denote the corresponding Seidel element. We also write $S_j = q_0\widetilde{S}_j$ with $\widetilde{S}_j \in QH^*(X)$ following Definition 2.7. Using the Seidel representation (see Remark 2.9), McDuff-Tolman [26] showed the following multiplicative relations in $QH(X)[q^{-d} : d \in \mathrm{NE}(X)_{\mathbb{Z}}]$:

$$\prod_{j=1}^{m} \widetilde{S}_j^{\langle D_j, d\rangle} = q^d \quad \text{for } d \in H_2(X;\mathbb{Z})\,. \tag{36}$$

4.3.2. Givental's mirror theorem. Givental [19] introduced the two cohomology-valued functions

$$I(y,z) = e^{\sum_{i=1}^r p_i \log y_i/z} \sum_{d\in \mathrm{NE}(X)_{\mathbb{Z}}} \prod_{i=1}^{m} \left(\frac{\prod_{k=-\infty}^{0}(D_i + kz)}{\prod_{k=-\infty}^{\langle D_i, d\rangle}(D_i+kz)} \right) y^d$$

$$J(q,z) = e^{\sum_{i=1}^r p_i \log q_i/z} \left(1 + \sum_j \sum_{d\in \mathrm{NE}(X)_{\mathbb{Z}}\setminus\{0\}} \left\langle \frac{\phi_j}{z(z-\psi)} \right\rangle_{0,1,d}^{X} \phi^j q^d \right)$$

called the *I-function* and the *J-function* respectively. Here we used a nef basis $\{p_1,\dots,p_r\} \subset H^2(X)$ in §4.1 and write

$$q^d = q_1^{\langle p_1, d\rangle}\cdots q_r^{\langle p_r, d\rangle}\,, \qquad y^d = y_1^{\langle p_1, d\rangle}\cdots y_r^{\langle p_r, d\rangle}\,,$$

and $\{\phi_j\}$ and $\{\phi^j\}$ are mutually dual basis of $H^*(X)$. The variables $y = (y_1,\dots,y_r)$ are called *mirror co-ordinates*, i.e. co-ordinates of the complex moduli of the mirror Landau-Ginzburg model. Givental [19] showed the following *mirror theorem*:

THEOREM 4.8 ([19]). *We have $I(y,z) = J(q,z)$ under a change of coordinates of the form $\log q_i = \log y_i + g_i(y)$, $i = 1,\dots,r$, $g_i(y) \in \mathbb{Q}[\![y_1,\dots,y_r]\!]$ with $g_i(0) = 0$. The functions $g_i(y)$ here are uniquely determined by the asymptotics:*

$$I(y,z) = e^{\sum_{i=1}^r p_i \log y_i/z}\left(1 + \sum_{i=1}^{r} g_i(y)\frac{p_i}{z} + o(z^{-1})\right).$$

The change of co-ordinates is called *mirror transformation* (or *mirror map*).

4.3.3. Batyrev elements and Seidel elements. In [21], we introduced *Batyrev elements* $\widetilde{D}_j$, $j = 1,\dots,m$. They are defined by

$$\widetilde{D}_j := \sum_{a=1}^{r} m_{aj}\tilde{p}_a\,, \quad \tilde{p}_a := \sum_{b=1}^{r} \frac{\partial \log q_b}{\partial \log y_a} p_b\,.$$

Note that $\widetilde{D}_j$ is an element corresponding to the vector field $\sum_{a=1}^r m_{aj} y_a \partial/\partial y_a$ whereas the genuine divisor class D_j corresponds to the vector field $\sum_{a=1}^r m_{aj} q_a \partial/\partial q_a$ (see (30)). Batyrev elements are determined by, and determine the Jacobi matrix $(\partial \log q_b/\partial \log y_a)$ of the mirror transformation. Using Givental's mirror theorem, we find that the Batyrev elements satisfy the multiplicative relations (see [21, Proposition 3.8])

$$\prod_{j=1}^{m} \widetilde{D}_j^{\langle D_j, d\rangle} = y^d \quad d \in H_2(X;\mathbb{Z})$$

in the quantum cohomology ring. These are very similar to the multiplicative relations (36) of Seidel elements, but note that co-ordinates q are replaced with mirror co-ordinates y. Moreover, the Batyrev elements satisfy the following *linear relations*:

$$\sum_{i=1}^{m} c_j \widetilde{D}_j = 0 \quad \text{whenever} \quad \sum_{i=1}^{m} c_j D_j = 0\,. \tag{37}$$

The linear relations are obvious from the definition. These multiplicative and linear relations show that $\widetilde{D}_j$ satisfy the relations of Batyrev's quantum ring [5]. It turns out that the Seidel elements are multiples of the Batyrev elements.

THEOREM 4.9 ([21, Theorem 1.1]). *Let $g_0^{(j)}(y)$ be the following hypergeometric series in mirror co-ordinates:*

$$g_0^{(j)}(y_1, \dots, y_r) = \sum_{\substack{\langle c_1(X), d\rangle = 0,\, \langle D_j, d\rangle < 0 \\ \langle D_i, d\rangle \geq 0 \text{ for all } i \neq j}} \frac{(-1)^{\langle D_j, d\rangle} \left(-\langle D_j, d\rangle - 1\right)!}{\prod_{i \neq j} \langle D_i, d\rangle!} y^d. \tag{38}$$

Then under the mirror transformation we have

$$\widetilde{S}_j = \exp\left(-g_0^{(j)}(y)\right)\widetilde{D}_j\,.$$

Conversely, one can recover the Batyrev elements from the Seidel elements in the following way.

THEOREM 4.10 ([21, Theorem 1.2]). *Given the Seidel elements $\widetilde{S}_1, \dots, \widetilde{S}_m$, the Batyrev elements $\widetilde{D}_j \in H^*(X) \otimes \mathbb{Q}[\![q_1, \dots, q_r]\!]$, $j = 1, \dots, m$ are uniquely characterized by the following conditions:*

(a) $\widetilde{D}_j = H_j \widetilde{S}_j$ *for some* $H_j \in \mathbb{Q}[\![q_1, \dots, q_r]\!]$;
(b) $\widetilde{D}_j = \widetilde{S}_j$ *if* b_j *is a vertex of the fan polytope*;
(c) $\widetilde{D}_j$ *satisfy the linear relations* (37).

In particular, the Seidel elements determine the mirror transformation $q \mapsto y$ and the functions $g_0^{(j)}(y)$.

4.4. Correction terms of potential functions and Seidel elements. Chan-Lau-Leung-Tseng [8] gave a conjecture relating the correction terms of the potential function and the Seidel elements for a semi-positive toric manifold.

CONJECTURE 4.11 ([8, Conjecture 5.2]). *For a semi-positive toric manifold, the correction term $f_j(q)$ of the potential function* (*Definition* 4.4) *coincides with* $\exp(g_0^{(j)}(y))$ *in Theorem 4.9 under mirror transformation.*

Originally Chan-Lau-Leung-Tseng [8] proved this conjecture under the convergence assumption for W using an isomorphism [14] of Jacobi ring and quantum cohomology. Recently they gave an alternative proof [9] which does not require the convergence assumption. They identified open Gromov-Witten invariants with certain closed Gromov-Witten invariants of the associated bundle E_i' given by the inverse $\mathbb{C}^\times$-action ρ_i^{-1}. They used the fact that a bordered stable map to (M, L) with boundary class $b_i \in N \cong H_1(L)$ can be completed to a holomorphic sphere in the associated bundle E'_i. This is closely related to the fact that the central fibre $\overline{\mathcal{E}}_0$ of the closing in §3.1 is the union of the two associated bundles E and E' which correspond to mutually inverse $\mathbb{C}^\times$-actions.

4.5. Degeneration formula for toric manifolds.

PROPOSITION 4.12. *Assumption* 3.18 *holds for a pair* (X, L) *equipped with the* $\mathbb{C}^\times$*-action* ρ_j *around the prime toric divisor* D_j *we considered in* §4.3.

PROOF. The statement (i) is shown in Proposition 4.1 and (ii), (iii) are obvious. To verify the statement (iv), it is enough to show that every stable map $u\colon C \to E_j$ representing a class $\sigma \in H_2^{\mathrm{sec}}(E_j)$ with $\langle c_1^{\mathrm{vert}}(E_j), \sigma\rangle = -1$ is contained in $\bigcup_{i=1}^m \widehat{D}_i$, where

$$\widehat{D}_i = D_i \times (\mathbb{C}^2 \setminus \{0\})\big/\mathbb{C}^\times$$

is a toric divisor of E_j. Let $C = \bigcup C_\alpha$ be an irreducible decomposition of C. If $u_*[C_\alpha]$ is a section class, we have $\langle c_1^{\mathrm{vert}}(E_j), u_*[C_\alpha]\rangle \geq -1$ by (3) and the semi-positivity of $c_1(X)$. If $u_*[C_\alpha]$ is not a section class, $u(C_\alpha)$ is contained in a fibre X and we have $\langle c_1^{\mathrm{vert}}(E_j), u_*[C_\alpha]\rangle = \langle c_1(X), u_*[C_\alpha]\rangle \geq 0$ again by the semi-positivity. Since $\langle c_1^{\mathrm{vert}}(E_j), \sigma\rangle = -1$, we have

$$\langle c_1^{\mathrm{vert}}(E_j), u_*[C_\alpha]\rangle = \begin{cases} -1 & \text{if } u(C_\alpha) \text{ is a section}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Suppose that $u(C) \not\subset \bigcup_{i=1}^m \widehat{D}_i$. Then we can find a component C_α such that $u(C_\alpha)$ is not a point and $u(C_\alpha) \not\subset \bigcup_{i=1}^m \widehat{D}_i$. Then $\langle \widehat{D}_i, u_*[C_\alpha]\rangle \geq 0$ for all i. Note that $\sum_{i=1}^m \widehat{D}_i$ is the Poincaré dual of $c_1^{\mathrm{vert}}(E_j)$. By the above calculation we see that $\langle \widehat{D}_i, u_*[C_\alpha]\rangle = 0$ for all i and $u(C_\alpha)$ is contained in a certain fibre X. Then $\langle D_i, u_*[C_\alpha]\rangle = 0$ for all i. A homology class $d \in H_2(X)$ satisfying $\langle D_i, d\rangle = 0$ for all i is zero. This is a contradiction. □

Recall from Remark 4.2 that the potential function $W = W(z_1, \ldots, z_m)$ of a toric manifold X is an element of

$$R := \mathbb{Q}[\![\mathrm{NE}(X)_{\mathbb{Z}} + (\mathbb{Z}_{\geq 0})^m]\!] \subset \Lambda^{\mathrm{op}}\,.$$

We also set

$$K := \mathbb{Q}[\![\mathrm{NE}(X)_{\mathbb{Z}}]\!] \subset \Lambda\,.$$

Then R is a K-algebra (cf. (34)). For $f \in R$, we write (following notation in Theorem 3.20):

$$df = \left(z_1\frac{\partial f}{\partial z_1}, \ldots, z_m\frac{\partial f}{\partial z_m}\right) \in \mathbb{Z}^m \otimes R \cong H_2(X, L) \otimes R\,.$$

In other words,

$$dz^\beta = \beta \otimes z^\beta$$

for $\beta \in H_2(X, L)$.

We apply Theorem 3.20 to the $\mathbb{C}^\times$-action ρ_j rotating around D_j. Note that the k-th term w_k of the potential W in Definition 4.4 corresponds to the boundary class $b_k \in N \cong H_1(L)$

and $w_k = W_{b_k}$ in the notation of Definition 2.2. Since the Seidel element $\widetilde{S}_j$ in §4.3 belongs to $H^2(X) \otimes K$, we have $\widetilde{S}_j^{(0)} = 0$ and $\widetilde{S}_j = \widetilde{S}_j^{(2)}$. By Proposition 4.12, we can define the lift

$$\widehat{S}_j \in H^2(X, L) \otimes K$$

of $\widetilde{S}_j = \widetilde{S}_j^{(2)}$ as in Definition 3.19. The class λ of an S^1-orbit on L is $-b_j \in H_1(L)$ and the maximal disc class α_0 is β_j. Hence we obtain:

THEOREM 4.13. *Assume that the degeneration formula (Conjecture 3.17) holds for (X, L) equipped with the $\mathbb{C}^\times$-action ρ_j around the toric divisor D_j (see §4.3). Then we have*

$$\langle \widehat{S}_j, dw_k \rangle = \delta_{jk} z_j \, . \tag{39}$$

In particular, we have $\langle \widehat{S}_j, dW \rangle = z_j$.

4.5.1. Example. Consider the second Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$, a compactification of $\mathcal{O}_{\mathbb{P}^1}(-2)$. The divisor matrix (30) is:

$$(m_{ai}) = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The column vectors give toric divisors classes D_1, D_2, D_3, D_4. Here D_1 is the ∞-section, D_2 is the zero-section (-2 curve) and D_3, D_4 are fibres. The potential function has been calculated by Auroux [4], Fukaya-Oh-Ohta-Ono [17] and Chan-Lau [7]:

$$W = z_1 + (1 + q_1) z_2 + z_3 + z_4 \, .$$

Therefore we have

$$\begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dw_4 \end{bmatrix} = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & (1 - q_1) z_2 & q_1 z_2 & q_1 z_2 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{bmatrix}$$

where we used $q_1 = z_2^{-2} z_3 z_4$ (see (34)) and $d(q_1 z_2) = [0, -q_1 z_2, q_1 z_2, q_1 z_2]$. Assuming the degeneration formula (39), we obtain

$$\left[\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4\right] = [D_1, D_2, D_3, D_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-q_1} & -\frac{q_1}{1-q_1} & -\frac{q_1}{1-q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is compatible with the calculations of $\widetilde{S}_j$ by McDuff-Tolman [26] and González-Iritani [21].

4.6. Kodaira-Spencer map. Recall from Definition 4.4 that $w_i = f_i(q)z_i$ for some $f_i(q) \in K$. We have (using (34))

$$z_i \frac{\partial w_j}{\partial z_i} = \left(\delta_{ij} + z_i \frac{\partial f_j(q)}{\partial z_i}\right) z_j = \left(\delta_{ij} + \sum_{a=1}^{r} m_{ai} q_a \frac{\partial f_j}{\partial q_a}(q)\right) z_j \in K z_j .$$

Therefore we have an isomorphism of K-modules:

$$\mathfrak{ks}\colon H^2(X, L) \otimes K \xrightarrow{\cong} \bigoplus_{j=1}^{m} K z_j , \quad D_i \longmapsto \left(z_i \frac{\partial w_1}{\partial z_i}, \dots, z_i \frac{\partial w_m}{\partial z_i}\right) .$$

The degeneration formula (39) says that $\mathfrak{ks}(\widehat{S}_i) = z_i$. For $\varphi \in H^1(L) = M$, we have

$$\begin{aligned}\mathfrak{ks}(\delta\varphi) &= \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle z_i \frac{\partial w_j}{\partial z_i} = \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle \left(z_i \delta_{ij} f_j(q) + z_i z_j \frac{\partial f_j(q)}{\partial z_i}\right) \\ &= \bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle w_j \in \bigoplus_{j=1}^{m} K z_j ,\end{aligned}$$

where $\delta\colon H^1(L) \cong M \to H^2(X, L) \cong \mathbb{Z}^m$ is a coboundary map. Hence $\mathfrak{ks}$ induces an isomorphism

$$\mathrm{ks}\colon H^2(X) \otimes K \xrightarrow{\cong} \bigoplus_{j=1}^{m} K z_j \Big/ \left\langle \textstyle\bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle w_j \ : \ \varphi \in M \right\rangle_K .$$

This satisfies $\mathrm{ks}(\widetilde{S}_i) = [z_i]$. Set $B_j := f_j(q)\widetilde{S}_j$, $j = 1, \dots, m$. Then $\mathrm{ks}(B_j) = f_j(q)[z_j] = [w_j]$, $j = 1, \dots, m$ satisfy the linear relations

$$\sum_{j=1}^{m} \langle \varphi, b_j \rangle [w_j] = 0$$

for all $\varphi \in M$. Consequently,

- $B_j = f_j(q)\widetilde{S}_j$;
- $f_j(q) = 1$ if b_j is a vertex of the fan polytope (Proposition 4.5);
- B_j, $j = 1, \dots, m$ satisfy the linear relations (by the injectivity of ks).

By the characterization of the Batyrev elements (see Theorem 4.10), we know that $B_j = \widetilde{D}_j$, i.e. $f_j(q) = \exp(g_0^{(j)}(y))$. This shows the conjecture of Chan-Lau-Leung-Tseng:

THEOREM 4.14. *Assume that the degeneration formula (Conjecture 3.17) holds for (X, L) equipped with the $\mathbb{C}^\times$-actions ρ_j, $j = 1, \dots, m$ in §4.3. Then Conjecture 4.11 holds.*

REMARK 4.15. Via the natural map $\bigoplus_{j=1}^{m} K z_j \to R$, the map ks induces the so-called Kodaira-Spencer map (cf. the discussion at the end of §3.3.3):

$$\mathrm{KS}\colon H^2(X) \otimes K \to \mathrm{Jac}(W)$$

where the Jacobi algebra $\mathrm{Jac}(W)$ is defined to be

$$\mathrm{Jac}(W) := R/R\langle H^1(L), dW\rangle\,.$$

Then we have $\mathrm{KS}(\widetilde{S}_i) = [z_i]$ and $\mathrm{KS}(\widetilde{D}_i) = [w_i]$. In other words, the Seidel elements are the inverses of $[z_i]$ and the Batyrev elements are the inverses of $[w_i]$.

4.7. Consistency check: computing equivariant Seidel elements. Here we give a consistency check concerning Chan-Lau-Leung-Tseng Conjecture 4.11 and our degeneration formula (39). We calculate the lifts $\widehat{S}_j$ of Seidel elements assuming Conjecture 4.11 and (39) and see that the result is compatible with our previous calculation [21]. The lifts $\widehat{S}_j$ here should be viewed as the $\mathbb{T}$-equivariant Seidel elements since $H^2_{\mathbb{T}}(X) \cong H^2(X, L)$.

LEMMA 4.16. *Suppose that Conjecture 4.11 holds. Then* $w_i = f_i(q)z_i$, $i = 1, \dots, m$ *satisfy the multiplicative relation*

$$\prod_{j=1}^m w_j^{\langle D_j, d\rangle} = y^d \quad \textit{for all } \ d \in H_2(X;\mathbb{Z})\,.$$

In other words, $y_a = \prod_{j=1}^m w_j^{m_{aj}}$, $a = 1, \dots, r$.

PROOF. Recall that the Seidel and the Batyrev elements satisfy the multiplicative relations with respect to the quantum product (§4.3):

$$\prod_{j=1}^m \widetilde{D}_j^{\langle D_j, d\rangle} = y^d\,, \quad \prod_{j=1}^m \widetilde{S}_j^{\langle D_j, d\rangle} = q^d\,.$$

Hence we have

$$\prod_{j=1}^m f_j(q)^{\langle D_j, d\rangle} = y^d/q^d\,.$$

Therefore

$$\prod_{j=1}^m w_j^{\langle D_j, d\rangle} = \prod_{j=1}^m \left(f_j(q)^{\langle D_j, d\rangle} z_j^{\langle D_j, d\rangle}\right) = (y^d/q^d)\cdot q^d = y^d\,.$$

□

THEOREM 4.17. *Assume Conjecture* 4.11 *and the degeneration formula* (39)*. The lifts* $\widehat{S}_j$ *of the Seidel elements are given by*

$$\widehat{S}_j = e^{-g_0^{(j)}(y)}\left(D_j - \sum_{i=1}^m D_i \sum_{\substack{c_1(X)\cdot d=0,\, D_i\cdot d<0,\\ D_k\cdot d\geq 0 \text{ for all } k\neq i.}} (-1)^{\langle D_i, d\rangle}\left\langle D_j, d\right\rangle \frac{(-\langle D_i, d\rangle - 1)!}{\prod_{k\neq i}\langle D_k, d\rangle!} y^d\right)$$

under the mirror transformation.

PROOF. Note that $(dw_1, \dots, dw_m)^{\mathrm{T}}$ can be viewed as the Jacobi matrix between the two co-ordinate systems $(w_1, \dots, w_m)$ and $(\log z_1, \dots, \log z_m)$ on the open-closed moduli space. The degeneration formula (39) implies that $(z_1^{-1}\widehat{S}_1, \dots, z_m^{-1}\widehat{S}_m)$ is the inverse Jacobi matrix, i.e.

$$z_j^{-1}\widehat{S}_j = \sum_{i=1}^{m} \frac{\partial \log z_i}{\partial w_j} D_i = w_j^{-1} \sum_{i=1}^{m} \frac{\partial \log z_i}{\partial \log w_j} D_i$$

in $H^2(X, L)$. Assuming Conjecture 4.11, we have $\log z_i = \log w_i - g_0^{(i)}(y)$. Hence

$$\begin{aligned}\widehat{S}_j &= \frac{z_j}{w_j} \sum_{i=1}^{m} \left(\delta_{ij} - w_j \frac{\partial g_0^{(i)}}{\partial w_j} \right) D_i \\ &= \exp\big(-g_0^{(j)}(y)\big) \left(D_j - \sum_{i=1}^{m} \sum_{a=1}^{r} m_{aj} y_a \frac{\partial g_0^{(i)}}{\partial y_a} D_i \right).\end{aligned}$$

In the second line, we used Lemma 4.16. The conclusion follows. □

Note that we did not use the lifts $\widehat{S}_j$ of the Seidel elements (but used only the original Seidel elements $\widetilde{S}_j$) in the proof of Theorem 4.14.

REMARK 4.18. This result is compatible with the calculation of $\widetilde{S}_j$ in our previous paper [21]. Note however that the formula in [21, Lemma 3.17] contains a mistake. It occurred from an erroneous cancellation between the factors $\langle D_j, d\rangle$ in the numerator and $\langle D_j, d\rangle!$ in the denominator.

REMARK 4.19. It is not difficult to generalize the computation in [21] to the $\mathbb{T}$-equivariant setting and to check the above computation of $\widehat{S}_j$ without using Conjecture 4.11 and the degeneration formula (39). Since Chan-Lau-Leung-Tseng's conjecture 4.11 was proved by themselves [9], it follows that the degeneration formula (39) holds true in toric case.

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