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NOTES ON 'INFINITESIMAL DERIVATIVE OF THE BOTT CLASS AND THE SCHWARZIAN DERIVATIVES'

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Abstract. The derivatives of the Bott class and those of the Godbillon–Vey class with respect to infinitesimal deformations of foliations, called infinitesimal derivatives, are known to be represented by a formula in the projective Schwarzian derivatives of holonomies [3], [1]. It is recently shown that these infinitesimal derivatives are represented by means of coefficients of transverse Thomas–Whitehead projective connections [2]. We will show that the formula can be also deduced from the latter representation.

Introduction. Given infinitesimal deformations of foliations, we can define the derivatives of secondary characteristic classes for foliations with respect to them. We call such derivatives *infinitesimal derivatives* for short. Infinitesimal derivatives of the Bott class for transversely holomorphic foliations, and those of the Godbillon–Vey class for real foliations are known to be represented by a formula in the projective Schwarzian derivatives of holonomies (Maszczyk [3] for $q = 1$ and [1] for general q, where q denotes the (complex) codimension of foliations). Both of the proofs consist of honest calculations so that their meanings are difficult to see. It is recently shown that these infinitesimal derivatives can be represented by means of coefficients of transverse Thomas–Whitehead projective connections [2]. We will show that the formula can be also deduced from the latter representation. This shows that the formula is indeed derived from transverse projective structures of foliations which are possibly non-holonomy invariant.

1. Definitions. We largely follow the notations in [2]. In particular, local coordinates in the transversal direction will be $(y^1, \dots, y^q)$ instead of $(z^1, \dots, z^q)$ in [1]. On the other hand, when we need notions related with Čech–de Rham complexes, we modify notations in accordance with [1]. For example, the product of a Čech–de Rham (r, s)-cochain a and a Čech–de Rham (t, u)-cochain b is denoted by $a \cup b$ and is defined by $(a \cup b)_{i_0,\dots,i_{r+t}} = (-1)^{st} a_{i_0,\dots,i_t} \wedge b_{i_t,\dots,i_{r+t}}$. We will assume that foliations are transversely holomorphic of complex codimension q, and deal with the Bott class. The arguments for the Godbillon–Vey class of real foliations of codimension q are parallel and omitted. Finally, the Einstein convention is used throughout the article.

We recall the projective Schwarzian derivatives.

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DEFINITION 1.1. Let γ be a local biholomorphic diffeomorphism of $\mathbb{C}^q$, and let $y = (y^1, \dots, y^q)$ and $\widehat{y} = (\widehat{y}^1, \dots, \widehat{y}^q)$ be the natural coordinates on the domain and the target of γ, respectively. We set

$$\begin{aligned}
\Sigma(\gamma)^l_{mn} &= \frac{\partial y^l}{\partial \widehat{y}^p}\frac{\partial^2 \widehat{y}^p}{\partial y^m \partial y^n} - \frac{\delta^l_n}{q+1}\frac{\partial \log J(\gamma)}{\partial y^m} - \frac{\delta^l_m}{q+1}\frac{\partial \log J(\gamma)}{\partial y^n}\,,\\
\Lambda(\gamma)_{mn} &= -\frac{1}{q+1}\frac{\partial^2 \log J(\gamma)}{\partial y^m \partial y^n}\\
&\quad - \frac{1}{(q+1)^2}\frac{\partial \log J(\gamma)}{\partial y^m}\frac{\partial \log J(\gamma)}{\partial y^n} + \frac{1}{q+1}\frac{\partial \log J(\gamma)}{\partial y^p}\frac{\partial y^p}{\partial \widehat{y}^l}\frac{\partial^2 \widehat{y}^l}{\partial y^m \partial y^n}\,,
\end{aligned}$$

where $J(\gamma) = \det D\gamma$ denotes the Jacobian of γ. The $(1, 2)$-tensor of which the coefficients are given by $\Sigma(\gamma)^l_{mn}$ is called the *projective Schwarzian derivative* of γ and denoted by $\Sigma(\gamma)$. The $(0, 2)$-tensor of which the coefficients are given by $\Lambda(\gamma)_{mn}$ is denoted by $\Lambda(\gamma)$, which is a kind of the curvature of $\Sigma(\gamma)$.

The following is a well-known

LEMMA 1.2. *We have the following.*

1) $\Sigma(\gamma)^p_{mn} = \Sigma(\gamma)^p_{nm}$.
2) $\Sigma(\gamma)^m_{mn} = \Sigma(\gamma)^m_{nm} = 0$.
3) Σ *is a cocycle in the sense that* $\Sigma(\gamma_2 \circ \gamma_1) = \gamma_1^*\Sigma(\gamma_2) + \Sigma(\gamma_1)$.
4) $\Lambda(\gamma)_{mn} = \Lambda(\gamma)_{nm}$.

In what follows, we fix a simple open covering $\{U_i\}$ of M such that each U_i is contained in a foliation chart. The coordinates in the transverse direction on U_i are denoted by $(y^1_{(i)}, \dots, y^q_{(i)})$. Let γ_{ij} be the transition function from U_j to U_i in the transversal direction. We fix a Bott connection on $\bigwedge^q Q(\mathcal{F})$ and let $\theta_i = f_{(i)l}dy^l_{(i)}$ be the connection form on U_i with respect to $\frac{\partial}{\partial y^1_{(i)}} \wedge \cdots \wedge \frac{\partial}{\partial y^q_{(i)}}$.

DEFINITION 1.3. We set

$$\begin{aligned}
\Sigma(\gamma_{ij})^l_m &= \Sigma(\gamma_{ij})^l_{mn}dy^n_{(j)}\,,\\
\Lambda(\gamma_{ij})_m &= \Lambda(\gamma_{ij})_{mn}dy^n_{(j)}\,,\\
\widetilde{\Lambda}(\gamma_{ij})_m &= (q+1)\Lambda(\gamma_{ij})_m\,,\\
H_{(ij)mn} &= \widetilde{\Lambda}(\gamma_{ij})_{mn} - f_{(j)l}\Sigma(\gamma_{ij})^l_{mn}\,,\\
H_{(ij)m} &= H_{(ij)mn}dy^n_{(j)} = \widetilde{\Lambda}(\gamma_{ij})_m - f_{(j)l}\Sigma(\gamma_{ij})^l_m\,.
\end{aligned}$$

We have $\delta H = 0$ by Lemma 1.4 below, and $dH_{(ij)mn} = -df_{(j)l}\Sigma(\gamma_{ij})^l_{mn}$ modulo I^q, where I^q denotes the ideal of $\Omega^*(M)$ locally generated by $dy^1_{(j)} \wedge \cdots \wedge dy^q_{(j)}$.

The following lemma can be shown by direct calculations.

LEMMA 1.4. 1) *We have*

$$\begin{aligned}
&f_{(k)l}\Sigma(\gamma_{jk})^l_m - f_{(k)l}\Sigma(\gamma_{ik})^l_m + f_{(j)l}\Sigma(\gamma_{ij})^l_r(D\gamma_{jk})^r_m \\
&\quad = (f_{(j)l} - f_{(k)p}(D\gamma_{kj})^p_l)\Sigma(\gamma_{ij})^l_r(D\gamma_{jk})^r_m \\
&\qquad + f_{(k)l}(\Sigma(\gamma_{jk})^l_m - \Sigma(\gamma_{ik})^l_m + (D\gamma_{kj})^l_p\Sigma(\gamma_{ij})^p_r(D\gamma_{jk})^r_m) \\
&\quad = \frac{\partial \log J_{kj}}{\partial y^l_{(j)}}\Sigma(\gamma_{ij})^l_r(D\gamma_{jk})^r_m\,.
\end{aligned}$$

2) *The cochain* $\Lambda(\gamma) = \{\Lambda(\gamma_{ij})\}$ *fails to be a cocycle. Indeed, we have*

$$\Lambda(\gamma_{jk})_m - \Lambda(\gamma_{ik})_m + \Lambda(\gamma_{ij})_r(D\gamma_{jk})^r_m = \frac{1}{q+1}\frac{\partial \log J_{kj}}{\partial y^l_{(j)}}\Sigma(\gamma_{ij})^l_r(D\gamma_{jk})^r_m\,.$$

3) *We have* $H_{(ij)k}(D\gamma_{ji})^k_m = -H_{(ji)m}$.

In what follows, $f_{(j)l}\Sigma(\gamma_{ij})^l_m$ is often denoted by $(f_j\Sigma_{ij})_m$.

COROLLARY 1.5. *We have*

$$\begin{aligned}
\Lambda(\gamma_{ji})_r(D\gamma_{ij})^r_m + \Lambda(\gamma_{ij})_m &= \frac{1}{q+1}\frac{\partial \log J_{ij}}{\partial y^l_{(j)}}\Sigma(\gamma_{ij})^l_m \\
&= \frac{1}{q+1}\frac{\partial \log J_{ji}}{\partial y^l_{(i)}}\Sigma(\gamma_{ji})^l_s(D\gamma_{ij})^s_m\,.
\end{aligned}$$

REMARK 1.6. The case where $q = 1$ is exceptional and we have

$$\begin{aligned}
&\Sigma(\gamma_{ij})^l_m = 0\,, \\
&\Lambda(\gamma_{ij})_r(D\gamma_{ji})^r_m + \Lambda(\gamma_{ji})_m = 0\,.
\end{aligned}$$

Moreover, we have

$$H_{(ij)1} = \widetilde{\Lambda}(\gamma_{ij})_1 = -2\left(\frac{1}{2}\frac{\gamma'''_{ij}}{\gamma'_{ij}} - \frac{3}{2}\left(\frac{\gamma''_{ij}}{\gamma'_{ij}}\right)^2\right)dy_{(j)}\,,$$

where the symbol '$'$' means the differentiation with respect to $y_{(j)}$. Therefore, both H and $\widetilde{\Lambda}$ are equal to the classical Schwarzian derivative.

LEMMA 1.7. *Let* $\alpha_k = \alpha_{kl} \wedge dy^l_{(j)}$, *where* α_{kl} *is a differential form. If* $(\dot{\omega}^m_{(j)})$ *is a* $\mathbb{C}^q$*-valued differential form, then,*

$$\begin{aligned}
&\alpha_{1l} \wedge \Sigma(\gamma_{ij})^l_m \wedge \dot{\omega}^m_{(j)} \wedge \alpha_2 \wedge \cdots \wedge \alpha_q \\
&\quad + \alpha_1 \wedge \dot{\omega}^m_{(j)} \wedge \alpha_{2l} \wedge \Sigma(\gamma_{ij})^l_m \wedge \alpha_3 \wedge \cdots \wedge \alpha_q \\
&\qquad + \cdots + \alpha_1 \wedge \dot{\omega}^m_{(j)} \wedge \alpha_2 \wedge \cdots \wedge \alpha_{ql} \wedge \Sigma(\gamma_{ij})^l_m \\
&= 0\,.
\end{aligned}$$

PROOF. The claim is shown by the following well-known equality, namely, if $a_1, \dots, a_q \in M_{1,q}(\mathbb{C})$ and if $B \in M_q(\mathbb{C})$, then

$$\det\begin{pmatrix} a_1 B \\ a_2 \\ \vdots \\ a_q \end{pmatrix} + \det\begin{pmatrix} a_1 \\ a_2 B \\ \vdots \\ a_q \end{pmatrix} + \cdots + \det\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_q B \end{pmatrix} = \det\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{pmatrix} \operatorname{tr} B\,.$$

If we set $B_n^l = (\Sigma(\gamma_{ij})_n^l)$, then $\operatorname{tr} B = \Sigma(\gamma_{ij})_l^l = 0$. □

DEFINITION 1.8. Let $a = \{a_{i_0 i_1 \cdots i_k}\}$ be a Čech–de Rham (k, r)-cochain which is not necessarily alternating. We define an alternating Čech–de Rham (k, r)-cochain $\operatorname{Alt}(a)$ by

$$\operatorname{Alt}(a)_{i_0 i_1 \cdots i_k} = \frac{1}{(k+1)!} \sum_{\rho \in \mathfrak{S}_{k+1}} (\operatorname{sgn} \rho) a_{\rho(i_0)\rho(i_1)\cdots\rho(i_k)}\,.$$

We also denote the cochain $\operatorname{Alt}(a)$ by $\operatorname{Alt}(a_{01\cdots k})$.

We have the following. The proof is easy and omitted.

LEMMA 1.9. *If a is a Čech–de Rham cochain, then* $\delta \operatorname{Alt}(a) = \operatorname{Alt}(\delta a)$.

Finally we will briefly explain infinitesimal derivatives. Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension q, of a manifold M. Then, the Bott class is defined by a Čech–de Rham $(0, 2q+1)$-form $\left(\frac{-1}{2\pi\sqrt{-1}}\right)^{q+1} \theta \cup (d\theta)^q$, where $\theta = \{\theta_i\}$ is a certain Čech–de Rham $(0, 1)$-form. Under a certain condition which is always fulfilled if we consider real transversely orientable foliations, we may assume that θ is a globally defined 1-form on M. Especially, the Godbillon–Vey class is usually defined by the $(2q+1)$-form $\theta \wedge (d\theta)^q$ up to multiplication of a non-zero constant. If we have a smooth 1-parameter family $\{\mathcal{F}_s\}$ of complex codimension-q foliations with $\mathcal{F}_0 = \mathcal{F}$, then the family $\left\{\frac{-1}{2\pi\sqrt{-1}}\theta_s \cup (d\theta_s)^q\right\}$ is differentiable and we can consider the derivative at $s = 0$. If we set $\dot{\theta} = \frac{\partial}{\partial s}\theta_s\big|_{s=0}$ and $\theta = \theta_0$, then $\dot{\theta}$ and $d\theta$ are globally well-defined and the derivative is represented by $\left(\frac{-1}{2\pi\sqrt{-1}}\right)^{q+1} (q+1)\,\dot{\theta} \wedge (d\theta)^q$. These derivatives can be generalized to the ones with respect to infinitesimal deformations. An *infinitesimal deformation* of $\mathcal{F}$ is by definition an element of $H^1(M; \Theta_{\mathcal{F}})$, where $\Theta_{\mathcal{F}}$ denotes the sheaf of germs of foliated sections to $Q(\mathcal{F})$. Let $E(\mathcal{F})$ be the vector bundle locally spanned by $T\mathcal{F}$ and $\frac{\partial}{\partial \bar{y}^1_{(i)}}, \dots, \frac{\partial}{\partial \bar{y}^q_{(i)}}$. In the real case, we set $E(\mathcal{F}) = T\mathcal{F}$. Then infinitesimal deformations are represented by $Q(\mathcal{F})^*$-valued 1-forms on $E(\mathcal{F})$. It is known that the representatives can always be extended to $Q(\mathcal{F})^*$-valued 1 forms and that the infinitesimal derivatives are independent of the extensions. Given an infinitesimal deformation, we can construct a derivative $\dot{\theta}$ of θ with respect to each representative of the deformation. The infinitesimal derivative of the Bott class with respect to the deformation is by definition the class in $H^{2q+1}(M; \mathbb{C})$ represented by $\left(\frac{-1}{2\pi\sqrt{-1}}\right)^{q+1} (q+1)\,\dot{\theta} \wedge (d\theta)^q$. An important remark is that we made a use of a representative σ for μ in [1], while we made use of a representative $\dot{\omega}$ for $-\mu$ in [2] because $\dot{\omega}$ corresponds to the derivative of a (fixed) family of local trivializations of $Q(\mathcal{F})^*$. In what follows, we will follow conventions in [2] and make use of $\dot{\omega} = -\sigma$.

2. A proof of the formula. The formula given in [3] and [1] for the infinitesimal derivatives of the Bott class and those of Godbillon–Vey class in the projective Schwarzian derivatives is as follows.

THEOREM A ([3], [1, Theorem 4.10]). *Let Λ be the foliated Čech 1-cochain defined by $\Lambda_{ij} = \Lambda(\gamma_{ij})_m \otimes dy^m_{(j)}$. If $\mu \in H^1(M; \Theta_{\mathcal{F}})$ is represented by σ and if we set $L_{ij}(\sigma) = \Lambda(\gamma_{ij})_{mn} dy^n_{(j)} \wedge \sigma^m_{(j)}$, then the infinitesimal derivative of the Bott class is represented by*

$$\frac{q+1}{(2\pi\sqrt{-1})^{q+1}(q-1)!} \sum_{\rho \in \mathfrak{S}_{q+1}} (\operatorname{sgn}\rho)((d\log J)^{q-1} \cup L(\sigma))_{\rho(0)\cdots\rho(q)}\,.$$

In the real case, the same formula also holds for the infinitesimal derivative of the Godbillon–Vey class.

By abuse of notations, we denote $L_{ij}(\sigma) = -L_{ij}(\dot{\omega})$ also by $-\Lambda_{ij} \wedge \dot{\omega}_j$. In [2], infinitesimal derivatives of the Bott class as well as those of the Godbillon–Vey classes are discussed, and it is shown that they can be represented by means of coefficients of transverse Thomas–Whitehead projective connections. We will present here a proof of Theorem A by continuing calculations in [2]. We remark that the proof given in [1] is in a slightly more general setting. Indeed, we can consider a family of locally defined Bott connections rather than globally well-defined ones.

Before the proof, we will make some remarks and recall some relevant facts from [2].

REMARK 2.1. On the page 406, line 7 of [1], we claimed that

$$(\mathcal{D}''\rho(k))_{0\cdots q} = (-1)^{q(q+1)/2} d\omega_k(\sigma_k)\Delta_k - \langle (d\log J)^q | \underline{\sigma}_k \rangle\,.$$

The last term should be read as

$$\langle \widehat{\partial}(d\log J)^q | \underline{\sigma}_k \rangle\,.$$

In addition, $-(2\pi\sqrt{-1})^{q+1}\langle (d\log J)^q | \underline{\sigma}_k \rangle$ in the line 9 should be read as

$$-(2\pi\sqrt{-1})^{-(q+1)}\langle \widehat{\partial}(d\log J)^q | \underline{\sigma}_k \rangle\,.$$

Let $N'_{(k)i} = df_{(k)i} - \frac{1}{q+1} f_{(k)i} f_{(k)j} dy^j_{(k)}$. The tensor N' is defined by coefficients of transverse Thomas–Whitehead projective connections [2]. If $\dot{\omega}$ is a representative of an infinitesimal deformation, then we can locally represent $\dot{\omega}$ as $\dot{\omega}^l_j \frac{\partial}{\partial y^l_{(j)}}$, where j is an index for coverings and we do not take contractions. We set $N'_j \wedge \dot{\omega}_j = N'_{(j)l} \wedge \dot{\omega}^l_j$. We define $\Lambda_{ij} \wedge \dot{\omega}_j$, $H_{ij} \wedge \dot{\omega}_j$, $f_j \Sigma_{ij} \wedge \dot{\omega}_j$ and $\frac{\partial \log J_{ij}}{\partial y_{(j)}} \Sigma_{kj} \wedge \dot{\omega}_j$ in a similar way. As $d\theta_i$ is globally well-defined, we will omit the indices in what follows. Then, the family of locally defined tensors $N' = \{N'_{(k)i}\}$ has the following properties.

LEMMA 2.2 ([2, Lemma 4.10]). *We have*

$$-\frac{1}{q}\dot{\theta} \wedge (d\theta)^q = d(N' \wedge \dot{\omega}) \wedge (d\theta)^{q-1}\,.$$

LEMMA 2.3 ([2, Lemma 4.5]). *We have*

$$N'_{(j)lt} D\gamma_{ji}{}^l_m (D\gamma_{ji})^t_s - N'_{(i)ms} = H_{(ji)ms}\,.$$

Actually Lemma 4.5 in [2] is shown in a slightly more general setting.

Lemma 2.2 shows that the infinitesimal derivative of the Bott class is cohomologous to $d(N' \wedge \dot{\omega} \wedge (d\theta)^{q-1})$ multiplied by $-(-2\pi\sqrt{-1})^{-(q+1)}q(q+1)$ in the Čech–de Rham complex, and Lemma 2.3 shows that $(\delta N')_{ij} = -H_{ij}$.

PROOF OF THEOREM A. If $c = \{c_{i_0,\dots,i_k}\}$ is a Čech–de Rham cochain, then we denote $c_{i_0,\dots,i_k}$ by $c_{0,\dots,k}$ for simplicity. As $\sigma = -\dot{\omega}$, it suffices to show that $\dot{\theta} \wedge (d\theta)^q$ is cohomologous to

$$\frac{(-1)^{\frac{q(q+1)}{2}}}{(q-1)!} \sum_{\rho \in \mathfrak{S}_{q+1}} (\operatorname{sgn}\rho) d\log J_{\rho(0)\rho(1)} \wedge \cdots \wedge d\log J_{\rho(q-2)\rho(q-1)} \wedge \Lambda_{\rho(q-1)\rho(q)} \wedge \dot{\omega}_q\,.$$

Let

$$c_0^{(0,2q)} = N'_{(0)l} \wedge \dot{\omega}_0^l \wedge (d\theta)^{q-1}\,.$$

We have $\frac{1}{q}\dot{\theta} \wedge (d\theta)^q + \mathcal{D}'' c^{(0,2q)} = 0$, and

$$\begin{aligned}(\delta c^{(0,2q)})_{01} &= c_1^{(0,2q)} - c_0^{(0,2q)}\\ &= H_{(10)m} \wedge \dot{\omega}_0^m \wedge (d\theta)^{q-1}\\ &= -H_{(01)m} \wedge \dot{\omega}_1^m \wedge (d\theta)^{q-1}\,.\end{aligned}$$

If $q = 1$, then these equalities together with Remark 1.6 show that $\dot{\theta} \wedge d\theta$ is cohomologous to $-\widetilde{\Lambda} \wedge \dot{\omega}$ and the proof is completed. We assume $q > 1$ in what follows. For $1 \le k \le q-1$, we set

$$\begin{aligned}\alpha_{0,\dots,k}^{(k,2q-k)} &= \theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge H_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k-1}\,,\\ \beta_{0,\dots,k}^{(k,2q-k)} &= \sum_{r=0}^{k-1} (\theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge f_r D\gamma_{rk} \Sigma_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k-1}\,,\\ \gamma^{(k,2q-k)} &= \alpha^{(k,2q-k)} + \frac{1}{q}\beta^{(k,2q-k)}\,,\\ c^{(k,2q-k)} &= (-1)^{\frac{(k+1)(k+2)}{2}} \operatorname{Alt}\gamma^{(k,2q-k)}\,,\end{aligned}$$

where $d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1}$ is formally set to be 1 if $k = 1$. We have

$$
\begin{aligned}
& d\alpha^{(k,2q-k)}_{0,\ldots,k} \\
& = d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge H_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k} \\
& \quad - (-1)^k \theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge df_k \wedge \Sigma_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k-1}\,, \\
& d\beta^{(k,2q-k)}_{0,\ldots,k} \\
& = \sum_{r=0}^{k-1} d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge f_r D\gamma_{rk} \Sigma_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k} \\
& \quad + (-1)^k \sum_{r=0}^{k-1} \theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge df_r \wedge D\gamma_{rk} \Sigma_{k-1,k} \wedge \dot{\omega}_k \wedge (d\theta)^{q-k-1}\,, \\
& \delta\alpha^{(k,2q-k)}_{0,\ldots,k+1} \\
& = d\log J_{01} \wedge \cdots \wedge d\log J_{k-2,k-1} \wedge d\log J_{k-1,k} \wedge H_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-1}\,, \\
& \delta\beta^{(k,2q-k)}_{0,\ldots,k+1} \\
& = \sum_{r=1}^{k} d\log J_{01} \wedge \cdots \wedge d\log J_{k-1,k} \wedge f_r D\gamma_{r,k+1} \Sigma_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-1} \\
& \quad + \sum_{r=0}^{k-1} (-1)^r (\theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\,\widehat{\log J_{r,r+1}} \wedge \cdots \wedge d\log J_{k-1,k} \\
& \qquad \wedge \frac{\partial \log J_{r,r+1}}{\partial y_{(r+1)}} D\gamma_{r+1,k} \Sigma_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-1})\,.
\end{aligned}
$$

By Lemma 1.7, we have

$$
\begin{aligned}
& \delta\gamma^{(k,2q-k)}_{0,\ldots,k+1} - d\gamma^{(k+1,2q-k-1)}_{0,\ldots,k+1} \\
& \quad = -\frac{1}{q} d\log J_{01} \wedge \cdots \wedge d\log J_{k-1,k} \wedge f_0 D\gamma_{0,k+1} \Sigma_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-1} \\
& \qquad + \frac{1}{q} \sum_{r=0}^{k-1} (-1)^r (\theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\,\widehat{\log J_{r,r+1}} \wedge \cdots \wedge d\log J_{k-1,k} \\
& \qquad\qquad \wedge \frac{\partial \log J_{r,r+1}}{\partial y_{(r+1)}} D\gamma_{r+1,k+1} \Sigma_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-1}) \\
& \qquad + (-1)^{k+1} \frac{q-k-1}{q} \theta_0 \wedge d\log J_{01} \wedge \cdots \wedge d\log J_{k-1,k} \\
& \qquad\qquad \wedge df_{k+1} \wedge \Sigma_{k,k+1} \wedge \dot{\omega}_{k+1} \wedge (d\theta)^{q-k-2} \\
& \quad = 0\,.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\mathcal{D}'(-1)^{\frac{(k+1)(k+2)}{2}}\gamma^{(k,2q-k)}_{0,\dots,k+1}+\mathcal{D}''(-1)^{\frac{(k+2)(k+3)}{2}}\gamma^{(k+1,2q-k-1)}\\
&\quad=(-1)^{\frac{(k+1)(k+2)}{2}}\delta\gamma^{(k,2q-k)}_{0,\dots,k+1}-(-1)^{\frac{(k+1)(k+2)}{2}}d\gamma^{(k+1,2q-k-1)}\\
&\quad=0
\end{aligned}$$

for $k\geq 1$ and

$$\begin{aligned}
\mathcal{D}''(-\gamma^{(1,2q-1)})&=d\gamma^{(1,2q-1)}\\
&=H_{01}\wedge\dot{\omega}_1\wedge(d\theta)^{q-1}+\theta_0\wedge df_1\wedge\Sigma_{01}\wedge\dot{\omega}_1\wedge(d\theta)^{q-2}\\
&\quad+\frac{1}{q}f_0D\gamma_{01}\Sigma_{01}\wedge\dot{\omega}_1\wedge(d\theta)^{q-1}\\
&\quad-\frac{1}{q}\theta_0\wedge df_0\wedge D\gamma_{01}\Sigma_{01}\wedge\dot{\omega}_1\wedge(d\theta)^{q-2}\\
&=H_{01}\wedge\dot{\omega}_1\wedge(d\theta)^{q-1}.
\end{aligned}$$

Hence $\dot{\theta}\wedge(d\theta)^q$ is cohomologous to $q\mathcal{D}'c^{(q-1,q+1)}$ in the Čech–de Rham complex. On the other hand, we have

$$\begin{aligned}
&\delta\gamma^{(q-1,q+1)}\\
&\quad=d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge H_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad+\frac{1}{q}\sum_{r=1}^{q-1}d\log J_{01}\wedge\cdots\wedge d\log J_{q-2,q-1}\wedge f_rD\gamma_{r,q}\Sigma_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad+\frac{1}{q}\sum_{r=0}^{q-2}(-1)^r(\theta_0\wedge d\log J_{01}\wedge\cdots\wedge d\,\widehat{\log J_{r,r+1}}\wedge\cdots\wedge d\log J_{q-2,q-1}\\
&\qquad\qquad\wedge\frac{\partial\log J_{r,r+1}}{\partial y_{(r+1)}}D\gamma_{r+1,q}\Sigma_{q-1,q}\wedge\dot{\omega}_q)\\
&\quad=d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad-d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge f_q\Sigma_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad+\frac{1}{q}\sum_{r=1}^{q-1}d\log J_{01}\wedge\cdots\wedge d\log J_{q-2,q-1}\wedge f_rD\gamma_{r,q}\Sigma_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad+\frac{1}{q}d\log J_{01}\wedge\cdots\wedge d\log J_{q-2,q-1}\wedge f_0D\gamma_{0,q}\Sigma_{q-1,q}\wedge\dot{\omega}_q\\
&\quad=d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad-\frac{1}{q}\sum_{r=0}^{q-1}d\log J_{01}\wedge\cdots\wedge d\log J_{q-2,q-1}\wedge\frac{\partial\log J_{rq}}{\partial y_{(q)}}\Sigma_{q-1,q}\wedge\dot{\omega}_q
\end{aligned}$$

$$
\begin{aligned}
&= d\log J_{01} \wedge \cdots \wedge d\log J_{q-3,q-2} \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-1,q} \wedge \dot{\omega}_q \\
&\quad - \frac{1}{q}\sum_{r=0}^{q-1} d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \wedge (\widetilde{\Lambda}_{qr} - \widetilde{\Lambda}_{q-1,r} + \widetilde{\Lambda}_{q-1,q}) \wedge \dot{\omega}_q \\
&= -\frac{1}{q}\sum_{r=0}^{q-1} d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{qr} \wedge \dot{\omega}_q \\
&\quad + \frac{1}{q}\sum_{r=0}^{q-2} d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-1,r} \wedge \dot{\omega}_q \,,
\end{aligned}
$$

where '$d\,\widehat{\log J_{r,r+1}}$' means that $d\log J_{r,r+1}$ is omitted. We have

$$
(2.4) \quad
\begin{cases}
\ d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \\
= d\log J_{01} \wedge \cdots \wedge \widehat{d\log J_{r-2,r-1}} \wedge \cdots \wedge d\log J_{q-2,q-1} \\
= d\log J_{01} \wedge \cdots \wedge d\log J_{r-3,r-2} \\
\qquad \wedge\, d\log J_{r-2,r} \wedge d\log J_{r-1,r+1} \wedge \cdots \wedge d\log J_{q-3,q-1} \wedge d\log J_{q-2,q-1} \,.
\end{cases}
$$

If $q-r$ is even, then the right hand side of (2.4) is equal to

$$
\begin{aligned}
&(-1)^{\epsilon} d\log J_{01} \wedge \cdots \wedge d\log J_{r-3,r-2} \\
&\quad \wedge d\log J_{r-2,r} \wedge d\log J_{r,r+2} \wedge \cdots \wedge d\log J_{q-4,q-2} \wedge d\log J_{q-2,q-1} \\
&\qquad \wedge d\log J_{q-1,q-3} \wedge \cdots \wedge d\log J_{r+1,r} \,,
\end{aligned}
$$

where $\epsilon = l^2$ if $q-r=2l$. If $q-r$ is odd, then the right hand side of (2.4) is equal to

$$
\begin{aligned}
&(-1)^{\epsilon} d\log J_{01} \wedge \cdots \wedge d\log J_{r-3,r-2} \\
&\quad \wedge d\log J_{r-2,r} \wedge d\log J_{r,r+2} \wedge \cdots \wedge d\log J_{q-3,q-1} \wedge d\log J_{q-1,q-2} \\
&\qquad \wedge d\log J_{q-2,q-4} \wedge \cdots \wedge d\log J_{r+1,r} \,,
\end{aligned}
$$

where $\epsilon = l^2 + l$ if $q-r = 2l+1$. Therefore, modulo alternations of indices, we have

$$
\begin{aligned}
&d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{qr} \wedge \dot{\omega}_r \\
&\quad \overset{\mathrm{Alt}}{=} -d\log J_{01} \wedge \cdots \wedge d\log J_{q-3,q-2} \wedge d\log J_{q-2,q} \wedge \widetilde{\Lambda}_{q-1,q} \wedge \dot{\omega}_q \\
&\quad = -d\log J_{01} \wedge \cdots \wedge d\log J_{q-3,q-2} \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-1,q} \wedge \dot{\omega}_q \\
&\qquad - d\log J_{01} \wedge \cdots \wedge d\log J_{q-3,q-2} \wedge d\log J_{q-1,q} \wedge \widetilde{\Lambda}_{q-1,q} \wedge \dot{\omega}_q \,,
\end{aligned}
$$

where the symbol '$\overset{\mathrm{Alt}}{=}$' means that the equality holds modulo alternations of indices. Similarly, we have

$$
\begin{aligned}
&d\log J_{01} \wedge \cdots \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-1,r} \wedge \dot{\omega}_r \\
&\quad \overset{\mathrm{Alt}}{=} d\log J_{01} \wedge \cdots \wedge d\log J_{q-3,q-2} \wedge d\log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-2,q-1} \wedge \dot{\omega}_{q-1} \,.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&\delta\gamma^{(q-1,q+1)}\\
&\quad\overset{\text{Alt}}{=} d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad + d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad + \frac{q-1}{q} d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-2,q-1}\wedge\dot{\omega}_{q-1}\,.
\end{aligned}$$

If we set $\zeta_{ij} = d\log J_{ij}\wedge\widetilde{\Lambda}_{ij}\wedge\dot{\omega}_j$, then we have

$$d(d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge\zeta_{q-2,q-1}) = 0$$

and

$$\begin{aligned}
&\delta(d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge\zeta_{q-2,q-1})\\
&\quad = d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad + d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-2,q-1}\wedge\dot{\omega}_{q-1}\\
&\qquad - d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q}\wedge\widetilde{\Lambda}_{q-2,q}\wedge\dot{\omega}_q\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\quad = d\log J_{01}\wedge\cdots\wedge d\log J_{q-4,q-3}\wedge d\log J_{q-3,q-1}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad - d\log J_{01}\wedge\cdots\wedge d\log J_{q-4,q-3}\wedge d\log J_{q-2,q-1}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\quad = d\log J_{01}\wedge\cdots\wedge d\log J_{q-4,q-3}\wedge d\log J_{q-3,q-1}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad - d\log J_{01}\wedge\cdots\wedge d\log J_{q-5,q-4}\wedge d\log J_{q-4,q-2}\\
&\qquad\qquad \wedge d\log J_{q-2,q-1}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\qquad + d\log J_{01}\wedge\cdots\wedge d\log J_{q-5,q-4}\wedge d\log J_{q-3,q-2}\\
&\qquad\qquad \wedge d\log J_{q-2,q-1}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\quad = \cdots\\
&\quad\overset{\text{Alt}}{=} (q-1)\, d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-2,q-1}\wedge\dot{\omega}_{q-1}\\
&\quad\overset{\text{Alt}}{=} -(q-1)\, d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q}\wedge\widetilde{\Lambda}_{q-2,q}\wedge\dot{\omega}_q\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\delta(d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge\zeta_{q-2,q-1})\\
&\quad\overset{\text{Alt}}{=} \frac{q+1}{q-1}\, d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-1,q}\wedge\widetilde{\Lambda}_{q-1,q}\wedge\dot{\omega}_q\\
&\quad\overset{\text{Alt}}{=} (q+1)\, d\log J_{01}\wedge\cdots\wedge d\log J_{q-3,q-2}\wedge d\log J_{q-2,q-1}\wedge\widetilde{\Lambda}_{q-2,q-1}\wedge\dot{\omega}_{q-1}\,.
\end{aligned}$$

Consequently, $\mathcal{D}' c^{(q-1,q+1)}$ is equal, modulo alternations, to

$$(-1)^{\frac{q(q+1)}{2}} \operatorname{Alt}(d \log J_{01} \wedge \cdots \wedge d \log J_{q-3,q-2} \wedge d \log J_{q-2,q-1} \wedge \widetilde{\Lambda}_{q-1,q} \wedge \dot{\omega}_q)\,.$$

Thus we are done. $\square$

REMARK 2.5. Actually, we have shown that

$$\mathcal{L}_{i_0 \cdots i_q} = \frac{q+1}{(2\pi\sqrt{-1})^{q+1}(q-1)!} \sum_{\rho \in \mathfrak{S}_{q+1}} (\operatorname{sgn} \rho)((d \log J)^{q-1} \cup \Lambda)_{\rho(0) \cdots \rho(q)}\,,$$

where $\mathcal{L}$ is defined in [1]. This is the formula in [1, Lemma 4.7].

REMARK 2.6. We can show that

$$\frac{(-1)^{\frac{q(q+1)}{2}}(q+1)^2 q}{(-2\pi\sqrt{-1})^{q+1}} \operatorname{Alt}(d \log J_{0q} \wedge \cdots \wedge d \log J_{q-2,q} \wedge \Lambda_{q-1,q} \wedge \dot{\omega}_q)$$

is also a representative for the infinitesimal derivative of the Bott class. It is more symmetric representation. Due to several non-trivial relations among $\Lambda(\gamma)$, $\Sigma(\gamma)$ and $d \log J(\gamma)$, a cocycle can have many expression. In addition, we can consider coboundaries so that we do not know if there is a canonical or natural choice of a representative such as so-called the Thurston cocycle for the Godbillon–Vey class (cf. [4]). We think however that the one given in Theorem A and the one as above are the simplest ones.

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