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ON PERIODIC MAPS OVER SURFACES WITH LARGE PERIODS

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Abstract. Kulkarni showed that, if g is greater than three, any periodic map on the oriented surface of genus g with period more than or equal to $4g$ is conjugate to a power of one of two types of periodic maps. In this paper, we show that, if g is greater than 12, any periodic map on the surface with period more than or equal to $3g$ is conjugate to a power of one of four types of periodic maps.

1. Introduction. Let Σ_g be a closed oriented surface of genus $g \geq 2$. The Nielsen-Thurston theory [10] classifies orientation preserving diffeomorphisms of Σ_g into the following three types: (1) periodic, (2) reducible and (3) pseudo-Anosov. For each type, there are important values describing conjugacy classes, for example, the *periods* for periodic maps and the *dilatations* for pseudo-Anosov maps. A natural problem is to what extent these values determine conjugacy classes. For periodic maps, Kulkarni [7] showed that the period determines the conjugacy classes when the genera and the periods are sufficiently large.

Wiman [11] showed that, if the genus g is at least 2, a period of a periodic map on Σ_g is at most $4g+2$.

We visualize periodic maps with periods $4g+2$ (see Figure 1 when $g=2$). We prepare two disks, divide each of them into $2g+1$ triangles, number each triangle on one disk by even integers $0, 2, \ldots, 4g$ clockwisely, and number each triangle on the other disk by odd integers $1, 3, \ldots, 4g+1$ clockwisely. We glue these disks along outer edges of triangles such that $2i$ is attached to $2i+2g+1$ for $i = 0, \ldots, g$ and $2i$ is attached to $2i-2g-1$ for $i = g+1, \ldots, 2g$, then we get Σ_g. The homeomorphism on Σ_g, which brings i to $i+1$ for $i = 0, \ldots, 4g$ and $4g+1$ to 0 is a periodic map whose period is $4g+2$. By a slight modification, we visualize periodic maps with periods $4g$ (see Figure 2 for $g=2$). At first, we glue these disks along outer edges of 0-th triangle and $(2g+1)$-st triangle, then we get a

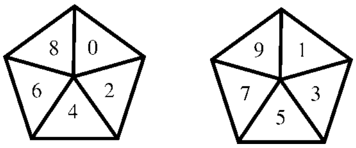

FIGURE 1. Period $10 = 4 \cdot 2 + 2$ on Σ_2.

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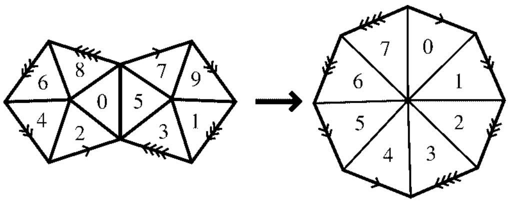

FIGURE 2. Period $8 = 4 \cdot 2$ on Σ_g.

$4g$-gon. We memorize which pairs of outer edges are glued in the previous construction, and erase all edges which divide this disk into $4g + 2$ triangles. We divide this $4g$-gon into $4g$ triangles, number these triangles by $0, 1, \dots, 4g$ clockwisely. We glue outer edges of these triangles according to our memory. Then we get Σ_g again, and the homeomorphism, which bring i to $i + 1$ for $i = 0, \dots, 4g - 1$ and $4g$ to 0 is a periodic map of period $4g$.

Almost one hundred years after Wiman's paper, Kulkarni [7] showed that, if $g > 3$, any periodic map on Σ_g with period at least $4g$ is conjugate to a power of one of two types of periodic maps explained above. In this paper, we show that, if $g > 12$, any periodic map on Σ_g with period at least $3g$ is conjugate to a power of one of four types of periodic maps.

For the early draft of this paper, Kulkarni pointed out that, for each periodic maps of order $4g + 2$ and $4g$, we can concretely construct hyperbolic structures on Σ_g such that these periodic maps act as isometries. (i) For the periodic map of order $4g + 2 = 2(2g + 1)$, the hyperbolic structure is constructed as follows. Let P be a regular $2(2g + 1)$-gon in the hyperbolic plane $\boldsymbol{H}^2$, with vertex-angle $2\pi/(2g + 1)$, and $e_1, \dots, e_{2(2g+1)}$ the edges on ∂P clockwisely. For $1 \leq i \leq 2g + 1$, we glue e_i with $e_{(2g+1)+i}$ by an isometry on $\boldsymbol{H}^2$. Then we obtain a hyperbolic structure on Σ_g, and the $2\pi/2(2g + 1)$ rotation on P induces an isometry and the periodic map of order $4g + 2$ on Σ_g. (ii) For the periodic map of order $4g$, the hyperbolic structure is constructed as follows. Let P be a regular $4g$-gon in the hyperbolic plane $\boldsymbol{H}^2$, with vertex-angle $2\pi/4g$, and $e_1, \dots, e_{4g}$ the edges on ∂P clockwisely. For $1 \leq i \leq 2g$, we glue e_i with e_{2g+i} by an isometry on $\boldsymbol{H}^2$. Then we obtain a hyperbolic structure on Σ_g, and the $2\pi/4g$ rotation on P induces an isometry and the periodic map of order $4g$ on Σ_g.

2. Nielsen's classification of periodic maps. An orientation preserving homeomorphism f from a surface Σ_g to itself is said to be a *periodic map*, if there is a positive integer n such that $f^n = \mathrm{id}_{\Sigma_g}$. The *period* of f is the smallest positive integer which satisfies the above condition. Two periodic maps f and f' on Σ_g are *conjugate*, if there is an orientation preserving homeomorphism h from Σ_g to itself such that $f' = h \circ f \circ h^{-1}$. In this section, we will review the classification of conjugacy classes of periodic maps on surfaces by Nielsen [8]. We follow a description by Smith [9] and Yokoyama [12].

Let f be a periodic map on Σ_g, whose period is n. A point p on Σ_g is a *multiple point* of f, if there is a positive integer k less than n such that $f^k(p) = p$. Let M_f be the set of multiple points of f. The orbit space Σ_g/f of f is defined by identifying x in Σ_g with $f(x)$. Let $\pi_f : \Sigma_g \to \Sigma_g/f$ be the quotient map. Then π_f is an n-fold branched covering ramified at $\pi_f(M_f)$. The set $\pi_f(M_f)$ is denoted by B_f, and each element of B_f is called a *branch point* of f. We choose a point x in $\Sigma_g/f - B_f$, and a point $\tilde{x}$ in $\pi_f^{-1}(x)$. We define a homomorphism $\Omega_f : \pi_1(\Sigma_g/f - B_f) \to \mathbf{Z}_n$ as follows: Let l be a loop in $\Sigma_g/f - B_f$ with the base point x, and $[l]$ the element of $\pi_1(\Sigma_g/f - B_f)$ represented by l. Let $\tilde{l}$ be the lift of l on Σ_g which begins from $\tilde{x}$. There is a positive integer r less than or equal to n such that the terminal point of $\tilde{l}$ is $f^r(\tilde{x})$. We define $\Omega_f([l]) = r \mod n$. Since $\mathbf{Z}_n$ is an Abelian group, the homomorphism Ω_f induces a homomorphism ω_f from the abelianization of $\pi_1(\Sigma_g/f - B_f)$ to $\mathbf{Z}_n$. The abelianization of $\pi_1(\Sigma_g/f - B_f)$ is $H_1(\Sigma_g/f - B_f)$, therefore ω_f is a homomorphism from $H_1(\Sigma_g/f - B_f)$ to $\mathbf{Z}_n$. For each point of $B_f = \{Q_1, \ldots, Q_b\}$, let D_i be a disk in Σ_g/f, which contains Q_i in its interior and is sufficiently small so that no other points of B_f are in D_i. Let S_{Q_i} be the boundary of D_i with clockwise orientation.

THEOREM 2.1 [8, §11]. *Two periodic maps f and f' on Σ_g are conjugate to each other if and only if the following three conditions are satisfied.*

(1) *The period of f is equal to the period of f'.*

(2) *The number of elements in B_f is equal to that of $B_{f'}$.*

(3) *After renumbering the elements of $B_{f'}$, we have $\omega_f(S_{Q_i}) = \omega_{f'}(S_{Q_i})$ for each i.*

Let $\theta_i = \omega_f(S_{Q_i})$ for each i. By the above Theorem, the data $[g, n; \theta_1, \ldots, \theta_b]$ determines a periodic map up to conjugacy. The following proposition shows a sufficient and necessary condition for a data $[g, n; \theta_1, \ldots, \theta_b]$ to correspond to a periodic map.

PROPOSITION 2.2. *There is a periodic map with the data $[g, n; \theta_1, \ldots, \theta_b]$ if and only if the following conditions are satisfied.*

(1) $\theta_1 + \cdots + \theta_b \equiv 0 \mod n$.

(2) *If Σ_g/f is a sphere, then* $\gcd\{\theta_1, \ldots, \theta_b\} \equiv 1 \mod n$.

(3) *Let g' be the genus of Σ_g/f and $n_i = n/\gcd\{\theta_i, n\}$, then*

$$2g - 2 = n\Big(2g' - 2 + \sum\Big(1 - \frac{1}{n_i}\Big)\Big),$$

where i runs through the branch points.

The necessity of the three conditions in the above proposition are shown as follows. (1) follows from the fact that ω_f is a homomorphism and $S_{Q_1} + \cdots + S_{Q_b}$ is null-homologous, (2) follows from the fact that ω_f is a surjection, and (3) is the Riemann-Hurwitz formula. The sufficiency of these conditions follows from the existence theorem of a branched covering space by Hurwitz [4]. The number n_i is called the *branching index* of Q_i.

In the following, we will use the expression $(n, \theta_1/n + \cdots + \theta_b/n)$ in place of $[g, n; \theta_1, \ldots, \theta_b]$. This data $(n, \theta_1/n + \cdots + \theta_b/n)$ is called the *total valency*, which is introduced by Ashikaga and Ishizaka [1]. In the above data, we call θ_i/n the *valency* of Q_i, and often rewrite this by an irreducible fraction. We remark that the denominator of the reduced θ_i/n is equal to the branching index of Q_i.

3. Main Result. The main result of this paper is:

THEOREM 3.1. *Let the genus g be greater than or equal to* 3. *If the period of a periodic map on Σ_g is greater than or equal to* $3g$, *then this map is conjugate to the power of one of the maps in Table* 1. *In Table* 1, g *is the genus*, $V = (n, a/b + \cdots)$ *is the total valency.*

REMARK 3.2. In the above theorem, the power k is an integer prime to n. If k satisfies this condition and $f = (n, m_1/n_1 + \cdots + m_b/n_b)$, then $f^k = (n, (k^* \cdot m_1)/n_1 + \cdots + (k^* \cdot m_b)/n_b)$ where k^* is an integer such that $k \cdot k^* \equiv 1 \bmod n$, and $k^* \cdot m_i$ is the remainder of $k^* m_i$ modulo n_i.

COROLLARY 3.3. *Let the genus* $g > 12$.

1. *If the period of a periodic map on Σ_g is greater than or equal to* $3g$, *then the period of this map is* $4g+2$, $4g$, $3g+3$ *or* $3g$.

TABLE 1.

	g	V
(1)	arbitrary	$\left(4g+2,\ \frac{1}{2} + \frac{g}{2g+1} + \frac{1}{4g+2}\right)$
(2)	arbitrary	$\left(4g,\ \frac{1}{2} + \frac{2g-1}{4g} + \frac{1}{4g}\right)$
(3)-(i)	$3k$	$\left(3g+3,\ \frac{2}{3} + \frac{k}{g+1} + \frac{1}{3g+3}\right)$
(3)-(ii)	$3k+1$	$\left(3g+3,\ \frac{1}{3} + \frac{2k+1}{g+1} + \frac{1}{3g+3}\right)$
(4)-(i)	$3k$ or $3k+1$	$\left(3g,\ \frac{1}{3} + \frac{2g-1}{3g} + \frac{1}{3g}\right)$
(4)-(ii)	$3k+2$	$\left(3g,\ \frac{2}{3} + \frac{g-1}{3g} + \frac{1}{3g}\right)$
(5)	4	$\left(12,\ \frac{3}{4} + \frac{1}{6} + \frac{1}{12}\right)$
(6)	6	$\left(20,\ \frac{3}{4} + \frac{1}{5} + \frac{1}{20}\right)$
(7)	9	$\left(28,\ \frac{1}{4} + \frac{5}{7} + \frac{1}{28}\right)$
(8)	12	$\left(36,\ \frac{3}{4} + \frac{2}{9} + \frac{1}{36}\right)$
(9)	10	$\left(30,\ \frac{4}{5} + \frac{1}{6} + \frac{1}{30}\right)$

2. *If the periods of two periodic maps f_1, f_2 on Σ_g are equal and greater than or equal to* $3g$, *then f_2 is conjugate to a power of f_1.*

In order to prove Theorem 3.1, we make an observation on branching indices. Since we assume that the period is greater than or equal to $3g$, by the result of Kasahara [6, Proposition 1.1 and Theorem 4.1], the orbit space of the periodic map is a 2-sphere with three branch points. Let n_1, n_2 and n_3 be the branching indices of these branch points, such that $n_1 \leq n_2 \leq n_3$.

LEMMA 3.4. *Under the condition of Theorem* 3.1, *the genus g, the period n and branching indices $I = (n_1, n_2, n_3)$ of the periodic map should be one of the following.*

(1) g *is arbitrary*, $n = 4g + 2$, $I = (2, 2g + 1, 4g + 2)$,
(2) g *is arbitrary*, $n = 4g$, $I = (2, 4g, 4g)$,
(3) $g + 1$ *is not a multiple of* 3, $n = 3g + 3$, $I = (3, g + 1, 3g + 3)$,
(4) g *is arbitrary*, $n = 3g$, $I = (3, 3g, 3g)$,
(5) $g = 4$, $n = 12$, $I = (4, 6, 12)$,
(6) $g = 6$, $n = 20$, $I = (4, 5, 20)$,
(7) $g = 9$, $n = 28$, $I = (4, 7, 28)$,
(8) $g = 12$, $n = 36$, $I = (4, 9, 36)$,
(9) $g = 10$, $n = 30$, $I = (5, 6, 30)$.

We prove Lemma 3.4 and then Theorem 3.1.

From the Riemann-Hurwitz formula (Proposition 2.2, (3)), we see,

$$2g - 2 = n\Big(1 - \Big(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\Big)\Big). \tag{3.1}$$

We have $n_1 \geq 2$ by the definition of branching index, and $n \geq 3g$ by our assumption. Hence we have $2 \leq n_1 \leq 8$ from the above equation. The following theorem is shown by Harvey.

THEOREM 3.5 [2, Theorem 4, (i)]. *For n_1, n_2, n_3 and n above, we have* $\mathrm{lcm}\{n_1, n_2\} = \mathrm{lcm}\{n_2, n_3\} = \mathrm{lcm}\{n_3, n_1\} = n$.

By the above theorem, Kulkarni pointed out the following fact in the beginning of the proof of [7, Proposition 4.5].

LEMMA 3.6. (1) *If n_1 divides n_2, then $n_2 = n_3 = n$.*
(2) *If n_1 is a power of a prime and n_1 does not divide n_2, then $n_3 = n$.*

We assume n_1 divides n_2, then, by Lemma 3.6, (1), $n_2 = n_3 = n$. From (3.1), we have $2g = n(1 - 1/n_1)$. If $n_1 = 2$, then $n = 4g$, $I = (2, 4g, 4g)$ (Lemma 3.4, (2)). If $n_1 = 3$, then $n = 3g$, $I = (3, 3g, 3g)$ (Lemma 3.4, (4)). If $n_1 \geq 4$, then $n \leq (8/3)g$. This contradicts our assumption $n \geq 3g$.

From here, we assume n_1 does not divide n_2.

If $n_1 = 2, 3, 5, 7$, then $\gcd\{n_1, n_2\} = 1$. By Theorem 3.5, the period n is equal to $\mathrm{lcm}\{n_1, n_2\} = n_1 n_2$, and by Lemma 3.6, (2), $n_3 = n$. From (3.1), we have $2g = (n_1 - 1)(n/n_1 - 1)$. When $n_1 = 2$, we have $n = 4g + 2$ and $I = (2, 2g + 1, 4g + 2)$ (Lemma

3.4, (1)). When $n_1 = 3$, we have $n = 3g + 3$ and $I = (3, g+1, 3g+3)$. Since n_1 does not divide n_2, $g+1$ is not a multiple of 3 (Lemma 3.4, (3)). When $n_1 = 5$, $n = (5/2)g + 5$. Since $n \geq 3g$, $10 \geq g$. On the other hand, by the condition $\gcd\{n_1, n_2\} = 1$, n_1 is not equal to n_2, so $n_1 < n_2 = n/n_1$. Therefore, $(5/2)g + 5 = n > n_1^2 = 25$, so $g > 8$. Since $n = (5/2)g + 5$ is an integer, g should be an even integer, hence $g = 10$. We have only one case, $g = 10$, $n = 30$, $I = (5, 6, 30)$ (Lemma 3.4, (9)). When $n_1 = 7$, from the same argument as when $n_1 = 5$, we see that there is no case to seek.

If $n_1 = 4, 8$, then n_1 is a power of a prime($= 2$), hence by Lemma 3.6, (2), $n_3 = n$. From (3.1), we have $2g - 1 = n(1 - 1/n_1 - 1/n_2)$. When $n_1 = 4$, n_2 should be $2k$ or k, where k is an odd integer. If $n_2 = 2k$, by Theorem 3.5, $n = \mathrm{lcm}\{n_1, n_2\} = 4k = 2n_2$, therefore $2g - 1 = (3/2)n_2 - 2$, so $n = (8g+4)/3$. As we assume $n \geq 3g$, we have $4 \geq g$. Since $8g + 4$ is a multiple of 3 and $g \geq 3$, g should be 4. Hence we have only one case, $g = 4$, $n = 12$, $I = (4, 6, 12)$ (Lemma 3.4, (5)). If $n_2 = k$, then $\gcd\{n_1, n_2\} = 1$, hence, from the same argument as the beginning of the last paragraph, we have $n = n_1 n_2$ and $2g = (n_1 - 1)(n/n_1 - 1) = (3/4)n - 3$, so $n = (8/3)g + 4$. As we assume $n \geq 3g$, we have $12 \geq g$. Since $n = (8/3)g + 4$ is an integer, g is a multiple of 3, hence $g = 3, 6, 9, 12$. The case $g = 3$ is excluded, since in this case $n_2 = 3 < 4 = n_1$. We have three cases, $g = 6$, $n = 20$, $I = (4, 5, 20)$ (Lemma 3.4, (6)), $g = 9$, $n = 28$, $I = (4, 7, 28)$ (Lemma 3.4, (7)) and $g = 12$, $n = 36$, $I = (4, 9, 36)$ (Lemma 3.4, (8)). When $n_1 = 8$, n_2 should be $4k$, $2k$ or k, where k is an odd integer. By the same argument as when $n_1 = 4$, we see that this is not the case.

Finally, we consider the case where $n_1 = 6$. From our assumption $n \geq 3g$ and (3.1), we have $1/n_2 + 1/n_3 \geq (g+4)/6g$. From the condition $n_1 \leq n_2 \leq n_3$, we have $6 \leq n_2 \leq 12g/(g+4)$. Therefore $6 \leq n_2 \leq 11$. Since we assume that n_1 does not divide n_2, we omit the case where $n_2 = 6$. If $n_2 = 7$, then, by Theorem 3.5, we have $n = \mathrm{lcm}\{n_1, n_2\} = \mathrm{lcm}\{6, 7\} = 6 \cdot 7 = 42$. Since $6 \cdot 7 = \mathrm{lcm}\{n_1, n_3\} = \mathrm{lcm}\{6, n_3\}$, n_3 should be a multiple of 7, and since $6 \cdot 7 = \mathrm{lcm}\{n_2, n_3\} = \mathrm{lcm}\{7, n_3\}$, n_3 should be a multiple of 6. Therefore $n_3 = 42$. From (3.1), we have $2g - 2 = 28$, hence $g = 15$. This contradicts our assumption $n \geq 3g$. If $n_2 = 8$, then, by Theorem 3.5, we have $n = \mathrm{lcm}\{n_1, n_2\} = \mathrm{lcm}\{6, 8\} = 24$. The branching index n_3 should be a divisor of $n = 24$ which is more than or equal to $n_2 = 8$, so $n_3 = 12, 24$. If $n_3 = 12$, then $\mathrm{lcm}\{n_3, n_1\} = \mathrm{lcm}\{12, 6\} = 12 \neq 24 = n$, which conflicts with Theorem 3.5. So, $n_3 = 24$. From (3.1), we have $2g - 2 = 16$, hence $g = 9$. This contradicts our assumption $n \geq 3g$. If $n_2 = 9, 10, 11$, from the argument as above, we come to a contradiction to our assumption $n \geq 3g$. Thus we finished the proof of Lemma 3.4.

In Lemma 3.4, we found the denominators of valencies. For the proof of Theorem 3.1, we have to find good numerators.

A branch point corresponds to a fixed point, if the branching index is equal to the period. Hence, in each case, there is a fixed point. We can take a proper power so that the numerator of the valency at the fixed point is equal to 1. Therefore we assume that, for one of the fixed points, the valency of the corresponding branch point is $1/n$. Cases (1) and (2) in the statement of Lemma 3.4 are treated by Kulkarni in [7, §5], so we discuss other cases.

For case (3), there are 2 candidates,

$$\text{(A)}\quad \frac{1}{3}+\frac{2g+1}{3g+3}+\frac{1}{3g+3}\,, \qquad \text{(B)}\quad \frac{2}{3}+\frac{g}{3g+3}+\frac{1}{3g+3}\,.$$

Since $n_2 = g+1$, if we rewrite the second fractions into irreducible fractions, then their denominators should be $g+1$. Since $g+1$ is not a multiple of 3, g is equal to $3k$ or $3k+1$. If $g = 3k$, then $2g+1 = 6k+1$ is not a multiple of 3, hence (A) is not the case. In (B), the second fraction is equal to an irreducible fraction $k/(3k+1) = k/(g+1)$. Therefore the valencies should be $2/3 + k/(g+1) + 1/(3g+3)$ when $g = 3k$. If $g = 3k+1$, then g is not a multiple of 3, hence (B) is not the case. In (A), the second fraction is equal to an irreducible fraction $(2k+1)/(3k+2) = (2k+1)/(g+1)$. Therefore the valencies should be $1/3 + (2k+1)/(g+1) + 1/(3g+3)$ when $g = 3k+1$.

For case (4), there are 2 candidates,

$$\text{(C)}\quad \frac{1}{3}+\frac{2g-1}{3g}+\frac{1}{3g}\,, \qquad \text{(D)}\quad \frac{2}{3}+\frac{g-1}{3g}+\frac{1}{3g}\,.$$

If $g = 3k$, then the second fractions $(2g-1)/3g$ of (C) and $(g-1)/3g$ of (D) are irreducible fractions, hence (C) and (D) are data of periodic maps we need. By the remark after Theorem 3.1, the $(g-1)^*$-th power of (C) is

$$\frac{g-1}{3}+\frac{(g-1)(2g-1)}{3g}+\frac{g-1}{3g}=\frac{2}{3}+\frac{1}{3g}+\frac{g-1}{3g}\,,$$

where we consider a denominator of a valency a/b as an element of $\boldsymbol{Z}/b\boldsymbol{Z}$. Therefore, (D) is the $(g-1)^*$-th power of (C). If $g = 3k+1$, then $g-1 = 3k$ is a multiple of 3, so the second fraction of (D) is not an irreducible fraction, which means $m_2 \neq 3g$ and is not the case. In (C), the second fraction $(2g-1)/3g$ is an irreducible fraction. Therefore, (C) is the data of periodic map we need. If $g = 3k+2$, then $2g-1 = 6k+3$ is a multiple of 3, so the second fraction of (C) is not an irreducible fraction. In (D), the second fraction $(g-1)/3g$ is an irreducible fraction. Therefor (D) is the data of periodic map we need.

For the cases (5) to (9), we find the numerators of valency data uniquely as shown in the statement in Theorem 3.1 easily.

We finished the proof of Theorem 3.1.

REMARK 3.7. 1. The periodic maps (1) and (2) in Theorem 3.1 commute with the hyperelliptic involution. The presentation of these maps as products of Dehn twists are investigated by Ishizaka [5]. Other periodic maps listed in Theorem 3.1 are not on the list in [5] of periodic maps which commute with the hyperelliptic involution.

2. Each periodic map (3) in Theorem 3.1 is a monodromy of $(3, g+1)$-torus knot, and their presentation as a product of Dehn twists are give in [3, §4.3] ((3)-(i) is "Phase I", and (3)-(ii) is "Phase III").

3. The periodic maps illustrated in Introduction are (1) and (2) in Theorem 3.1. In Introduction, we visualized the relationship between (1) and (2). There is a same kind of relationship between (3) and (4). In Figure 3, we illustrate the relationship when $g = 3$.

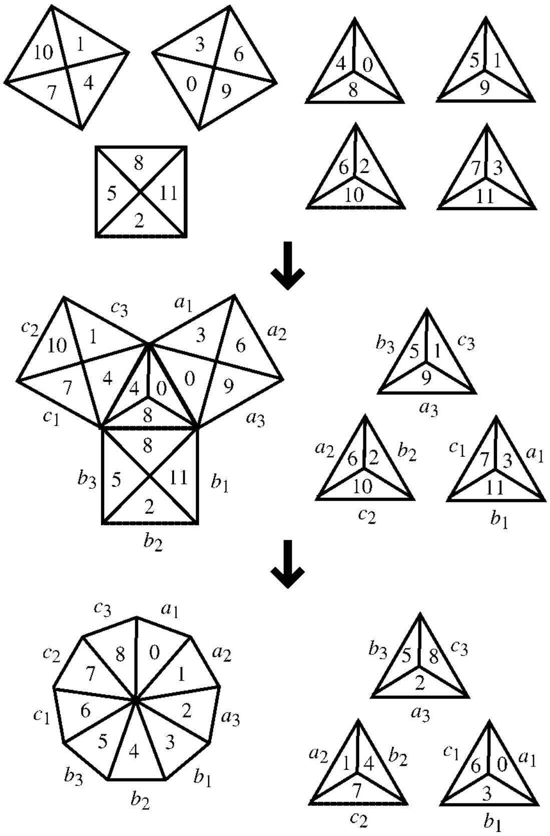

FIGURE 3. The top figure illustrates $(12, 2/3+1/4+1/12)$. The bottom figure illustrates $(9, 1/3+5/9+1/9)$.

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