NOTE ON DIRICHLET SERIES (V) ON THE INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES (I)

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1. Introduction. Let us put

$$(1.1)\quad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \dots < \lambda_n \to +\infty).$$

Let (1.1) be uniformly convergent in the whole plane, i.e. for any given σ $(-\infty < \sigma < +\infty)$, (1.1) be uniformly convergent for $\sigma \leqq \Re(s)$. Then (1.1) defines the integral function, and for any given σ, $\sup_{-\infty<t<+\infty} |F(\sigma + it)|$ has the finite value $M(\sigma)$. After J. Ritt ([1], pp. 18–19) we can define the order and type of (1.1) as follows:

DEFINITION. *The order of* (1.1) *is defined by*

$$(1.2)\quad \rho = \overline{\lim_{\sigma\to-\infty}}(-\sigma)^{-1} \cdot \log^+\log^+ M(\sigma),$$

where $M(\sigma) = \sup_{-\infty<t<+\infty} |F(\sigma + it)|$, $\log^+ x = \mathrm{Max}(0, \log x)$. *If* $0 < \rho < +\infty$, *the type* k *of* (1.1) *is defined by*

$$(1.3)\quad k = \overline{\lim_{\sigma\to-\infty}} 1/\exp((-\sigma)\rho) \cdot \log^+ M(\sigma).$$

J. Ritt [2], S. Izumi [3] and K. Sugimura [4] have given formulas determining ρ and k in terms of $\{a_n\}$ $(n = 1, 2, \dots)$ under some additional conditions imposed upon $\{\lambda_n\}$ $(n = 1, 2, \dots)$. In this note, we shall establish more general formulas determining ρ and k in terms of $\{a_n\}$ $(n = 1, 2, \dots)$.

2. Theorem. The main theorem reads as follows:

MAIN THEOREM. *Let* (1.1) *be uniformly convergent in the whole plane. Then we have*

$$(2.1)\quad \overline{\lim_{x\to+\infty}} (x \log x)^{-1} \cdot \log T_x = -\rho_u^{-1},$$

where

$$\begin{cases} \text{(i)} & T_x = \sup_{-\infty<t<+\infty} \Big| \sum_{[x]\leqq\lambda_n<x} a_n \exp(-i\lambda_n t) \Big|^{*}, \\ \text{(ii)} & M_u(\sigma) = \sup_{\substack{-\infty<t<+\infty \\ 1\leqq k<+\infty}} \Big| \sum_{n=1}^{k} a_n \exp(-\lambda_n(\sigma + it)) \Big|, \\ \text{(iii)} & \rho_u = \overline{\lim_{\sigma\to-\infty}} (-\sigma)^{-1} \cdot \log^+\log^+ M_u(\sigma) \qquad (\geqq 0). \end{cases}$$

* $[x]$ means the greatest integer contained in x.

If furthermore $0<\rho_u<+\infty$, *then we get*

$$\overline{\lim_{x\to+\infty}}\,(x^{-1}\cdot\log T_x+\rho_u^{-1}\cdot\log x)=\rho^{-1}\cdot\log(e\,\rho_u\,k_u), \tag{2.2}$$

where $$k_u=\overline{\lim_{\sigma\to-\infty}}\,1/\exp((-\sigma)\rho_u)\cdot\log^+M_u(\sigma).$$

REMARK. (1) By M. Kuniyeda's theorem ([5], pp. 8-9), the uniform convergence-abscissa σ_u of (1.1) is given by

$$-\infty=\sigma_u=\overline{\lim_{x\to+\infty}}\,x^{-1}\cdot\log T_x.$$

(2) Since $M(\sigma)\leqq M_u(\sigma)$, this main theorem can not give the exact value of ρ and k in terms of $\{a_n\}$ $(n=1,2,\dots.)$.

From this main theorem follow next theorems, whose proof we shall give later.

THEOREM I. *Let* (1.1) *be uniformly convergent in the whole plane. Then we have*

$$-1/\rho_c\leqq-1/\rho\leqq-1/\rho_u\leqq-1/\rho_c+\overline{\lim_{x\to+\infty}}(x\log x)^{-1}\cdot\log^+N(x) \tag{2.3}$$

where
$$\begin{cases}\text{(i)} & -1/\rho_c=\overline{\lim_{n\to+\infty}}(\lambda_n\log\lambda_n)^{-1}\cdot\log|a_n|,\\ \text{(ii)} & N(x)=\sum\limits_{[x]\leqq\lambda_n<x}1.\end{cases}$$

REMARK. By a lemma ([6], p. 50) we have

$$0\leqq\sigma_s-C\leqq\overline{\lim_{n\to+\infty}}\lambda_n^{-1}\cdot\log n, \tag{2.4}$$

where
$$\begin{cases}\text{(i)} & \sigma_s:\text{simple convergence-abscissa of (1.1)},\\ \text{(ii)} & C=\overline{\lim_{n\to+\infty}}\,\lambda_n^{-1}\cdot\log|a_n|.\end{cases}$$

Therefore, by (2.4) and $\sigma_s=-\infty$, we get $C=-\infty$, so that we can put $\rho_c\geqq0$.

THEOREM II. *Let* (1.1) *with* $\Re(a_n)\geqq0$ $(n=1,2,\dots.)$ *be uniformly convergent in the whole plane. Then we get*

$$-1/\rho_u+\Delta_1\leqq-1/\rho\leqq-1/\rho_u, \tag{2.5}$$

where $$\Delta_1=\underline{\lim_{n\to+\infty}}(\lambda_n\log\lambda_n)^{-1}\cdot\log(\cos\theta_n),\quad\theta_n=\arg(a_n).$$

THEOREM III. *Let* (1.1) *be uniformly convergent in the whole plane. If* $\overline{\lim_{x\to+\infty}}(x\log x)^{-1}\cdot\log^+N(x)=0$, *and* $0<\rho<+\infty$, *then*

$$k_c\leqq k\leqq k_u\leqq k_c\exp\{\rho\cdot\overline{\lim_{x\to+\infty}}x^{-1}\cdot\log^+N(x)\}, \tag{2.6}$$

where
$$\begin{cases}\text{(i)} & \rho^{-1}\cdot\log(e\rho\,k_c)=\overline{\lim_{n\to+\infty}}\{\lambda_n^{-1}\cdot\log|a_n|+\rho^{-1}\cdot\log\lambda_n\},\\ \text{(ii)} & \rho^{-1}\cdot\log(e\,\rho\,k_u)=\lim_{x\to+\infty}\{x^{-1}\cdot\log T_x+\rho^{-1}\cdot\log x\}.\end{cases} \tag{2.7}$$

REMARK. On account of Theorem 1, we obtain $\rho_c=\rho_u=\rho$, so that we can define k_c and k_u by (2.7).

THEOREM IV. *Let* (1.1) *with* $\Re(a_n) \geqq 0$ $(n = 1, 2, \ldots.)$ *be uniformly convergent in the whole plane. If* $\Delta_1 = 0$, *and* $0 < \rho < +\infty$, *then*

$$k_u \exp(\rho\Delta_2) \leqq k \leqq k_u, \tag{2.8}$$

where
$$\Delta_2 = \lim_{n\to+\infty} \lambda_n^{-1} \cdot \log(\cos\theta_n), \quad \theta_n = \arg(a_n).$$

REMARK. By Theorem 2, we have $\rho_u = \rho$. Hence we can define k_u by (2.7), (ii).

3. Lemmas. To prove these theorems, we need some lemmas.

LEMMA I. *Let* (1.1) *be uniformly convergent in the whole plane. Suppose that*

$$M_u(\sigma) < A \exp\{\beta \exp((-\sigma)\alpha)\}, \tag{3.1}$$

for sufficiently large $-\sigma(\sigma < 0)$, *where*

$$\text{(i)} \quad M_u(\sigma) = \sup_{\substack{-\infty < t < +\infty \\ 1 \leqq k < +\infty}} \left| \sum_{n=1}^{k} a_n \exp(-\lambda_n(\sigma + it)) \right|,$$

(ii) A, α, β: *positive constants.*

Then we have

$$\begin{cases} \overline{\lim_{x\to+\infty}} (x \log)^{-1} \cdot \log T_x \leqq -\alpha^{-1}, \\ \overline{\lim_{x\to+\infty}} (x^{-1} \cdot \log T_x + \alpha^{-1} \cdot \log x) \leqq \alpha^{-1} \cdot \log(\alpha\beta e). \end{cases} \tag{3.2}$$

PROOF. Let us denote by $\{\lambda_{j,m}\}$ $(m = 1, 2, \ldots, r(x))$ λ_n's contained in $[x] = j \leqq \lambda_n < x$, and by $\{a_{j,m}\}$ its coefficients. Setting

$$S_{j,m}(\sigma, t) = \sum_{\lambda_1 \leqq \lambda_n \leqq \lambda_{j,m}} a_n \exp(-\lambda_n(\sigma + it)) \qquad (\sigma < 0),$$

by Abel's transformation we get

$$\begin{aligned} \sum_{[x]\leqq\lambda_n<x} a_n \exp(-it\lambda_n) &= \sum_{m=1}^{r} a_{j,m} \exp(-it\lambda_{j,m}) \\ &= \sum_{m=1}^{r-1} S_{j,m}(\sigma, t)\{\exp(\sigma\lambda_{j,m}) - \exp(\sigma\lambda_{j,m+1})\} \\ &\quad + S_{j,r}(\sigma, t) \exp(\sigma\lambda_{j,r}) - S_{j,0}(\sigma, t) \exp(\sigma\lambda_{j,1}), \end{aligned}$$

where
$$S_{j,0}(\sigma, t) = \sum_{\lambda_1 \leqq \lambda_n < \lambda_{j,1}} a_n \exp(-\lambda_n(\sigma + it)).$$

Hence $\left| \sum_{[x]\leqq\lambda_n<x} a_n \exp(-it\lambda_n) \right| \leqq 2M_u(\sigma) \exp(\sigma\lambda_{j,1}) \leqq 2M_u(\sigma) \exp(\sigma[x])$.

Since the right-hand side is independent of t, we get

$$T_x \leqq 2M_u(\sigma) \exp(\sigma[x]),$$

so that, by (3.1), for sufficiently large $-\sigma$ ($\sigma < 0$),

(3.3) $$T_x < 2A\exp\{\beta\exp((-\sigma)\alpha) + [x]\sigma\}.$$

If we let the righ-hand side of (3.3) take its minimum, we get easily

$$T_x < 2A\exp\{-[x]/\alpha\cdot\log([x]/\alpha\beta e)\},$$

from which (3.2) immediately follows.

LEMMA II. *Let* (1.1) *be uniformly convergent in the whole plane. Assume that*

(3.4) $$T_x < \exp\{-([x]+1)/\alpha\cdot\log(([x]+1)/\alpha\beta e)\}$$

for sufficiently large $x(>0)$, *where* α *and* β *are positive constants. Then, for sufficiently large* $-\sigma(\sigma<0)$ *we get*

(3.5) $$M_u(\sigma) \leqq A\exp\{\beta\exp((-\sigma)\alpha) + (-\sigma)\alpha\},$$

where A is a suitable constant.

PROOF. On account of (3.4), we have

(3.6) $$\left|\sum_{[x]\leqq\lambda_n<x} a_n\exp(-it\lambda_n)\right| \leqq T_x < \exp\{-([x]+1)/\alpha\cdot\log(([x]+1)/\gamma)\}$$

for arbitrary t ($-\infty < t < +\infty$) and $[x] > X$, where $\gamma = \alpha\beta e$ and X are sufficiently large constants. Let us denote by $\{\lambda_{j,m}\}$ ($m = 1, 2, \ldots r(j)$) λ_n's contained in $j \leqq \lambda_n < j+1$, and by $a_{j,m}$ its coefficients. Put

$$S_{j,m}(t) = \sum_{k=1}^{m} a_{j,k}\exp(-it\lambda_{j,k}), \qquad S_{j,0}(t) = 0.$$

Then, by (3.6)

(3.7) $$|S_{j,m}(t)| < \exp\{-(j+1)/\alpha\cdot\log((j+1)/\gamma)\}$$

for $m = 1, 2, , \ldots r(j), j > X$.

Putting $[\lambda_\nu] = N$, $\lambda_\nu = \lambda_{N,s_1}$, and $[\lambda_\mu] = M$, $\lambda_\mu = \lambda_{M,s_2}$, ($\nu < \mu$), by Abel's transformation, we obtain

$$\sum_{n=\nu}^{\mu} a_n\exp(-\lambda_n(\sigma+it)) = \sum_{m=s_1}^{r(N)-1} S_{N,m}(t)\{\exp(-\sigma\lambda_{N,m}) - \exp(-\sigma\lambda_{N,m+1})\}$$
$$+ S_{N,r(N)}(t)\exp(-\sigma\lambda_{N,r(N)}) - S_{N,s_1-1}(t)\exp(-\sigma\lambda_{N,s_1-1})$$
$$+ \sum_{j=N+1}^{M-1}\sum_{m=1}^{r(j)-1} S_{j,m}(t)\{\exp(-\sigma\lambda_{j,m}) - \exp(-\sigma\lambda_{j,m+1})\} + S_{j,r(j)}(t)\exp(-\sigma\lambda_{j,r(j)})$$
$$+ \sum_{m=1}^{s_2-1} S_{M,m}(t)\{\exp(-\sigma\lambda_{M,m}) - \exp(-\sigma\lambda_{M,m+1}) + S_{M,s_2}(t)\exp(-\sigma\lambda_{M,s_2}).$$

Hence, by (3.7) and simple computations, we have

(3.8) $$\left|\sum_{n=\nu}^{\mu} a_n\exp(-\lambda_n(\sigma+it))\right.$$

$$< 3 \sum_{[\lambda_\nu]+1}^{[\lambda_\mu]+1} \exp(-j/\alpha \cdot \log(j/\gamma) - j\sigma) < 3 \sum_{j=1}^{\infty} \exp(-j/\alpha \cdot \log(j/\gamma) - j\sigma)$$

for $\sigma < 0$ and $[\lambda_\nu] > X$.

Now we can easily prove that for $\sigma < 0$,

$$\begin{cases} \text{(i)} \quad \operatorname*{Max}_{1 \leqq j < +\infty} \exp(-j/\alpha \cdot \log(j/\gamma) - j\sigma) \leqq \exp(\exp((-\sigma)\alpha)), \\ \text{(ii)} \quad \exp(-j/\alpha \cdot \log(j/\gamma) - j\sigma) < \exp(-j/\alpha) \\ \qquad\qquad\qquad \text{for } j > j(\sigma) = \exp((-\sigma)\alpha + 2). \end{cases}$$

Accordingly, putting

$$I = \sum_{j=1}^{\infty} \exp(-j/\alpha \cdot \log(j/\gamma) - j\sigma) = \sum_{j=1}^{j(\sigma)} + \sum_{j=j(\sigma)+1}^{\infty} = I_1 + I_2,$$

we get

$$I_1 < j(\sigma) \exp\{\exp((-\sigma)\alpha)\} = \alpha \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha + 2\},$$

$$I_2 < \sum_{j=0}^{\infty} \exp(-j/\alpha) = \{1 - \exp(-j/\alpha)\}^{-1},$$

so that, for sufficiently large $-\sigma(\sigma < 0)$,

$$I < 2\alpha \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha + 2\}.$$

Hence, (3.8) yields

$$\left| \sum_{n=\nu}^{\mu} a_n \exp(-\lambda_n(\sigma + it)) \right| < 6\alpha \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha + 2\}, \tag{3.9}$$

for $\mu > \nu$, $[\lambda_\nu] > X$, where μ is arbitrary, but ν is fixed.

On the other hand, for sufficiently large $-\sigma(\sigma < 0)$, we have evidently

$$\left| \sum_{\substack{n=1 \\ 1 \leqq k < \nu}}^{k} a_n \exp(-\lambda_n(\sigma + it)) \right| \leqq \sum_{n=1}^{\nu-1} |a_n| \exp(-\lambda_n \sigma) \tag{3.10}$$

$$< \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha\}.$$

Hence, by (3.9) and (3.10)

$$\left| \sum_{n=1}^{k} a_n \exp(-\lambda_n(\sigma + it)) \right| < \{6\alpha e^2 + 1\} \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha\},$$

for arbitrary k $(1 \leqq k < +\infty)$, t $(-\infty < t < +\infty)$ and sufficiently large $-\sigma$ $(\sigma < 0)$, so that immediately follows

$$M_u(\sigma) \leqq A \exp\{\exp((-\sigma)\alpha) + (-\sigma)\alpha\}, \quad A = (6\alpha e^2 + 1),$$

for sufficiently large $-\sigma(\sigma < 0)$, which proves Lemma 2.

4. Proof of Theorems.

PROOF OF MAIN THEOREM. By definition of ρ_u, for any given $\varepsilon(>0)$, there exist constants A and B depending only on ε such that

$$M_u(\sigma) < A \exp\{\exp((\rho_u + \varepsilon)(-\sigma))\}$$

for $\sigma < B < 0$. Hence, applying Lemma 1, in which $\beta = 1, \alpha = \rho_u + \varepsilon$, we get

$$-1/\rho_u^* \leqq -1/(\rho_u + \varepsilon),$$

where $-1/\rho_u^* = \overline{\lim_{x\to+\infty}} (x) \log x)^{-1} \cdot \log T_x$ $(\rho_u^* \geqq 0)$. Letting $\varepsilon \to 0$,

$$\rho_u^* \leqq \rho_u. \tag{4.1}$$

Since $-1/\rho_u^* = \overline{\lim_{x\to+\infty}} \{([x]+1)\cdot\log([x]+1)\}^{-1}\cdot \log T_x$, for any given $\varepsilon(>0)$, we have

$$T_x < \exp\{-([x]+1)/(\rho_u^* + \varepsilon)\cdot\log([x]+1)\}$$

for $[x] > X(\varepsilon)$. Accordingly, by Lemma 2, in which $\alpha = \rho_u^* + \varepsilon$, $\alpha\beta e = 1$, we get

$$M_u(\sigma) \leqq A \exp\{1/e(\rho_u^* + \varepsilon)\cdot \exp((-\sigma)(\rho_u^* + \varepsilon))\}$$

for sufficiently large $-\sigma$ $(\sigma < 0)$. Therefore, $\rho_u \leqq \rho_u^* + \varepsilon$. Letting $\varepsilon \to 0$,

$$\rho_u \leqq \rho_u^*. \tag{4.2}$$

Combining (4.1) with (4 2), we obtain $\rho_u = \rho_u^*$, which proves the first part of main theorem.

Arguing quite similarly, the second part of main theorem is also proved.

PROOF OF THEOREM I. Since $M(\sigma) \leqq M_u(\sigma)$, we get immediately

$$\rho \leqq \rho_u. \tag{4.3}$$

By definition of ρ_c, for any given $\varepsilon(>0)$, there exists $X(\varepsilon)$ such that

$$|a_n| < \exp(-\lambda_n \log\lambda_n/(\rho_c + \varepsilon)) \text{ for } \lambda_n > X(\varepsilon).$$

Hence

$$T_x \leqq \sum_{[x]\leqq\lambda_n<x} |a_n| < N(x)\ \exp\{-[x]\log[x]/(\rho_c+\varepsilon)\}$$

for $[x] > X(\varepsilon)$. Accordingly, by (2.1)

$$-1/\rho_u \leqq -1/(\rho_c + \varepsilon) + \overline{\lim_{x\to+\infty}} (x\log x)^{-1}\cdot\log^+ N(x),$$

Letting $\varepsilon \to 0$,

$$-1/\rho_u \leqq -1/\rho_c + \overline{\lim_{x\to+\infty}} (x\log x)^{-1}\cdot\log^+ N(x). \tag{4.4}$$

Taking account of Hadamard's theorem ([7], p. 15) and the uniform convergence in the whole plane of (1.1),

$$a_n = \lim_{T\to+\infty} T^{-1}\int_0^T F(\sigma + it)\ \exp(\lambda_n(\sigma+it))dt \qquad n = 1, 2, \dots,$$

so that

$$|a_n| \leqq M(\sigma)\ \exp(\sigma\lambda_n) \qquad (n = 1, 2, \dots). \tag{4.5}$$

By definition of ρ, we have, for any given $\varepsilon(>0)$,

$$M(\sigma) < \exp\{\exp((\rho + \varepsilon)(-\sigma))\}$$

for sufficiently large $-\sigma (\sigma < 0)$. Therefore, by (4.5),

$$|a_n| < \exp\{\exp((\rho + \varepsilon)(-\sigma)) - (-\sigma)\lambda_n\} \quad (n = 1, 2, \ldots) \tag{4.6}$$

for sufficiently large $-\sigma$. If we make minimum the right-hand side of (4.6), we get easily

$$|a_n| \leqq \exp\{-\lambda_n/(\rho + \varepsilon) \cdot \log(\lambda_n/(\rho + \varepsilon)e)\}$$

for sufficiently large n, so that

$$-1/\rho_c \leqq -1/(\rho + \varepsilon).$$

Letting $\varepsilon \to 0$,

$$-1/\rho_c \leqq -1/\rho. \tag{4.7}$$

On account of (4.3), (4.4) and (4.7), we get (2.3).

PROOF OF THEOREM II. Let us put

$$f(s) = \sum_{n=1}^{\infty} \Re(a_n) \exp(-\lambda_n s), \quad \Re(a_n) \geqq 0, \tag{4.8}$$

which is evidently absolutely convergent in the whole plane. Since

$$M(\sigma) = \sup_{-\infty < t < +\infty} |F(\sigma + it)| \geqq |F(\sigma)| \geqq f(\sigma) = \sup_{-\infty < t < +\infty} |f(\sigma + it)| = M_r(\sigma),$$

we have

$$\rho \geqq \rho_r, \tag{4.9}$$

where $\rho_r = \varlimsup_{\sigma \to \infty} (-\sigma)^{-1} \cdot \log^+ \log^+ M_r(\sigma)$. Since, by $\Re(a_n) \geqq 0$,

$$\sup_{\substack{-\infty < t < +\infty \\ 1 \leqq k < +\infty}} \left| \sum_{n=1}^{k} \Re(a_n) \exp(-\lambda_n(\sigma + it)) \right| = M_r(\sigma),$$

applying main theorem to $f(s)$, we obtain

$$\begin{aligned} -1/\rho_r &= \varlimsup_{x \to +\infty} (x \log x)^{-1} \cdot \log\left\{ \sup_{-\infty < t < +\infty} \left| \sum_{[x] \leqq \lambda_n < x} \Re(a_n) \exp(-it\lambda_n) \right| \right\} \\ &= \varlimsup_{x \to +\infty} (x \log x)^{-1} \cdot \log\left\{ \sum_{[x] \leqq \lambda_n < x} \Re(a_n) \right\}. \end{aligned} \tag{4.10}$$

On the other hand, we get easily

$$\sum_{[x] \leqq \lambda_n < x} \Re(a_n) = \sum_{[x] \leqq \lambda_n < x} |a_n| \cos\theta_n \geqq \cos\theta_{n(x)} \cdot \sum_{[x] \leqq \lambda_n < x} |a_n| \geqq \cos\theta_{n(x)} \cdot T_x,$$

where $\cos\theta_{n(x)} = \operatorname{Min}_{[x] \leqq \lambda_n < x}\{\cos\theta_n\}$, $T_x = \sup_{-\infty < t < +\infty} \left| \sum_{[x] \leqq \lambda_n < x} a_n \exp(-it\lambda_n) \right|$.

Hence, by (4.10) and (2.1), we obtain

$$\begin{aligned} -1/\rho_r &\geqq -1/\rho_u + \varliminf_{x \to +\infty} (x \log x)^{-1} \cdot \log\{\cos\theta_{n(x)}\} \\ &\geqq -1/\rho_u + \varliminf_{n \to +\infty} (\lambda_n \log \lambda_n)^{-1} \cdot \log\{\cos\theta_n\}, \end{aligned}$$

so that, by (4.9)

(4.11) $$-1/\rho \geqq -1/\rho_r \geqq -1/\rho_u + \Delta_1.$$

By (4.3) and (4.11), we have (2.5).

PROOF OF THEOREM III. Taking account of $\overline{\lim}_{x\to+\infty} (x\log x)^{-1}\cdot\log^+ N(x) = 0$ and Theorem 1, we have

$$1/\rho = -1/\rho_c = -1/\rho_u = \overline{\lim_{n\to+\infty}}(\lambda_n\log\lambda_n)^{-1}\cdot\log|a_n|.$$

Hence, by (2.2), we can define k_c and k_u by (2.7). Since $M(\sigma)\leqq M_u(\sigma)$, $\rho=\rho_u$, we get immediately

(4.12) $$k\leqq k_u.$$

By definition of k_c, there exists $X(\varepsilon)$ for any given $\varepsilon(>0)$, such that

$$|a_n| < \exp\{-\lambda_n/\rho\cdot\log(\lambda_n/e\rho(k_c+\varepsilon))\} \qquad \text{for } \lambda_n > X(\varepsilon).$$

Accordingly

$$T_x \leqq \sum_{[x]\leqq\lambda_n<c} |a_n| < N(x)\exp\{-[x]/\rho\cdot\log([x]/e\rho(k_c+\varepsilon))\}$$

for $[x] > X(\varepsilon)$, so that

$$\overline{\lim_{x\to+\infty}}\left(x^{-1}\cdot\log T_x + \rho^{-1}\cdot\log x\right) = \rho^{-1}\cdot\log(e\rho k_u)$$

$$\leqq \rho^{-1}\cdot\log(e\rho(k_c+\varepsilon)) + \overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log^+ N(x).$$

Letting $\varepsilon\to 0$,

$$\rho^{-1}\cdot\log(e\,\rho\,k_u)\leqq\rho^{-1}\cdot\log(e\,\rho\,k_c) + \overline{\lim_{x\to+\infty}}\cdot x^{-1}.\ \log^+ N(x), \text{ i. e.}$$

(4.13) $$k_u \leqq k_c\,\exp\{\rho\cdot\overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log^+ N(x)\}.$$

By definition of k, we have, for any given $\varepsilon(>0)$,

$$M(\sigma) < \exp\{(k+\varepsilon)\exp((-\sigma)\rho)\}$$

for sufficiently large $-\sigma(\sigma<0)$. Therefore, by (4.5)

(4.14) $$|a_n| < \exp\{(k+\varepsilon)\exp((-\sigma)\rho) - (-\sigma)\lambda_n\} \qquad (n=1,2,\ldots.)$$

for sufficiently large $-\sigma$. If we make minimum the right-hand side of (4.14), we have

$$|a_n| \leqq \exp\{-\lambda_n/\rho\cdot\log(\lambda_n/e\,\rho(k+\varepsilon))\}$$

for sufficiently large n, so that

$$\overline{\lim_{n\to+\infty}}(\lambda_n^{-1}\cdot\log|a_n| + \rho^{-1}\cdot\log\lambda_n) = \rho^{-1}\cdot\log(e\,\rho\,k_c) \leqq \rho^{-1}\log(e\,\rho\,(k+\varepsilon)).$$

Letting $\varepsilon\to 0$, $\rho^{-1}\cdot\log(e\,\rho\,k_c)\leqq\rho^{-1}\cdot\log(e\,\rho\,k)$. Hence,

(4.15) $$k_c\leqq k$$

By virtue of (4.12), (4.13) and (4.15), we obtain (2.6).

PROOF OF THEOREM IV. On account of $\Delta_1 = 0$ and Theorem 2, we get

$$-1/\rho = -1/\rho_u = \overline{\lim_{x\to+\infty}}(x\log x)^{-1}\cdot\log T_x.$$

Hence, by (2.2), we can put

$$\rho^{-1}\cdot\log(e\,\rho\,k_u) = \overline{\lim_{x\to+\infty}}(x^{-1}\cdot\log T_x + \rho^{-1}\cdot\log x), \tag{4.16}$$

where $k_u = \overline{\lim_{\sigma\to-\infty}} 1/\exp((-\sigma)\rho)\cdot\log^+ M_u(\sigma)$. Accordingly, on account of $M(\sigma)\leqq M_u(\sigma)$, $\rho=\rho_u$, we have easily

$$k \leqq k_u. \tag{4.17}$$

Using the same notations as in the proof of Theorem 2, and $\Delta_1 = 0$ and (4.11), we have $\rho = \rho_u = \rho_r$. Therefore, applying the main theorem to $f(s) = \sum_{n=1}^{\infty}\Re(a_n)\exp(-\lambda_n s)$ with $\Re(a_n)\geqq 0$, we get

$$\rho^{-1}\cdot\log(e\,\rho\,k_r) = \lim_{x\to+\infty}\left\{x^{-1}\cdot\log\Big(\sum_{[x]\leqq\lambda_n<x}\Re(a_n)\Big) + \rho^{-1}\cdot\log x\right\}, \tag{4.18}$$

where (i) $k_r = \overline{\lim_{\sigma\to-\infty}}\ 1/\exp((-\sigma)\rho)\cdot\log^+ M_r(\sigma)$,

(ii) $M_r(\sigma) = \operatorname*{Sup}_{-\infty<t<+\infty}|f(\sigma+it)| = f(\sigma)$.

Hence, by $M_r(\sigma)\leqq M(\sigma)$, $\rho=\rho_r$, we have

$$k_r \leqq k. \tag{4.19}$$

In the proof of Theorem 2, we have proved that

$$\sum_{[x]\leqq\lambda_n<x}\Re(a_n) \geqq \cos\theta_{n(x)}\cdot T_x,$$

where $\cos\theta_{n(x)} = \operatorname*{Min}_{[x]\leqq\lambda_n<x}\{\cos\theta_n\}$. Hence, by (4.18) and (4.16),

$$\begin{aligned}\rho^{-1}\cdot\log(e\,\rho\,k_r) &\geqq \overline{\lim_{x\to+\infty}}(x^{-1}\cdot\log T_x + \rho^{-1}\cdot\log x) + \underline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log\{\cos\theta_{n(x)}\}\\ &\geqq \rho^{-1}\cdot\log(e\,\rho\,k_u) + \underline{\lim_{n\to+\infty}}\,\lambda_n^{-1}\cdot\log\{\cos\theta_n\}\\ &= \rho^{-1}\cdot\log(e\rho k_u) + \Delta_2,\end{aligned}$$

so that

$$k_r \geqq k_u\exp(\rho\Delta_2). \tag{4.20}$$

By virtue of (4.17) (4.19) and (4.20), the required question (2.8) is completely established.

5. Corollaries. From Theorem 1, we get immediately

COROLLARY I. *Let* (1.1) *be uniformly convergent in the whole plane. If* $\lim_{x\to+\infty}(x\log x)^{-1}\cdot\log^+ N(x) = 0$, $N(x) = \sum_{[x]\leqq\lambda_n<x} 1$, *then its order* ρ *is given by*

$$-1/\rho = \lim_{n\to+\infty}(\lambda_n\log\lambda_n)^{-1}\cdot\log|a_n|. \tag{5.1}$$

As its special case, we obtain

COROLLARY II. (J. Ritt. [2]) *Let* (1.1) *be simply convergent in the whole plane. If* $\overline{\lim\limits_{n\to+\infty}}\, \lambda_n^{-1}\cdot\log n < +\infty$, *then* (5.1) *holds.*

REMARK. J. Ritt supposed the absolute convergence in the whole plane, but it is a consequence of $\sigma_s = -\infty$ and $\overline{\lim\limits_{n\to+\infty}}\, \lambda_n^{-1}\cdot\log n < +\infty$.

PROOF. By the similar arguments as a lemma in [6] p. 50, we get

$$0 \leqq \sigma_a - \sigma_s \leqq \overline{\lim_{x\to+\infty}}\; x^{-1}\cdot\log^+ N(x) \leqq \overline{\lim_{n\to+\infty}}\, \lambda_n^{-1}\log n, \tag{5.2}$$

where $\sigma_a(\sigma_s)$ is the absolute (simple) convergence-abscissa of (1.1). Hence, on account of $\sigma_s = -\infty$ and $\overline{\lim\limits_{n\to+\infty}}\, \lambda_n^{-1}\cdot\log n < +\infty$, $\sigma_a = -\infty$. A fortiori, (1.1) is uniformly convergent in the whole plane. By (5.2) and $\overline{\lim\limits_{n\to+\infty}}\, \lambda_n^{-1}\cdot\log n < +\infty$, we get evidently $\lim\limits_{x\to+\infty}(x\log x)^{-1}\cdot\log^+ N(x) = 0$, so that Corollary 2 is a special case of Corollary 1.

COROLLARY III. (K. Sugimura [4]) *Let* (1.1) *be simply convergent in the whole plane. If* $\sum\limits_{\lambda_n<x} 1 = O(x^{\delta x})$ *for any given* $\delta > 0$, *and* $0 \leqq \rho_c < +\infty$, where

$-1/\rho_c = \overline{\lim\limits_{n\to+\infty}}\,(\lambda_n\log\lambda_n)^{-1}\cdot\log|a_n|$, *then* (5.1) *holds.*

REMARK. K. Sugimura have not assumed $0 \leqq \rho_c < +\infty$ explicitly, but he assumed it implicitly.

PROOF. By hypothesis, we get easily

$$N(x) = \sum_{[x]\leqq\lambda_n<x} 1 < \sum_{\lambda_n<[x+1]} 1 = O(([x]+1)^{\delta([x]+1)}). \tag{5.3}$$

Hence $0 \leqq \overline{\lim\limits_{x\to+\infty}}\,(x\log x)^{-1}\cdot\log^+ N(x) \leqq \delta$. Letting $\delta\to 0$,

$$\lim_{x\to+\infty}(x\log x)^{-1}\cdot\log^+ N(x) = 0. \tag{5.4}$$

On account of hypothesis, we can determine $X(\varepsilon)$ for any given $\varepsilon\,(>0)$, such that, for $\lambda_n > X(\varepsilon)$

$$|a_n| < \exp\{-\lambda_n\log\lambda_n/(\rho_c+\varepsilon)\}.$$

Hence, by (5.3),

$$\begin{aligned}\sum_{[x]\leqq\lambda_n<x}|a_n| &< N(x)\,\exp\{-([x]+1)\cdot\log(([x]+1)/(\rho_c+\varepsilon))\}\\ &< 0(([x]+1)^{\delta([x]+1)})\cdot\exp\{-([x]+1)\cdot\log(([x]+1)/(\rho_c+\varepsilon))\},\end{aligned}$$

for $[x] > X(\varepsilon)$. Therefore,

$$\overline{\lim_{x\to+\infty}}\; x^{-1}\cdot\log\Big\{\sum_{[x]\leqq\lambda_n<x}|a_n|\Big\} \leqq \lim_{x\to+\infty}\log([x]+1)\cdot\{\delta-1/(\rho_c+\varepsilon)\}.$$

Since $0 \leqq \rho_c < +\infty$, taking sufficiently small $\delta(>0)$, we can assume that $\delta - 1/(\rho_c + \varepsilon) < 0$. Hence,

$$\overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log\Big\{\sum_{[x]\leqq\lambda_n<x}|a_n|\Big\} \leqq -\infty,$$

which proves $\sigma_a = -\infty$. A fortiori, (1.1) is uniformly convergent in the whole plane. Thus, by (5.4) and Corollary 1, Corollary 3 is established.

From Theorem 2 immediately follows

COROLLARY IV. *Let* (1.1) *be uniformly convergent in the whole plane. If* $\Re(a_n) \geqq 0$ *and* $\lim\limits_{n\to+\infty}(\lambda_n\log\lambda_n)^{-1}\cdot\log(\cos\theta_n) = 0$, $\theta_n = \arg(a_n)$, *then its order* ρ *is given by*

$$-1/\rho = \lim_{x\to+\infty}(x\log x)^{-1}\cdot\log T_x.$$

As a corollary of Theorem 3, we get a generalization of S. Izumi's theorem [3].

COROLLARY V. (S. Izumi) *Let* (1.1) *with* $\overline{\lim\limits_{x\to+\infty}}\, x^{-1}\cdot\log^+ N(x) = 0$ *be simply (necessary absolutely) convergent in the whole plane. If* $0 < \rho < +\infty$, *then its type* k *is given by*

$$\rho^{-1}\cdot\log(e\,\rho\,k) = \overline{\lim_{n\to+\infty}}\{\lambda_n^{-1}\cdot\log|a_n| + \rho^{-1}\cdot\log\lambda_n\}.$$

PROOF. By (5.2) and hypothesis, we have $\sigma_a = \sigma_s = -\infty$. A fortiori, (1.1) converges uniformly in the whole plane. From $\overline{\lim\limits_{x\to+\infty}}\, x^{-1}\cdot\log^+N(x) = 0$ we get evidently $\overline{\lim\limits_{x\to+\infty}}\,(x\log x)^{-1}\cdot\log^+N(x) = 0$. Hence, by Theorem 2, $k_r = k = k_u$, which proves Corollary 5.

As a special case of Theorem 4, we have

COROLLARY IV. *Let* (1.1) *with* $\Re(a_n)\geqq 0$, $\lim\limits_{n\to+\infty}\lambda_n^{-1}\cdot\log(\cos\theta_n) = 0$, $\theta_n = \arg(a_n)$ *be simply (necessarily absolutely) convergent in the whole plane. If* $0<\rho<+\infty$, *then its type* k *is determined by*

$$\rho^{-1}\cdot\log(e\,\rho\,k) = \overline{\lim_{x\to+\infty}}(x^{-1}\cdot\log T_x + \rho^{-1}\cdot\log x).$$

PROOF. We have easily

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| \geqq \Big|\sum_{[x]\leqq\lambda_n<x}\Re(a_n)\Big| = \sum_{[x]\leqq\lambda_n<x}|a_n|\cos\theta_n \geqq \cos\theta_{n(x)}\cdot\sum_{[x]\leqq\lambda_n<x}|a_n|,$$

where $\cos\theta_{n(x)} = \operatorname*{Min}\limits_{[x]\leqq\lambda_n<x}\{\cos\theta_n\}$. Hence, by T. Kojima's theorem [8]

$$(5.5)\qquad -\infty = \sigma_s = \overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x}a_n\Big|$$
$$\geqq \lim_{x\to+\infty} x^{-1}\cdot\log\{\cos\theta_{n(x)}\} + \overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log\Big\{\sum_{[x]\leqq\lambda_n<x}|a_n|\Big\}$$

$$= \overline{\lim_{x\to+\infty}}\, x^{-1}\cdot\log\{\cos\theta_{n(x)}\} + \sigma_a.$$

On the other hand, from $\lim_{n\to+\infty}\lambda_n^{-1}\cdot\log(\cos\theta_n)=0$, we have easily

$$\lim_{x\to+\infty} x^{-1}\cdot\log\{\cos\theta_{n(x)}\}=0,$$

so that, by (5.5), $\sigma_a = -\infty$. A fortiori, (1.1) converges uniformly in the whole plane. Thus, by Theorem 4 and $\lim_{n\to+\infty}\lambda_n^{-1}\cdot\log(\cos\theta_n)=0$, we get easily $k=k_u$, which proves Corollary 4.

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