ON COVERING SURFACES OF A CLOSED RIEMANN SURFACE OF GENUS $p \geqq 2$

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1. Let F be a closed Riemann surface of genus $p \geqq 2$ spread over the z-plane. We cut F along p disjoint ring cuts C_i $(i = 1, 2, \cdots, p)$ and let F_0 be the resulting surface. We take infinitely many same samples as F_0 and connect them along the opposite shores of C_i as in the well known way, then we obtain a covering surface $F^{(\infty)}$ of F, which is of planar character. Hence by Koebe's theorem, we can map $F^{(\infty)}$ conformally on a schlicht domain D on the ζ-plane, whose boundary E is a non-dense perfect set, which is the singular set of a certain linear group of Schottky type. Myrberg[1] proved:

THEOREM 1. *E is of positive logarithmic capacity.*

In another paper[2], I have proved:

THEOREM 2. *Every point of E is a regular point for Dirichlet problem.*

Hence $F^{(\infty)}$ is of positive boundary and its Green's function $G(z, z_0)$ tends to zero, when z tends to the ideal boundary of $F^{(\infty)}$. Now instead of cutting F along p ring cuts, we cut F along q $(1 \leqq q \leqq p)$ ring cuts C_i $(i=1,2,\cdots,q)$ and let F_0 be the resulting surface. We take infinitely many same samples as F_0 and connect them along the opposite shores of C_i $(i=1,2,\cdots,q)$, then we obtain a covering surface $F^{(\infty)}_{(q)}$ of F. Then I have proved in another paper[3] the following extension of Theorem 1.

THEOREM 3. *$F^{(\infty)}_{(1)}$ is of null boundary, while if $q \geqq 2$, $F^{(\infty)}_{(q)}$ is of positive boundary and there exists a non-constant bounded harmonic funtion on $F^{(\infty)}_{(q)}$, whose Dirichlet integral is finite.*

In this paper, we shall prove the following extension of Theorem 2.

THEOREM 4. *The Green's function $G(z, z_0)$ of $F^{(\infty)}_{(q)}$ $(q \geqq 2)$ tends to zero, when z tends to the ideal boundary of $F^{(\infty)}_{(q)}$.*

2. Let Φ be a non-compact subsurface of $F^{(\infty)}_{(q)}$ $(q \geqq 2)$ whose compact boundary

1) P. J. MYRBERS: Die Kapaziät der singulären Menge der linearen Grupen, Ann. Acad. Fenn. Series A. Math.-Phys. 10(1941).

M. TSUJI: On the uniformization of an algebraic function of genus $p \geqq 2$, Tôhoku Math. Journ. 3(1951).

2) M. TSUJI: On the capacity of a general Cantor set, Jourual Math. Soc. Japan, 5(1953).

3) M. TSUJI Theory: of meromorphic functions on an open Riemann surface with null boundary, Nagoya Math. Journ. 6(1953).

consists of one analytic Jordan curve Γ_0. We exhaust Φ by a sequence of compact Riemann surfaces:

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_n \subset \to \Phi,$$

where $\Gamma_0 + \Gamma_n$ is the boundary of Φ_n and Γ_n consists of a finite number of analytic Jordan curves. Let $u_n(z)$ be harmonic in Φ_n, such that $u_n(z) = 1$ on Γ_0, $u_n(z) = 0$ on Γ_n, then $u_n(z)$ increases with n, so that we put

$$\lim_{n\to\infty} u_n(z) = u_\Phi(z).$$

Then as I have proved[4]), $u_\Phi(z) \not\equiv \text{const.}$, so that $0 < u_\Phi(z) < 1$ in Φ and $u_\Phi(z) = 1$ on Γ_0.

It is easily seen that Theorem 4 is equivalent to

THEOREM 5. *For any Φ, $u_\Phi(z)$ tends to zero, when z tends to the ideal boundary of Φ.*

Next we shall prove Theorem 5.

PROOF. Let C_i^+, C_i^- be the both shores of C_i. Instead of $C_1^+, C_1^-, \cdots, C_q^+, C_q^-$ we write $\Gamma_1, \Gamma_2, \cdots, \Gamma_{2q}$. In the following, F_j, $F_{ji_1}, \cdots$ denote the same sample as F_0, which will be defined as follows.

We connect F_j $(j = 1, 2, \cdots, 2q)$ to F_0 along Γ_j and let $\Gamma_j + \sum_{i_1=1}^{2q-1} \Gamma_{ji_1}$ be its boundary. We connect $F_{ji_1} (i_1 = 1, 2, \cdots, 2q-1)$ to F_j along Γ_{ji_1} and let $\Gamma_{ji_1} + \sum_{i_2=1}^{2q-1} \Gamma_{ji_1i_2}$ be its boundary. Similarly we define $F_{ji_1\cdots i_n}$ and let $\Gamma_{ji_1\cdots i_n} + \sum_{i_{n+1}=1}^{2q-1} \Gamma_{ji_1\cdots i_{n+1}}$ be its boundary.

We put

$$F_j + \sum_{i_1}^{1,\cdots,2q-1} F_{ji_1} + \cdots + \sum_{i_1,\cdots,i_n}^{1,\cdots,2q-1} F_{ji_1\cdots i_n} + \cdots = (F_{(q)}^{(\infty)})_j \tag{1}$$

and

$$\Phi_n = F_j + \sum_{i_1}^{1,\cdots,2q-1} F_{ji_1} + \cdots + \sum_{i_1,\cdots,i_n}^{1,\cdots,2q-1} F_{ji_1\cdots i_n}, \tag{2}$$

then

$$\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_n \to (F_{(q)}^{(\infty)})_j.$$

The boundary of Φ_n is $\Gamma_j + \Gamma_{(n)}$, where

$$\Gamma_{(n)} = \sum_{i_1,\cdots,i_{n+1}}^{1,\cdots,2q-1} \Gamma_{ji_1\cdots i_{n+1}} \quad . \tag{3}$$

Let $u_j^{(n)}(z)$ be harmonic in $\Phi_{n,1}$, such that $u_j^{(n)}(z) = 1$ on Γ_j, $u_j^{(n)}(z) = 0$ on $\Gamma_{(n)}$. Then $u_j^{(n)}(z)$ increases with n, so that let

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4) M. TSJUI, loc. cit. 3)

$$\lim_{n\to\infty} u_j^{(n)}(z) = u_j(z). \tag{4}$$

Then as remarked above, $u_j(z) \not\equiv$ const., so that $0 < u_j(z) < 1$ in $(F_{(q)}^{(\infty)})_j$ and $u_j(z) = 1$ on Γ_j.

To prove our theorem, it suffices to prove that

$$\lim u_j(z) = 0, \tag{5}$$

when z tends to the ideal boundary of $(F_{(q)}^{(\infty)})_j$.

Let

$$\underset{z\in\Gamma_{ji_1}}{\mathrm{Max}}\ u_j(z) = \lambda_{ji_1}, \quad \underset{1\leqq j\leqq 2q, 1\leqq i_1\leqq 2q-1}{\mathrm{Max}} \lambda_{ji_1} = \lambda, \tag{6}$$

then

$$0 < \lambda < 1. \tag{7}$$

We put

$$F_{ji_1} + \sum_{i_2}^{1,\dots,2q-1} F_{ji_1i_2} + \cdots + \sum_{i_2,\cdots,i_n}^{1,\dots,2q-1} F_{ji_1i_n} + \cdots = \left(F_{(q)}^{(\infty)}\right)_{ji_1}. \tag{8}$$

We define $u_{ji_1}(z)$ for $(F_{(q)}^{(\infty)})_{ji_1}$ similarly we defined $u(z_j)$ for $(F_{(q)}^{(\infty)})_j$, then by the maximum principle, we have easily

$$\frac{u_j(z)}{\lambda} \leqq u_{ji_1}(z) \text{ in } F_{ji_1},$$

so that

$$\underset{z\in\Gamma_{ji_1i_2}}{\mathrm{Max}} \frac{u_j(z)}{\lambda} \leqq \underset{z\in\Gamma_{ji_1i_2}}{\mathrm{Max}}\ u_{ji_1}(z).$$

Since as easily seen from the definition of λ,

$$\underset{z\in\Gamma_{ji_1i_2}}{\mathrm{Max}}\ u_{ji_1}(z) \leqq \lambda,$$

we have

$$\underset{z\in\Gamma_{ji_1i_2}}{\mathrm{Max}}\ u_j(z) \leqq \lambda^2.$$

Similarly we have

$$\underset{z\in\Gamma_{ji_1\cdots i_n}}{\mathrm{Max}}\ u_j(z) \leqq \lambda^n. \tag{9}$$

From this we conclude that $\lim u_j(z) = 0$, when z tends to the ideal boundary of $(F_{(q)}^{(\infty)})_j$. Hence our theorem is proved.

3. We shall extend Theorem 3 as follows. Let F be a closed Riemann surface of genus $p \geqq 2$ spread over the z-plane and C_i, C_i' $(i = 1,2,\cdots,p)$ be conjugate ring cuts, such that C_i, C_j' $(i,j = 1,2,\cdots,p, i \neq j)$ are disjoint and C_i, C_i' have one common point z_i and we connect z_i to a point z_0 by a curve l_i, which are disjoint each other and if we cut F along C_i, C_i', l_i $(i=1,2,\cdots,p)$, then we obtain a simply connected surface. Now we cut F along C_i $(i = 1,2,\cdots,q)$ and C_{q+j}, C'_{q+j}, l_{q+j} $(j=1,2,\cdots,r)$, $1 \leqq q+r \leqq p)$ and let F_0 be the resulting surface. We take infinitely many same samples as F_0 and connect them along the opposite shores of these cuts, then we obtain a covering surface $F_{(q,r)}^{(\infty)}$ of F. Then

THEOREM 5. *$F^{(\infty)}_{(1,0)}$ and $F^{(\infty)}_{(0,1)}$ are of null boundary, while if $q+r\geqq 2$, $F^{(\infty)}_{(q,r)}$ is of positive boundary and there exists a non-constant bounded harmonic function on $F^{(\infty)}_{(q,r)}$, whose Dirichlet integral is finite.*

PROOF. By Theorem 3, $F^{(\infty)}_{(1,0)} = F^{(\infty)}_{(1)}$ is of null boundary and we can prove similarly as Theorem 3, that if $q+r\geqq 2$, then $F^{(\infty)}_{(q,r)}$ is of positive boundary and there exists a non-constant bounded harmonic function on $F^{(\infty)}_{(q,r)}$, whose Dirichlet integral is finite. Hence it remains only to prove that $F^{(\infty)}_{(0,1)}$ is of null boundary. We put $C = C_1$ and we assume that C, C' do not contain $z=\infty$ and branch points of F. We cut F along C, C' and let F_0 be the resulting surface. F_0 is bounded by a single closed curve, which consists of C^+, C^-, C'^+, C'^-. Hence we can represent F_0 topologically by a square with p-1 handles. We denote $C^+, C^-, C'^+, C^{-\prime}$ as the sides of F_0 and call the points of F_0 vertices, which correspond to the vertices of the square. We take infinitely many same samples F_i as F_0. We connect 8 F_i's to F_0, which have common sides or vertices with F_0 and let G_1 be the sum of these 8 F_i's and put $F^{(1)} = F_0 + G_1$. Next we connect 16 F_i's to $F^{(1)}$, which have common sides or vertices with $F^{(1)}$ and let G_2 be the sum of these 16 F_i's and put $F^{(2)} = F_0 + G_1 + G_2$. Similarly we define G_n, which consists of $8n$ F_i's and put

$$F^{(n)} = F_0 + G_1 + \cdots + G_n. \tag{1}$$

We take a schlicht circular disc $\varDelta_0$ in F_0, whose boundary is Γ_0. Then

$$\varDelta_0 \subset F^{(1)} \subset F^{(2)} \subset \cdots \subset F^{(n)} \to F^{(\infty)}_{(0,1)} \quad . \tag{2}$$

Let Γ_n be the boundy of $F^{(n)}$, which consists of one closed curve.

In virtue of the hypothesis on C, C', we can cover Γ_n by kn equal schlicht discs, whose radius is independent of n and k is a constant independent of n and any two of these discs overlap at most once.

Hence by Nevanlinna's theorem [5], $F^{(\infty)}_{(0,1)}$ is of null boundary.

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5) R. Nevanlinna: Ein Satz über offene Riemannsche Flächen. Ann. Acad. Sci. Fenn, A. I, 54. Nr. 3. (1940).