CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, II

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1. Introduction. The theory of integration on a measure space has been generalized to a W^*-algebra by Segal [10] and Dixmier [2] as a non-commutative extension of it. Applying their theory, some parts of the probability theory may be described in a certain W^*-algebra. In the paper of Dixmier [2], he has proved the existence of a mapping $x \to x^e$ defined on a semi-finite W^*-algebra A acting on a Hilbert space H into its W^*-subalgebra A_1 with the similar properties of the Dixmier's trace (= natural mapping) in the finite W^*-algebra, A being semi-finite provided every non-zero projection in A contains a non-zero finite projection in A (cf. [5]). In the previous paper [11], we have discussed for a σ-finite finite W^*-algebra A (with the faithful normal trace μ with $\mu(I) = 1$) that the mapping $x \to x^e$ is defined on $L^1(A)$ and valued on $L^1(A_1)$ and it has the likewise properties with the conditional expectation in the usual probability space, and we have also called it the conditional expectation relative to the W^*-subalgebra A_1, where $L^1(A)$ being a Banach space of all integrable operators on H in the sense of Segal (cf. [10]) which coincides with that in the sense of Dixmier (cf. [2]) as Banach space. Nakamura-Turumaru have also given a very simple proof of the characterization theorem of the conditional expectation in A (cf. [8]).

If A is a commutative W^*-algebra with a faithful normal trace μ, then there exists a probability space $(\Omega, \mathbf{B}, \nu)$ such that, considering the space B of all bounded random variables as the multiplication algebra on a Hilbert space $L^2(\Omega, \mathbf{B}, \nu)$, B is isomorphic with A by the canonical mapping ϕ satisfying $\mu(x) = \int_\Omega (\phi^{-1}(x))(\omega) d\nu(\omega)$ for every $x \in A$. Conversely, let $(\Omega, \mathbf{B}, \nu)$ be a probability space. Then the multiplication algebra B is a W^*-algebra on $L^2(\Omega, \mathbf{B}, \nu)$ and μ, defined by the above equation, is a faithful normal trace on it. Furthermore, the canonical mapping ϕ defines an isomorphism between $L^r(A)$ and $L^r(\Omega, \mathbf{B}, \nu)$ as Banach spaces ($r \geqq 1$), $L^r(A)$ being the Banach space defined by Dixmier (cf. [2]). For any W^*-subalgebra A_1 of A, there corresponds a σ-subfield $\mathbf{B}_1$ of $\mathbf{B}$, and A_1, $L^r(A_1)$ are isomorphic with B_1, $L^r(\Omega, \mathbf{B}_1, \nu)$ respectively, where B_1 being the multiplication algebra of the bounded random variables on $(\Omega, \mathbf{B}_1, \nu)$. The conditional expectation defined for the commutative algebra A (relative to the A_1) is transformed to the one defined for the corresponding probability space $(\Omega, \mathbf{B}, \nu)$ (relative to the $\mathbf{B}_1$) by the canonical mapping (cf. [7] and [11]).

In the probability theory, the martingales have been investigated by many authors, particularly by Doob, Lévy and Ville (cf. [3]), which is defined by a linear system of the conditional expectations. The concept of the martingale can be extended to a non-commutative W^*-algebra as the generalized conditional expectation.

In the present paper, we shall begin with a characterization theorem of the Dixmier mapping in a semi-finite W^*-algebra (cf. Theorem 1 of §2). This is a generalization of the characterization theorem of the conditional expectation of Moy (cf. Theorem 3 of [7], also cf. Nakamura-Turumaru [8]), and we shall call the Dixmier mapping to be the conditional expectation (cf. §2 below). In § 3, we shall give the definition of the M-net in a semi-finite or finite W^*-algebra A with respect to a given gage μ (cf. [10] or [2], and cf. § 2 below). If A is commutative and μ is a faithful normal trace, then any M-net is transformed to a martingale in the corresponding probability space $(\Omega, \mathbf{B}, \nu)$. In Theorem 2 and its Corollary, for a σ-finite finite W^*-algebra A with a faithful normal trace μ we shall prove that for an M-net to be simple (cf. Def. in § 3) and to converge in L^1-mean, are equivalent to the weak* conditional compactness of it, or to the uniform μ-integrabilities of the real and imaginary parts and L^1-uniform boundedness, and moreover that if an M-net is uniformly bounded then it is simple and converges strongly to a bounded operator in A. If the directed set D is decreasing (cf. § 3), then any M-net with finite integral in semi-finite A necessarily converges strongly, and if the M-net belongs to $L^2(A)$ then it converges in the L^2-mean. These facts can be applied to a convergence of a sequence of bounded operators (cf. In I_∞ or II_∞-factor, Theorem 6 of [9]), which was introduced by von Neumann and we can show that it is a simple M-net. I want to thank Mr. Sakai for his valuable remarks.

2. Let A be a semi-finite W^*-algebra on a Hilbert space H with a regular gage μ in the sense of Segal (cf. [10]) which is considered as the "normale, fidèle, éssentielle et maximale" trace in the sense of Dixmier (cf. [2]). Let $L^1(A)$ and $L^2(A)$ be the space of all integrable and square integrable operators with respect to μ in the sense of Segal respectively (cf. [10] and [2]). Denote the set of all μ-integrable operators belonging to A by J which is a two-sided ideal of A and is dense in $L^1(A)$ and $L^2(A)$ relative to the respective norm $\| \ \|_1$ (= L^1-norm) and $\| \ \|_2$ (= L^2-norm). Dixmier has proved the following theorem (cf. Théorème 8 of [2]).

THEOREM D. [1] *Let A_1 be a W^*-subalgebra of A. Then there exist a maximal central projection p_μ in A_1 and a linear mapping $x \to x^e$ from A into itself such that the range $A^e = p_\mu A_1$ and for all $x \in A$*

(D. 1) $\| x^e \|_\infty \leqq \| x \|_\infty$, $\| x \|_\infty$ being operator bound.

(D. 2) $x^{ee} = x^e$ and $x^{*e} x^e \leqq (x^* x)^e$.

1) Dixmier has proved more strict conditions i. e. (D. 8)$'$: $\| x^e \|_r \leqq \| x \|_r$ for $x \in J_1{}^r$ $(r \geqq 1)$ and (D. 9)$'$: (D. 9) holds for $x \in J^{1/r_1}$ and $y \in (J \cap A_1)^{1/r_2}$ $(1/r_1 + 1/r_2 = 1)$, where the power $1/r$ and the norm $\| \cdot \|_r$ are notations of him (cf. [2]). But we can see their equivalences such as (D. 9) implies (D. 8)$'$, (D. 8)$'$ implies (D. 9)$'$ (by (D. 5) and by Hölder's inequality of Dixmier, (cf. [2] and Proposition 5 of [2]), and (D. 9)$'$ implies (D. 9) by (D. 7).

$(D.3)$ *$x \geqq 0$ implies $x^e \geqq 0$.* $(D.3')$ $x^{*e} = x^{e*}$.

$(D.4)$ *$x \geqq 0$ and $x^e = 0$ imply $p_\mu x p_\mu = 0$.*

$(D.5)$ *$(yx^e y') = yx^e y'$ and $(p_\mu x p_\mu)^e = x^e$ for $y, y' \in A_1$.*

$(D.6)$ *$(xy)^e = (yx)^e$ for every $y \in A \cap A_1'$.*

$(D.7)$ *The mapping $x \to x^e$ is strongest and weakest continuous.*

$(D.8)$ *$\|x^e\|_1 \leqq \|x\|_1$ and $\|x^e\|_2 \leqq \|x\|_2$ for every $x \in J$.*

$(D.9)$ *$\mu(xy^e) = \mu(x^e y)$ for every $x \in A$ and $y \in J$.*

It is clear that $I^e = p_\mu$, and the mapping $x \to x^e$ is extensible uniquely onto $L^1(A)$ and $L^2(A)$ by (D. 8). In the case that the gage μ is finite, regular and normal i. e. $\mu(I) = 1$ (we shall call such a μ to be a faithful normal trace), the Dixmier's mapping $x \to x^e$ satisfies $I^e = I$. In the previous paper [11], we called such a mapping $x \to x^e$ from $L^1(A)$ into $L^1(A_1)$ with respect to the faithful normal trace μ to be a conditional expectation relative to A_1. If A is commutative, it coincides with the conditional expectation in the usual probability sense on the corresponding probability space $(\Omega, \mathbf{B}, \nu)$. In the present paper, we shall also call *the Dixmier's mapping $x \to x^e$* from $L^1(A)$ (or $L^2(A)$) into itself to be *a conditional expectation relative to A_1*, and x^e denotes it.

Firstly we shall prove a characterization theorem of the conditional expectation in the semi-finite case. Let A be a semi-finite W^*-algebra and let μ be a regular gage on A. Then

THEOREM 1. *Let $x \to x^\varepsilon$ be a linear mapping from A into itself satisfying the conditions $(D.2)$, $(D.3)$, $(D.9)$ and $I^\varepsilon \leqq I$. Then for any $x \in A$, x^ε coincides with the conditional expectation x^e relative to the W^*-subalgebra A_1 which is the direct sum of $A^\varepsilon = \{x^\varepsilon;\ x \in A\}$ and $\{\lambda(I - I^\varepsilon) : \lambda$ complex numbers$\}$.*

PROOF. The linearity of x^ε and (D. 3) imply obviously (D. 3'). Since for any $x \in A$, $0 \leqq x^* x \leqq \|x^* x\|_\infty I$, by (D. 2), (D. 3) we have

$$(1) \qquad 0 \leqq x^{*\varepsilon} x^\varepsilon \leqq (x^* x)^\varepsilon \leqq \|x^* x\|_\infty I^\varepsilon \leqq \|x\|_\infty^2 I$$

and $\|x^\varepsilon\|_\infty \leqq \|I^\varepsilon\|_\infty^{\frac{1}{2}} \|x\|_\infty \leqq \|x\|_\infty$, so we have (D. 1). Let $\{x_\gamma\}_D \subset A$ be a uniformly $\|\ \|_\infty$-bounded directed set converging weakly to $x \in A$, D being a directed set, then $\mu(x_\gamma y) \underset{\gamma}{\to} \mu(xy)$ for any $y \in J$. By (D. 9), $J^\varepsilon \subset J$ and

$$(2) \qquad \mu(x_\gamma^\varepsilon y) = \mu(x_\gamma y^\varepsilon) \to \mu(x y^\varepsilon) = \mu(x^\varepsilon y)$$

for every $y \in J$. Since $\{x_\gamma^\varepsilon\}$ is uniformly $\|\ \|_\infty$-bounded, (2) implies the weak convergence of x_γ^ε to x^ε on H by Dixmier's Theorem (cf. Corollary 2 of [2]).

We shall now prove that A^ε is a weakly closed self-adjoit subalgebra*) of A. If $x \in A$ then $x^{\varepsilon*} = x^{*\varepsilon} \in A^\varepsilon$, i. e. A^ε and similarly J^ε are self-adjoint. While, by (D. 2), for any $x \in J^\varepsilon$

$$(3) \qquad x^* x = x^{*\varepsilon} x^\varepsilon \leqq (x^* x)^\varepsilon.$$

As $I^\varepsilon \leqq I$, for any $x \in J^{+}$[2)]

*) In this paper, by a weakly closed self-adjoint algebra we mean a *-algebra which is closed in the weak operator topology, not necessarily having the identity operator; and by a W^*-algebra we mean a weakly closed *-algebra which has the identity operator.

(4) $$\mu(x^{\epsilon}) = \mu(xI^{\epsilon}) \leqq \mu(x).$$

Therefore, putting $y = (x^*x)^{\epsilon} - x^*x$, since $y \in J^+$ and by (3), (4)

$$0 \leqq \mu(y) = \mu((x^*x)^{\epsilon}) - \mu(x^*x) \leqq 0.$$

This implies $y = 0$ and $x^*x = (x^*x)^{\epsilon}$ which belongs to J^{ϵ}. For any $x, y \in J^{\epsilon}$, xy can be expressed by $\sum_{j=1}^{4} \lambda_j z_j^* z_j$ for some $z_j \in J^{\epsilon}$ and complex numbers λ_j ($j = 1, 2, 3, 4$). Consequently,

$$(xy)^{\epsilon} = \sum_{j=1}^{4} \lambda_j (z_j^* z_j)^{\epsilon} = \sum_{j=1}^{4} \lambda_j z_j^* z_j = xy$$

and $xy \in J^{\epsilon}$. Therefore J^{ϵ} is a self-adjoint subalgebra of A. Next we shall prove $A^{\epsilon} = \bar{J}^{\epsilon}$ [2]. For $x \in \bar{J}^{\epsilon}$, there is $\{x_\gamma\}_D \subset J^{\epsilon}$ such that $\|x_\gamma\|_\infty \leqq \|x\|_\infty$ and x_γ converges weakly to x by a Kaplansky's Theorem (cf. [6]). Hence $x_\gamma = x_\gamma^{\epsilon}$ converges weakly to $x = x^{\epsilon}$ and $x \in A^{\epsilon}$, i. e. $\bar{J}^{\epsilon} \subset A^{\epsilon}$. Conversely, since $\bar{J} = A$, for $x \in A^{\epsilon}$ we can take $\{x_\gamma\}_D \subset J$ converging weakly to x and $\|x_\gamma\|_\infty \leqq \|x\|_\infty$, and obtain that x_γ^{ϵ} converges weakly to $x^{\epsilon} = x$, i. e. $x \in \bar{J}^{\epsilon}$. Therefore $A^{\epsilon} = \bar{J}^{\epsilon}$, and A^{ϵ} is a self-adjoint weakly closed subalgebra of A.

Further, we prove that I^{ϵ} is a self-adjoint unit element in the algebra A^{ϵ}. For any $x \in J^{\epsilon}$ and $y \in A^{\epsilon}$, since $xy \in J$ by the above fact,

(5) $$\mu(yxI^{\epsilon}) = \mu((yx)^{\epsilon} I) = \mu(yx).$$

Hence, for any complex number λ and for any $y \in A^{\epsilon}$,

$$\mu((y + \lambda I)xI^{\epsilon}) = \mu((y + \lambda I)x).$$

This implies $\mu(zxI^{\epsilon}) = \mu(zx)$ for every $x \in J^{\epsilon}$ and $z \in A_1$. Therefore we have $xI^{\epsilon} = x$ and similarly $= I^{\epsilon}x$ for every $x \in J^{\epsilon}$. Since $I^{\epsilon *} = I^{\epsilon}$ is clear and since $A^{\epsilon} = \bar{J}^{\epsilon}$, I^{ϵ} is a self-adjoint unit element in A^{ϵ}. Consequently A_1 is the direct sum of A^{ϵ} and $\{\lambda(I - I^{\epsilon})$; λ complex numbers$\}$, and I^{ϵ} is a maximal central projection in A_1.

Finally, in order to prove $x^{\epsilon} = x^{e}$ for all $x \in A$, x^{e} being the conditional expectation relative to A_1, we show $J^{\epsilon} = J \cap A_1$. Each $x \in J \cap A_1$ is expressed by $x' + \lambda(I - I^{\epsilon})$ for some $x' \in J^{\epsilon}$ and λ, and hence for every $y \in J^{\epsilon}$

$$\mu(xy) = \mu((x' + \lambda(I - I^{\epsilon}))y) = \mu(x'y) + \lambda\mu((I - I^{\epsilon})y) = \mu(x'y).$$

Since $x \in J$ and $x' \in J$, $\mu(x(y + \lambda' I)) = \mu(x'(y + \lambda' I))$ for every $y \in J$ and complex numbers λ'. This implies easily $\mu(xz) = \mu(x'z)$ for all $z \in A_1$ and $x = x'$ which belongs to J^{ϵ}. Since $J^{\epsilon} \subset J \cap A_1$ is clear, we obtain $J^{\epsilon} = J \cap A_1$. Therefore, for every $x \in A$ and $y \in J \cap A_1$ $(= J^{\epsilon})$,

$$\mu(x^{\epsilon}y) = \mu(xy^{\epsilon}) = \mu(xy) = \mu(x^{e}y),$$

i. e. $x^{\epsilon} = x^{e}$ for all $x \in A$.

Using a method of Nakamura and Turumaru (cf. Cor. of [8]), the

2) For any subset S in A, $\bar{S}$ denote the weak closure (as operator on H) of S which coincides with the strong closure when S is convex, S^+ denotes the set of all non-negative operators in S.

conditional expectation satisfying the condition $I^e = I$ can be characterized as the following:

COROLLARY 1.1. *Let $x \to x^e$ be a linear mapping from A into itself satisfying* (*D.* 2), (*D.* 3), $I^e = I$ *and*

(6) $$\mu(x^e y) \leqq \mu(x y^e) < +\infty \text{ for every } x \in A^+ \text{and } y \in J^+.$$

Then the range A^e is a W^*-subalgebra and x^e coincides with the conditional expectation relative to A^e.

PROOF. Taking $y \in J^+$ such that $\mu(y^e) = 1$ and putting $\sigma_y(x) = \mu(xy^e)$ for $x \in A$, then by (6) and (D. 2) $\sigma_y(x^e) \leqq \sigma_y(x)$ for every $x \in A^+$. Since $I^e = I$ and $\sigma_y(I) = \mu(y^e) = 1$, by the proof of the Nakamura-Turumaru's Theorem we have that $\sigma_y(x^e) = \sigma_y(x)$ for every $x \in A$. This implies that $\mu(x^e y) = \mu(xy^e)$ for every $y \in J^+$ and hence for every $y \in J$. Further, since the strong continuity of x^e (on bounded part) is followed from (b) (cf. Remark 1.2 below), and since (D. 9) holds for $x, y \in J$ (by(6)), we obtain (4) and complete the proof.

REMARK 1.1. We shall give in the last section (cf. § 4) an example of the conditonal expectation satisfying $I^e = I$ in a semi-finite W^*-algebra. When μ is a faithful normal trace, i. e. A is a σ-finite, finite W^*-algebra, Theorem 1 holds and Corollary 1.1 also characterizes the conditional expectation. These characterizations are analytical and somewhat simple when compared with the Theorem 2 in the preceding paper (cf. [11])[3].

REMARK 1.2. *In Corollary* 1.1, *if $I^e \leqq I$ then $x \to x^e$ is strongly continuous on the unit sphere of A and A^e is a self-adjoint weakly closed subalgebra of A.* The first part will follow from the fact that $J^e \subset J$ (by (6)) and for every $x \in A$ and $y \in J$

(7) $$\| x^e y \|_2^2 = \mu(y^* x^{*e} x^e y) \leqq \mu(y^*(x^*x)^e y) = \mu((x^*x)^e yy^*) \leqq \mu((x^*x)(yy^*)^e).$$

The second part follows by the similar way of the proof of Theorem 1. Further we remark that *if μ is a faithful normal trace and the mapping $x \to x^e$ satisfies $I^e = I$ and a weaker condition than* (6):

(8) $$\mu(x^e) \leqq \mu(x) \qquad \text{for every } x \in A^+,$$

then A^e is a W^*-subalgebra of A. Indeed, for the present A and μ, $J = A$ and

$$\| x^e \|_2 = \mu(x^{*e} x^e) \leqq \mu((x^*x)^e) \leqq \mu(x^*x) = \| x \|_2^2 \qquad \text{for every } x \in A.$$

This implies the strong continuity of $x \to x^e$ on the unit sphere, and hence by the similar way of the first part in this Remark, we obtain the required ones.

Let A be again a semi-finite W^*-algebra and let μ be a regular gage on A. For any W^*-subalgebra A_1 of A, we shall also denote the contracted gage on A_1 by μ. Then the space $L^r(A_1)$ is considered as a closed subspace of $L^r(A)$, $r = 1, 2$. For any self-adjoint operator x in $L^r(A)$ ($r = 1$,

3) In this case we have assumed (D. 3)′ and (D. 5) but not (D. 3).

2), let $x = \int \lambda\, dE_\lambda(x)$ be the spectral resolution of x. Then each $E_\lambda(x)$ belongs to A. Denote by $W(x)$ the W^*-subalgebra of A generated by $\{E_\lambda(x);\ \lambda\}$. If x is not self-adjoint, then it can be uniquely expressed by $x = x^{(1)} + ix^{(2)}$ where $x^{(1)}$ and $x^{(2)}$ are the real and imaginary parts respectively. Then there correspond the W^*-subalgebras $W(x^{(1)})$ and $W(x^{(2)})$ to $x^{(1)}$ and $x^{(2)}$ respectively. Let $W(x)$ be the W^*-subalgebra generated by $W(x^{(1)})$ and $W(x^{(2)})$. Then $W(x)$ is a minimal W^*-subalgebra of A containing the resolutions of identities $E_\lambda(x^{(1)})$ and $E_\lambda(x^{(2)})$ of $x^{(1)}$ and $x^{(2)}$ respectively. Under these notations we have

PROPOSITION. *For any subset S of $L^r(A)$ ($r = 1$ or 2 resp.) there corresponds uniquely a minimal W^*-subalgebra $W(S)$ of A such that $S \subset L^r(W(S))$. The operation $S \to W(S)$ has the properties that, $W(L^r(W(S))) = W(S)$; $S \subset A$ implies $W(S) = S''$*[4]*; $S_1 \subset S_2$ implies $W(S_1) \subset W(S_2)$; and further for S_1 and S_2 having the same closed linear hull in $L^r(A)$ ($r = 1$, or 2 resp.), $W(S_1) = W(S_2)$.*

PROOF. Let $S \subset L^r(A)$ ($r = 1$ or 2 resp.) and let $W(S)$ be a W^*-subalgebra of A generated by $\{W(x);\ x \in S\}$. Since for any $x \in S$ $x^{(1)}, x^{(2)} \in L^r(A)$ and the projections $E_\lambda(x^{(1)})$, $E_\lambda(x^{(2)})$ belongs to $W(S)$, $x^{(1)}$ and $x^{(2)}$ are measurable with respect to $W(S)$ in the sense of Segal (cf. [10]) and hence they belong to $L^r(W(S))$. For a W^*-subalgebra W of A such that $S \subset L^r(W)$, $W(S) \subset W$ follows from [10]. Hence $W(S)$ is minimal and uniquely determined by S. Now we prove $W(L^r(W(S))) = W(S)$. Since $S \subset L^r(W(S))$, $W(S) \subset W(L^r(W(S)))$ (because $S_1 \subset S_2$ implies clearly $W(S_1) \subset W(S_2)$). Conversely, for $x = x^*$ in $L^r(W(S))$, $E_\lambda(x) \in W(S)$ and hence $W(L^r(W(S))) \subset W(S)$. The other parts in this proposition will easily follow from these facts.

The following corollary contains a generalization of a half part of a theorem of Bahadur (cf. [1]).

COROLLARY 1.2. (1°) *Let A be a semi-finite W^*-algebra with a regular gage μ, and put $L = L^2(A)$. Let $x \to x^\varepsilon$ be a projection in L such that $x^{*\varepsilon} = x^{\varepsilon *}$. Then the following conditions are equivalent:*

(1′) *$x \to x^\varepsilon$ coincides with a conditional expectation x^e (on L) relative to a certain W^*-subalgebra A_1 of A.*

(2′) *$L^\varepsilon = L^2(W(L^\varepsilon))$.*

(2°) *If μ is a faithful normal trace, then* (1′) *and* (2′) *are equivalent to the following each condition:*

(3′) *L^ε contains a self-adjoint subalgebra B of A such that $I \in B$ and B is L^2-dense in L^ε.*

(4′) *$A^\varepsilon = W(L^\varepsilon)$.*

PROOF. (1°). (1′) $\to$ (2′): Since $L^e = L^2(A_1)$ and J^e is dense in L^e, by Theorem 1, $J^e = J \cap A_1 \supset J \cap W(J^e) \supset J^e$ and $J^e = J \cap W(J^e)$. Further, by the preceding Proposition, $W(J^e) = W(L^e)$. Hence we obtain $L^e = L^2(W(L^e))$ and (2′) holds.

4) For any subset S of bounded operators on H, S' denotes the set of all bounded operators on H which with all operators in e S. S'' denotes $(S')'$. S'' is a W^*-algebra generated by S.

$(2')\to(1')$: $<,>$ denotes the inner product in L. Since $x\to x^{\epsilon}$ is a projection in L, for $x, y\in L$

$$\mu(x^{\epsilon}y) = <x^{\epsilon}, y^{*}> = <x, y^{*\epsilon}> = <x, y^{\epsilon *}> = \mu(xy^{\epsilon}).$$

Let x^{e} be the conditional expectation relative to $W(L^{\epsilon})$. Then, for every $x\in L$, $x^{\epsilon e} = x^{\epsilon}$ and $x^{e\epsilon} = x^{e}$, and for every $y\in J$

$$\mu(x^{\epsilon}y) = \mu(x^{\epsilon e}y) = \mu(x^{\epsilon}y^{e}) = \mu(xy^{e\epsilon}) = \mu(xy^{e}) = \mu(x^{e}y). \tag{9}$$

This implies that $x^{\epsilon} = x^{e}$ for every $x\in L$. (2°) will be followed immediately from (1°) as its corollary.

By a slight modification of the proof of Cor. 1. 2 it also holds that: In Cor. 1. 2, (1°), put $L = L^{1}(A)$, and let $x\to x^{\epsilon}$ be a bounded linear mapping from L into itself satisfying $x^{\epsilon}{}^{\epsilon} = x^{\epsilon}$ and (D. 9) for every $x, y\in J$. Then the condition $(1')$ is equivalent to

$(2'')$ $L^{\epsilon} = L^{1}(W(L^{\epsilon}))$ and $J^{\epsilon}\subset A$.

3. As the preceding section we shall consider a semi-finite W^{*}-algebra A on a Hilbert space H, with a regular gage μ. Let $\{x_{\alpha}, \alpha\in D\}$ be a family of operators in A (or $L^{1}(A)$ or $L^{2}(A)$ resp.), D being directed set. Let A_{α} be W^{*}-subalgebra of A generated by $\{W(x_{\gamma}); \gamma\leqq\alpha\}$, $W(x_{\gamma})$ being the W^{*}-subalgebra given in the Proposition in § 2. Then $A_{\alpha}\subseteqq A_{\beta}$ if and only if $\alpha\leqq\beta$. If $\{x_{\alpha}, \alpha\in D\}$ satisfies the conditions $x_{\alpha} = x_{\beta}^{e_{\alpha}}$ for every $\alpha, \beta\in D$ $(\alpha\leqq\beta)$, where $x^{e_{\alpha}}$ denotes the conditional expectation relative to A_{α}, then we shall call the family of operators $\{x_{\alpha}, \alpha\in D\}$ to be an *M-net (with respect to the gage μ), and $\{A_{\alpha}, \alpha\in D\}$ the family of W^{*}-subalgebras associated to the M-net.* We shall call an M-net to be *increasing* or *decreasing*, whenever: for any $\alpha, \beta\in D$ there exists $\gamma\in D$ such that $\alpha, \beta\leqq\gamma$ or $\gamma\leqq\alpha, \beta$ respectively.

An example of M-net is given such as: Let $\{B_{\alpha}, \alpha\in D\}$ be a family of W^{*}-subalgebras of A and suppose that $B_{\alpha}\subset B_{\beta}$ if and only if $\alpha\leqq\beta$. Let $\{x_{\alpha}, \alpha\in D\}$ be a family of operators in $L^{1}(A)$ or $L^{2}(A)$ such that

$$x_{\alpha} = x_{\beta}^{\epsilon_{\alpha}} \quad \text{for every } \alpha, \beta\in D \ (\alpha\leqq\beta) \tag{10}$$

where $x^{\epsilon_{\alpha}}$ denotes the conditional expectation relative to B_{α}. Then $\{x_{\alpha}, \alpha\in D\}$ is an M-net[5]. We denote such an M-net by $\{x_{\alpha}, B_{\alpha}, \alpha\in D\}$. Further, for any $x\in L^{1}(A)$ or $L^{2}(A)$ putting $x_{\alpha} = x^{\epsilon_{\alpha}}$ $(\alpha\in D)$, $\{x_{\alpha}, B_{\alpha}, \alpha\in D\}$ is also an M-net. Such an M-net $\{x_{\alpha}, B_{\alpha}, \alpha\in D\}$ is called to be *simple.* Any finite M-net is clearly simple. The sequence of bounded operators in I_{∞} or II_{∞}-factor given by von Neumann (cf. p. 118 of [9] and cf. § 4 in this paper) is an example of simple M-net.

If A is a σ-finite commutative W^{*}-algebra with faithful normal trace μ, then any M-net $\{x_{\alpha}, \alpha\in D\}$ in $L^{1}(A)$ is transformed to a martingale on the corresponding probability space by the canonical mapping.

By the definition of M-net and the properties of the conditional expectations the following conditions are equivalent for a given family of operators

5) That is, taking the corresponding family of W^{*}-subalgebras $\{A_{\alpha}, \alpha\in D\}$, it satisfies that $x_{\alpha} = x_{\beta}^{e_{\alpha}}$ for every $\alpha, \beta\in D$ $(\alpha\leqq\beta)$.

$\{x_\alpha, \alpha \in D\}$ in $L^1(A)$ or $L^2(A)$:

(i) $\{x_\alpha, \alpha \in D\}$ is an M-net.

(ii) $\mu(yx_\alpha) = \mu(yx_\beta)$ for every $\alpha, \beta \in D$ $(\alpha \leqq \beta)$ and $y \in J \cap A_\alpha$, A_α being the W^*-subalgebra given at the first paragraph in this section.

(iii) $x_\alpha = x_\beta^{e(\gamma,\alpha)}$ for every $\alpha, \beta, \gamma \in D$ such that $\gamma \leqq \alpha \leqq \beta$, where $e(\gamma, \alpha)$ denotes the conditional expectation relative to the W^*-subalgebra $W(x_\gamma, x_\alpha)$.

If $\{x_\alpha, A_\alpha, \alpha \in D\}$ and $\{y_\alpha, A_\alpha, \alpha \in D\}$ are M-nets, then $\{x_\alpha^*, A_\alpha, \alpha \in D\}$, $\{\lambda x_\alpha, A_\alpha, \alpha \in D\}$ and $\{x_\alpha + y_\alpha, A_\alpha, \alpha \in D\}$ are also M-nets, λ being any complex number. We shall say an M-net $\{x_\alpha, \alpha \in D\}$ to be *real* or *positive* if $x_\alpha^* = x_\alpha$ for every $\alpha \in D$ or $x_\alpha \geqq 0$ for every $\alpha \in D$. Any M-net $\{x_\alpha, \alpha \in D\}$ can be decomposed into two real M-nets in an obvious way, that is, $\{x_\alpha^{(1)}, \alpha \in D\}$ and $\{x_\alpha^{(2)}, \alpha \in D\}$ where $x_\alpha^{(1)} = \frac{1}{2}(x_\alpha + x_\alpha^*)$ and $x_\alpha^{(2)} = \frac{1}{2i}(x_\alpha - x_\alpha^*)$. In an M-net $\{x_\alpha, \alpha \in D\}$, for any directed subset D' of D, $\{x_\alpha, \alpha \in D'\}$ is also an M-net.

Besides, we shall define a subset $S \subset L^1(A)$ to be *uniformly μ-integrable* if for any $\varepsilon > 0$ there is a positive number $\delta > 0$ such that $\mu(p) < \delta$ (p being projection in $L^1(A)$) implies $\mu(p|x|) < \varepsilon$ for all $x \in S$.

With these terminologies, we shall prove

THEOREM 2.[6] *Let A be a σ-finite, finite W^*-algebra on a Hilbert space H with a faithful normal trace μ, and let D be an increasing directed set. Then, for a given M-net $\{x_\alpha, \alpha \in D\}$, the following conditions are equivalent*:

(2.1) *Both $\{x_\alpha^{(1)}, \alpha \in D\}$, and $\{x_\alpha^{(2)}, \alpha \in D\}$ are uniformly μ-integrable and uniformly bounded in L^1-norm.*

(2.2) *$\{x_\alpha, \alpha \in D\}$ is weakly* conditional compact in $L^1(A)$.*

(2.3) *$\{x_\alpha, \alpha \in D\}$ is simple.*

(2.4) *There exists $x \in L^1(A)$ such that $\|x_\alpha - x\|_1 \to 0$.*

PROOF. If the M-net $\{x_\alpha, \alpha \in D\}$ is finite, the proof is trivial. Hence we consider the case that it is infinite. Let $\{A_\alpha, \alpha \in D\}$ be the family of the W^*-subalgebras of A associated to $\{x_\alpha, \alpha \in D\}$. Let $A_0 = \bigcup_{\alpha \in D} A_\alpha$ and let A_1 be the weak closure of A_0. Let $z^{e\alpha}$ be the conditional expectation of $z \in L^1(A)$ relative to A^α.

Firstly we prove that (2.1)$\to$(2.2): Each $x_\alpha^{(1)}$ is uniquely expressed by $x'_\alpha - x''_\alpha$ such that $x'_\alpha, x''_\alpha \in L^1(A)$, $x'_\alpha, x''_\alpha \geqq 0$ and $x'_\alpha x''_\alpha = 0$ for every $\alpha \in D$. Since $\{x_\alpha^{(1)}, \alpha \in D\}$ is uniformly μ-integrable and $|x_\alpha^{(1)}| = x'_\alpha + x''_\alpha$, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mu(p) < \delta$ implies for all $\alpha \in D$

$$\mu(px'_\alpha) + \mu(px''_\alpha) = \mu(p(x'_\alpha + x''_\alpha)) = \mu(p|x_\alpha|) < \varepsilon/2. \tag{11}$$

Since $\mu(px'_\alpha), \mu(px''_\alpha) \geqq 0$, both $< \varepsilon/2$ for all $\alpha \in D$, and hence $\{x'_\alpha, \alpha \in D\}$ and $\{x''_\alpha, \alpha \in D\}$ are uniformly μ-integrable. Putting $\sigma'_\alpha(y) = \mu(yx'_\alpha)$ and $\sigma''_\alpha(y) = \mu(yx''_\alpha)$ for all $y \in A$ and $\alpha \in D$, σ'_α and σ''_α belong to the conjugate

6) This theorem contains a generalization of L^1-mean convergence of a martingale in a probability space (cf. Theorem 1.4 of [3]).

Banach space $A^\wedge$ of A. Let T' and T'' be the weak closures of $\{\sigma'_\alpha, \alpha \in D\}$ and $\{\sigma''_\alpha, \alpha \in D\}$ as functional on A respectively. Let $\sigma \in T'$ be a limiting point of $\{\sigma'_\alpha, \alpha \in D\}$ which is a positive linear functional on A. We take a sequence of projections $\{p_j, j = 1, 2, \dots\}$ in A such that $p_j \perp p_k$ $(j \neq k)$. For $\varepsilon > 0$, taking $\delta > 0$ as (11), there exists an integer $k_0 > 0$ such that $\mu\left(\sum_{j=k_0}^{\infty} p_j\right) < \delta$, and since $\sum_{j=k_0}^{\infty} p_j$ is a projection in A, by (11)

$$\sigma'_\alpha\left(\sum_{j=k_0}^{\infty} p_j\right) = \mu\left(\left(\sum_{j=k_0}^{\infty} p_j\right) x'_\alpha\right) < \varepsilon/2.$$

Choosing σ'_α such that

$$\left|\sigma'\left(\sum_{j=k_0}^{\infty} p_j\right) - \sigma'_\alpha\left(\sum_{j=k_0}^{\infty} p_j\right)\right| < \varepsilon/2,$$

then for any integer $k \geqq k_0$

$$\sigma'\left(\sum_{j=k}^{\infty} p_j\right) \leqq \sigma'\left(\sum_{j=k_0}^{\infty} p_j\right) \leqq \left|\sigma'\left(\sum_{j=k_0}^{\infty} p_j\right) - \sigma'_\alpha\left(\sum_{j=k_0}^{\infty} p_j\right)\right| + \sigma'_\alpha\left(\sum_{j=k_0}^{\infty} p_j\right)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore σ is countably additive and by Dye-Radon-Nikodym's Theorem (cf. [4]), there exists $x' \in L^1(A)$ such that $x' \geqq 0$ and

$$\sigma'(y) = \mu(yx') \qquad \text{for all } y \in A. \tag{12}$$

Hence every σ' in T' (and similarily every σ'' in T'') are represented as (12) (and $\sigma''(y) = \mu(yx'')$ for some $x'' \in L^1(A)$). x' and x'' are uniquely determined by σ' and σ'' in $L^1(A)$ respectively. Therefore the weak* closures of $\{x'_\alpha, \alpha \in D\}$ and $\{x''_\alpha, \alpha \in D\}$ in $L^1(A)$ are weak* compact in $L^1(A)$ and so is the weak* closure of $\{x'_\alpha - x''_\alpha, \alpha \in D\}$, i. e. $\{x^{(1)}_\alpha, \alpha \in D\}$ is weakly* conditionally compact. Since for $\{x^{(2)}_\alpha, \alpha \in D\}$ the same fact can be proved by the same way and $x_\alpha = x^{(1)}_\alpha + ix^{(2)}_\alpha$, $\{x_\alpha, \alpha \in D\}$ is weakly* conditionally compact.

Secondly we prove that (2. 2) $\rightarrow$ (2. 3): Put $S = \{\sigma_\alpha, \alpha \in D\}$ $\sigma_\alpha(y) = \mu(yx_\alpha)$, and S^{γ_1} and S^{γ_0} the weak* closures in $A_1^\wedge$ of S with respect to A_1 and A_0 respectively, that is, the closures with respect to the weak* topologies on $L^1(A_1)$ defined by the neighborhoods:

$$U(x_0\,; z_1, \dots, z_n, \varepsilon > 0) = \{x \in L^1(A_1)\,;\ |\mu((x_0 - x)z_j)| < \varepsilon,\ j = 1, 2, \dots, n\},$$

z_j belonging to A_1 or A_0 respectively. Then by (2.2) S^{γ_1} is weakly* compact. Since the weak* topology on $L^1(A_1)$ with respect to A_1 is stronger than the one with respect to A_0, the canonical mapping from S^{γ_1} to S^{γ_0} is continuous, and it is also one-to-one. For, since A_0 is strongly dense in A_1 (as operator on H), $\mu(x_1 z) = \mu(x_2 z)$ for x_1, $x_2 \in L^1(A_1)$ and for all $z \in A_0$ thus *a fortiori*, for all $z \in A_1$. Therefore S^{γ_1} is compact (and hence closed) in S^{γ_0}, and $S \subset S^{\gamma_1}$

mplies $S^{b_0} \subset S^{b_1 b_0} = S^{b_1}$. Further, by the definition of the M-net, $\lim_\alpha \mu(yx_\alpha)$ $(= \sigma(y)$ say) always exists for every $y \in A_0$, which belongs to S^{b_0} and hence S^{b_1}. Consequently, there is an $x \in L^1(A_1)$ such that $\sigma(y) = \mu(yx)$ for every $y \in A_0$. Since $\mu(yx) = \lim \mu(yx_\alpha)$ for every $y \in A_0$, for any fixed $\alpha \in D$ and for any $y \in A_\alpha$,

$$\mu(yx) = \lim_\beta \mu(yx_\beta) = \lim_\beta \mu(yx_\beta^{e_\alpha}) \ (\alpha \leqq \beta) = \mu(yx_\alpha).$$

Hence we obtain $x_\alpha = x^{e_\alpha}$ for every $\alpha \in D$.

Next we shall show the equivalence of (2.3) and (2.4). Assume (2.3). For any $z \in A$, putting $z_\alpha = z^{e_\alpha}$ for all $\alpha \in D$ and $z_1 = z^{e_1}$, $\{z_\alpha, \alpha \in D\}$ is a simple M-net satisfying $\|z^{e_\alpha}\|_2 \leqq \|z^{e_1}\|_2$ and

$$(14) \qquad \|z_\alpha - z_1\|_2^2 = \mu(z_1^* z_1) - \mu(z_1^* z_\alpha) \underset{\alpha}{\rightarrow} 0$$

and hence $\|z_\alpha - z_1\|_1 \leqq \|z_\alpha - z_1\|_2 \rightarrow 0$. Let $x \in L^1(A)$ be $x_\alpha = x^{e_\alpha}$ $(\alpha \in D)$ and take $\{z_n\} \subset A_1$ such that $\|x^{e_1} - z_n\|_1 < 1/3n$ $(n = 1, 2, \ldots.)$. Moreover for each n taking $\alpha_n \in D$ such that $\|z_n - z_n^{e_\alpha}\|_1 < 1/3n$ for all $\alpha \in D$, $(\alpha_n \leqq \alpha)$,

$$\|x^{e_1} - x_\alpha\|_1 \leqq \|x^{e_1} - z_n\|_1 + \|z_n - z_n^{e_\alpha}\|_1 + \|(z_n - x)^{e_\alpha}\|_1 < 1/n$$

and $\|x^{e_1} - x_\alpha\|_1 \rightarrow 0$. This implies (2.4). Conversely, assuming (2.4), $\lim_\alpha \mu(yx_\alpha)$ $= \mu(yx)$ for all $y \in A$, and if $y \in A_{\alpha_0}$ then $\mu(yx_{\alpha_0}) = \mu(yx_\alpha)$ for all $\alpha \in D\,(\alpha_0 \leqq \alpha)$. These facts imply that $\mu(yx_{\alpha_0}) = \mu(yx)$ for all $y \in A_{\alpha_0}$.

Finally we shall prove that (2.3) and (2.4) imply (2.1). Let $x \in L^1(A)$ be $x_\alpha = x^{e_\alpha}$ and $\|x_\alpha - x_1\| \underset{\alpha}{\rightarrow} 0$. The expressions $x = x^{(1)} + ix^{(2)}$ and $x^{(1)} = x' - x''$ (x', $x'' \geqq 0$ and $x'x'' = 0$) are unique. Putting $x'_\alpha = x'^{e_\alpha}$ and $x''_\alpha = x''^{e_\alpha}$ $(\alpha \in D)$, $x_\alpha^{(1)} = x'_\alpha - x''_\alpha$ and $\{x'_\alpha, \alpha \in D\}$ and $\{x''_\alpha, \alpha \in D\}$ are positive simple M-sets satisfying $\|x'_\alpha - x'\|_1$, $\|x''_\alpha - x''\|_1 \rightarrow 0$. If $\{x_\alpha^{(1)}, \alpha \in D\}$ is not uniformly μ-integrable, then so is at least one $\{x'_\alpha, \alpha \in D\}$ or $\{x''_\alpha, \alpha \in D\}$. Indeed, if both are uniformly μ-integrable, then let $|x_\alpha| = v_\alpha x$ being canonical decomposition, v_α being partially isometric operator,

$$\begin{aligned}\mu(p\,|x_\alpha^{(1)}|) &= \mu(pv_\alpha x_\alpha^{(1)}) = \mu(pv_\alpha x'_\alpha) - \mu(pv_\alpha x''_\alpha) \\ &\leqq \mu(px'_\alpha)^{1/2}\mu(v_\alpha^* v_\alpha x'_\alpha)^{1/2} + \mu(px''_\alpha)^{1/2}\mu(v_\alpha^* v_\alpha x''_\alpha)^{1/2} \\ &\leqq \|x'\|_1^{1/2}\,\mu(px'_\alpha) + \|x''\|_1^{1/2}\mu(px''_\alpha)^{1/2},\end{aligned}$$

because $x'_\alpha = x'^{e}_\alpha$ and $x''_\alpha = x''^{e_\alpha}$. This implies the uniformly μ-integrability of $\{x_\alpha, \alpha \in D\}$. Now if $\{x'_\alpha, \alpha \in D\}$ is not uniformly μ-integrable, then there exist an $\mathcal{E} > 0$, sequences of projections $\{p_n\} \subset A$ and indices $\{\alpha_n\} \subset D$ such that $\mu(p_n) < 1/n$, $\mu(p_n x')$ and $\alpha_n \leqq \alpha_{n+1}$ $(n = 1, 2, \ldots.)$. Let B be a W^*-subalgebra of A generated by $\{A_{\alpha_n}, n = 1, 2, \ldots\}$ and let x'^{ϵ} be the conditional expectation of x' relative to B. Then $\|x'_{\alpha_n} - x'^{\epsilon}\|_1 \rightarrow 0$ $(n \rightarrow \infty)$. Therefore

$$\mathcal{E} < \mu(p_n x'^n) \leqq |\mu(p_n(x'_{\alpha_n}{}^n - x'^{\epsilon}))| + \mu(p_n x'^{\epsilon})) \leqq \|x'_{\alpha_n} - x'^{\epsilon}\|_1 + \mu(p_n x'^{\epsilon}) \rightarrow 0 \, (n \rightarrow \infty).$$

This is a contradiction. The uniform μ-integrability of $\{x_\alpha^{(2)}, \alpha \in D\}$ also follows in the same way as for case of $\{x_\alpha^{(1)}, \alpha \in D\}$. Q. E. D.

For the M-nets in $L^2(A)$ and A we have the following:

COROLLARY 2.1. *Let A and μ have the same meanings as Theorem 2. Let*

$\{x_\alpha,\ \alpha \in D\}$ be an increasing M-net in $L^2(A)$ (in A, resp.). Then the following three conditions are equivalent:

(2.1)′ $\{x_\alpha,\ \alpha \in D\}$ is uniformly bounded in L^2-norm ($\|\cdot\|_\infty$-norm resp.).

(2.3)′ $\{x_\alpha,\ \alpha \in D\}$ is simple for $x \in L^2(A)$ (for $x \in A$ resp.).

(2.4)′ There exists an $x \in L^2(A)$ ($x \in A$ resp.) such that x_α converges to x in the L^2-mean (in the strong operator topology on H resp.).

In the proof of this corollary, we shall use the notations in the proof of Theorem 1.

PROOF FOR THE L^2(A)-CASE. (2.1)′ → (2.3)′: Since $\|x_\alpha\|_2^2 = \|x_\alpha^{(1)}\|_2^2 + \|x_\alpha^{(2)}\|_2^2$ $(\alpha \in D)$ and for any projection $p \in A$

$$\mu(p\,|x_\alpha^{(j)}|) \leqq \mu(p)^{1/2}\mu(x_\alpha^{(j)*}x_\alpha^{(j)})^{1/2} \leqq \mu(p)^{1/2}\|x_\alpha^{(j)}\|_2 \quad (j = 1, 2),$$

Theorem 1 (2.1) is satisfied. Hence $\{x_\alpha^{(j)},\ \alpha \in D\}$ are simple in $L^1(A)$, i.e. $x_\alpha^{(j)} = x^{(j)e_\alpha}$ and $\|x_\alpha^{(j)} - x^{(j)}\|_1 \to 0$ for some $x^{(j)} \in L^1(A)$, and

$$|\mu(yx_\alpha^{(j)})| \leqq \mu(y^*y)^{1/2}\mu(x_\alpha^{(j)})^{1/2*}x_\alpha^{(j)}) \leqq \|x_\alpha^{(j)}\|_2\|y\|_2$$

for every $y \in A$. Consequenly, $|\mu(yx^{(j)})| \leqq \sup_\alpha \|x^{(j)}\|_2\|y\|_2$ and $x^{(j)} \in L^2(A)$ $(j = 1, 2)$. (2.3)′ → (2.4)′: As in the proof of Theorem 2 (cf. the part (2.3) → (2.4)), taking $\{z_\alpha,\ \alpha \in D\} \subset A$ and $z_1 \in A$, (2.4)′ holds (see (14)). Since A is dense in $L^2(A)$, we can show $\|x_\alpha - x^{e_1}\|_2 \to 0$ by the same method of the proof of Theorem 1, if we take the L^2-norm instead of L^1-norm. (2.4)′ → (2.3)′ and (2.3)′ → (2.1)′ are obvious by Theorem 1 and by the fact that the x belonging to $L^2(A)$.

PROOF FOR THE A-CASE. (2.1)′ → (2.3)′: Since $\mu(x_\alpha^* x_\alpha) \leqq \|x_\alpha\|_\infty^2$, (2.1)′ holds for L^2-norm and (2.3)′ holds for $x \in L^2(A)$. Hence, by Cor. 2.1. (2.4)′, for every $y \in A$

$$\|xy\|_2^2 = \mu(x^*xyy^*) = \lim_\alpha \mu(x^*xyy^*) \leqq \sup_\alpha \|x_\alpha\|_\infty^2\|y\|_2^2$$

and $x \in A$. (2.3)′ → (2.4)′: We can assume that $x \in A$ satisfies $x_\alpha = x_\alpha^e$ and $\|x_\alpha - x_2\| \to 0$. Whence for any $y \in A$ $\|(x_\alpha - x)y\|_2 \to 0$. Since $\|x_\alpha\|_\infty \leqq \|x\|_\infty$ and A is dense in $L^2(A)$, $x_\alpha \to x$ strongly on the Hilbert space $L^2(A)$. This fact implies the strong convergence of x_α to x on H. It is clear that (2.4)′ → (2.3)′ and (2.3)′ → (2.1)′ for A.

REMARK 2.1. In Theorem 1, the condition (2.1) implies (2.2) for arbitrary set S in $L^1(A)$ and the converse case holds for S consisting of positive operators in $L^1(A)$. Indeed, the former follows from the proof of Theorem 1, and the latter will be obtained by the last part of its proof, because we can take the weak* convergence in the place of the L^1-mean convergence in that part of the proof.

As the final part in this section, we shall discuss a decreasing M-nets in a semi-finite W^*-algebra A on H with a regular gage μ:

THEOREM 3. *Let $\{x_\alpha,\ \alpha \in D\}$ be a decreasing M-net in $L^2(A)$ (in $L^2(A) \cap A$ resp.). Then x_α converges to an operator $x \in L^2(A)$ (in $L^2(A) \cap A$ resp.) in L^2-mean (strongly as operator on H resp.). In particular, if A is σ-finite finite* *and μ is a faithful normal trace and $\{x_\alpha, \alpha \in D\} \subset L^1(A)$, then x_α converges to an $x \in L^1(A)$ in L^1-mean.*

PROOF. Let $\{A_\alpha, \alpha \in D\}$ be the family of the W^*-subalgebras associated to $\{x_\alpha, \alpha \in D\}$ (cf. The first paragraph of § 3). Let $A_1 = \bigcap_{\alpha \in D} A_\alpha$ which is a W^*-subalgebra of A and let y^{e_1} be the conditional expectation of y relative to A_1.

L^2-CASE: Since for any $\alpha, \beta \in D$ $(\alpha \leqq \beta)$, $0 \leqq x_\alpha^* x_\alpha = x_\beta^{*e_\alpha} x_\beta^{e_\alpha} \leqq (x_\beta^* x_\beta)^{e_\alpha}$,

$$0 \leqq \mu(x_\alpha^* x_\alpha) \leqq \mu((x_\beta^* x_\beta)^{e_\alpha}) \leqq \mu(x_\beta^* x_\beta)$$

and $\lim_\alpha \mu(x_\alpha^* x_\alpha)$ exists (uniquely, $= \lambda$ say). Therefore

$$\|x_\alpha - x_\beta\|_2 = \mu((x_\alpha - x_\beta)^*(x_\alpha - x_\beta)) = \mu(x_\beta^* x_\beta) - \mu(x_\alpha^* x_\alpha) \to \lambda - \lambda = 0,$$

and there exists an $x \in L^2(A)$ such that $\|x_\alpha - x\|_2 \to 0$.

$L^2 \cap A$-CASE: Since $\{x_\alpha, \alpha \in D\} \subset L^2(A)$, it converges to $x \in L^2(A)$ in the L^2-mean. While for any $y \in J$ and any fixed $\alpha_0 \in D$, $|\mu(xy)| = |\lim_{\alpha \leqq \alpha_0} \mu(x_\alpha y)|$ $\leqq \|x_{\alpha_0}\|_\infty \|y\|_1$, which implies $x \in A$. Hence $x \in L^2(A) \cap A$ and $\|(x_\alpha - x)y\|_2 \leqq \|y\|_\infty \|x_\alpha - x\|_2 \to 0$ for every $y \in J$, and $\|(x_\alpha - x)y\|_2 \to 0$ for every $y \in L^2(A)$, because J is dense in $L^2(A)$. Therefore x_α converges strongly to x on H.

Finally we prove the last part. For fixed $\alpha_0 \in D$, taking $\{y_n\} \subset A_{\alpha_0}$ such that $\|y_n - x_{\alpha_0}\| \to 0$ $(n \to \infty)$,

$$\|y_n^{e_\alpha} - y_n^{e_\beta}\|_1 \leqq \|y_n^{e_\alpha} - y_n^{e_\beta}\|_2 \to 0$$

and

$$\|y_n^{e_\alpha} - x_\alpha\|_1 \leqq \|y_n - x_{\alpha_0}\|_1 \to 0 \qquad (n \to \infty, \ \alpha \leqq \alpha_0).$$

Therefore for any $\varepsilon > 0$ there are α_ε and n such that for every $\alpha, \beta \leqq \alpha_\varepsilon, \alpha_0$

$$\|x_\alpha - x_\beta\|_1 \leqq \|x_\alpha - y_n^{e_\alpha}\|_1 + \|y_n^{e_\alpha} - y_n^{e_\beta}\|_1 + \|y_n^{e_\beta} - x_\beta\|_1 < \varepsilon$$

and x_α converges to some $x \in L^1(A)$ in the L^1-mean.

REMARK 2.2. In the above proof, each limit operator belongs to $L^2(A_1)$, $L^2(A) \cap A_1$ or $L^1(A_1)$ respectively. For, let x be the limit operator, then by the above proof, we find $\mu(yx) = \mu(yx_\alpha)$ for every $y \in J \cap A_1$ and for every $\alpha \in D$.

4. In this section we shall show that a sequence of bounded operators defined by von Neumann (cf. p. 118 of [9]) is a simple M-set, and apply the preceding consideration to the convergence theorem of it (cf. Theorem 6 of [9]). Firstly, we show a lemma:

LEMMA 1. (Misonou). *Let W be a W^*-algebra on H and let p be a projection in W. For any $x \in W$, put*

$$x^{|p} = pxp + (1-p)x(1-p)^{7)}. \tag{15}$$

Then the range $W^{|p}$ of the mapping $x \to x^{|p}$ is a W^*-subalgebra of W and the mapping is linear and satisfies the conditions $(D.1)-(D.5)$, $(D.7)$ (in

7) These notations were introduced by von Neumann (cf. [9; p. 118]).

Theorem D) and $I^{\varepsilon} = I$.

PROOF. We prove only (D. 5), since the others are almost obvious. For any $x, y \in W$

(16) $(x^{|p}y)^{|p} = ((pxp + (1-p)x(1-p))y)^{|p} = pxpyp + (1-p)x(1-p)y(1-p)$

and $x^{|p}y^{|p} = (pxp + (1-p)x(1-p))(pyp + (1-p)y(1-p))$ which equals to the right side of (16). This implies $(x^{|p}y)^{|p} = x^{|p}y^{|p}$ and similarly $= (xy^{|p})^{|p}$. Since $y^{|p} = y$ for every $y \in W^{|p}$, $(yxy')^{|p} = y(xy')^{|p} = yx^{|p}y'$ for $y, y' \in W^{|p}$.

For any projections in W of finite number $p_1, \ldots, p_n$, we denote $(x^{|p_1})^{|p_2}$, $((x^{|p_1})^{|p_2})^{|p_3}$ and $((\ldots((x^{|p_1})^{|p_2})^{|p_3})\ldots)^{|p_n}$ by $x^{|p_1|p_2}$, $x^{|p_1|p_2|p_3}$, and $x^{|p_1|p_2\cdots|p_n}$ respectively.

LEMMA 2 (*von Neumann* [9]). *If the projections $p_1, \ldots p_n$ in W commute with each others, then for any permutation $(1', 2', \ldots, n')$ of $(1, 2, \ldots, n)$*

$$x^{|p_{1'}|p_{2'}\cdots|p_{n'}} = x^{|p_1|p_2|\cdots|p_n}.$$

This lemma was proved by von Neumann for the I_∞ or II_∞ factor (cf. [9]), which is valid for the present case.

Let $W = A$ be a semi-finite W^*-algebra on H and let μ be a regular gage. Let $\{p_n\}$ be a finite or infinite sequence of projections in S commuting with each others. For any $x \in A$, put

(17) $$x^{\varepsilon_n} = x^{|p_1|p_2\cdots|p_n} \qquad n = 1, 2 \ldots.$$

Under these notations we obtain

THEOREM 4. *$A_{-n} = \{x^{\varepsilon_n};\ x \in A\}$ is a W^*-subalgebra of A for each $n = 1, 2, \ldots,$ and the mapping $x \to x^{\varepsilon_n}$ transforms A onto A_{-n} and is the conditional expectation relative to A_{-n} satisfying $I^{\varepsilon_n} = I$. Putting $x_{-n} = x^{\varepsilon_n}$ for each n, $\{x_{-n}, n = 1, 2, \ldots\}$ is a decreasing simple M-net.*

PROOF. By Lemmas 1 and 2, each A_{-n} is obviously a W^*-subalgebra satisfying

(18) $$A_{-1} \supset A_{-2} \supset \ldots \supset A_{-n} \supset \ldots.$$

For any fixed projection $p \in A$ and for $x \in J$, $x^{|p}$ belongs to $J \cap A$ and satisfying

(19) $\mu(x^{|p}) = \mu(pxp + (1-p)x(1-p)) = \mu(px) + \mu((1-p)x) = \mu(x)$.

Hence by Lemma 1 $\mu(y^{|p}x) = \mu((y^{|p}x)^{|p}) = \mu((yx^{|p})^{|p}) = \mu(yx^{|p})$ for every $y \in A$ and $x \in J$, and by Theorem 1 the mapping $x \to x^{|p}$ is the conditional expectation relative to $A^{|p}$. Similarly, $\mu(x^{|p_1|p_2}) = \mu(x)$ and by Lemmas 1 and 2 $(x^{|p_1|p_2}y)^{|p_1|p_2} = x^{|p_1|p_2}y^{|p_1|p_2}$ holds. Hence by the same way for $x^{|p}$, the mapping $x \to x^{|p_1|p_2}$ is the conditional expectation relative to $A^{|p_1|p_2} (= A_{-2})$. By the inductive method and by (18) these facts hold for every ε_n. It follows from the definition of M-net and $I^{\varepsilon_n} = I$ that $\{x_{-n}, n = 1, 2, \ldots\}$ is a decreasing simple, M-set.

For $x \in L^2(A) \cap A$ and each $n = 1, 2, \ldots,$

$$\mu(x_{-n}^* x_{-n}) = \mu(x^{*\varepsilon_n} x^{\varepsilon_n}) \leqq \mu((x^*x)^{\varepsilon_n}) = \mu((x^*x)$$

and hence $x_{-n} \in L^2(A) \cap A_{-n}$. Then by Theorems 3 and 4, x_{-n} converges strongly to an operator $x_{-\infty}$ in $L^2(A) \cap A_{-\infty}$ which is also a limit in L^2-mean, where $A_{-\infty} = \bigcap_{n=1}^{\infty} A_{-n}$. This implies the Theorem of von Neumann:

THEOREM 5. *For any $x \in L^2(A) \cap A$, $\{x^{\epsilon_n}\}$ belongs to $L_2(A) \cap A_{-n}$, and converges strongly as operator on H and in L^2-mean to an operator $x_{-\infty}$ in $L^2(A) \cap A_{-\infty}$.*

Put $x^\epsilon = x_{-\infty}$ for $x \in L^2(A) \cap A$. Since $\|x^{\epsilon_n}\|_2 = \|x_{-n}\|_2 \leqq \|x\|_2$, we have

(20) $\|x^\epsilon\|_2 \leqq \|x\|_2$ for every $x \in L^2(A) \cap A$.

While for every $x, y \in L^2(A) \cap A$, $x^{*\epsilon} = \lim_{n\to\infty} x^{*\epsilon_n} = \lim_{n\to\infty} x^{\epsilon_n *} = x^{\epsilon *}$ and

$$(x^\epsilon y)^\epsilon = \lim_{n\to\infty} (x^\epsilon y)^{\epsilon_n} = \lim_{n\to\infty} x^\epsilon y^{\epsilon_n} = x^\epsilon y^\epsilon = \lim_{n\to\infty} x^{\epsilon_n} y^\epsilon = (xy^\epsilon)^\epsilon,$$

where the limit is that with respect to the weak operator topology, and for every $x, y \in J$

$$\mu(x^\epsilon y) = \lim_{n\to\infty} \mu(x^{\epsilon_n} y) = \lim_{n\to\infty} \mu(xy^{\epsilon_n}) = \mu(xy^\epsilon).$$

The linearity and idempotency ($x^{\epsilon\epsilon} = x^\epsilon$) of the mapping $x \to x^\epsilon$ (defined on $L^2(A) \cap A$) are clear. Since $L^2(A) \cap A$ is dense in $L^2(A)$, by (20) it is uniquely extended on the whole space $L^2(A)$. Further, since $x^\epsilon = x$ for every $x \in L^2(A) \cap A_{-\infty}$, $x \to x^\epsilon$ satisfies the condition (2′) in Corollary 1.2. Therefore, we obtain

COROLLARY 5.1 *The mapping $x \to x_{-\infty}$ ($x \in L^2(A) \cap A$) is uniquely extended to the conditional expectation $x \to x^\epsilon$ relative to $A_{-\infty}$.*

From Theorem 5 and this Corollary it follows that for every $x \in A$, $I^\epsilon x_{-n}$ converges weakly to the x^ϵ, I^ϵ being the maximal central projection in the W^*-subalgebra $A_{-\infty}$ of A.

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