ON THE AUTOMORPHISM GROUP OF A CERTAIN CLASS OF ALGEBRAIC MANIFOLDS

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(Recived December, 18, 1958)

1. Introduction. The main purpose of the present paper is to prove the following

THEOREM. *Let M be a compact complex manifold whose first Chern class $c_1(M)$ is negative definite. Then the group G of all holomorphic transformations of M is finite.*

For the case dim $M = 1$ (i. e., a compact Riemann surface of genus>1), this theorem has been known for a long time. In fact,

(A) H. A. Schwarz proved that G is discrete. [16].

(B) In 1882, F. Klein proved, in his letter to H. Poincaré, that G is finite. (cf. pp. 15-17 of [14]).

(C) A. Hurwitz proved that the order of G is not greater than $84(p-1)$, where p is the genus of M. [9].

For the case dim $M = 2$, we have the following theorem of A. Andreotti.1)

(D) If M is a non-singular irrational algebraic surface of linear genus>1, then G is finite. [2].

Observe that the assumption $c_1(M) < 0$ is stronger than that of (D).

For the case of an arbitrary dimension, a special case of our theorem has been proved by Bochner [5], Hawley [8] and Sampson [15]. Namely,

(E) If M is a compact complex manifold whose universal covering space is a bounded domain in C^n, then G is finite.

In 1946, Bochner proved that

(F) If M is a compact Kaehler manifold whose Ricci tensor is negative definite, then G is discrete. [4].

Making use of a result of Akizuki-Nakano [1], Nakano generalized the result of Bochner as follows. (cf. p. 386 [12]).

(G) If M is a compact complex manifold with $c_1(M) < 0$, then G is dis-

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*) The author was supported by the National Science Foundation.

1) For algebraic surfaces, Andreotti gave estimates on the order of G and on the periods of elements of G. Cf. Sopra le superficie algebriche che posseggono transformazioni birazionali in sè, Univ. Roma Inst. Naz. Alta Mat. Rend. Mat. e Appl. 9(1950)255-279.

crete.

Our theorem, therefore, strengthens the theorem of Nakano.

A. Andreotti wrote us that he has another proof for our theorem. He proves that G is algebraic and, by virtue of the theorem of Nakano, derives the fact that G is finite. He has pointed out that the fact that G is an algebraic group follows also from a result of Matsusaka. [13].

We prove first that G is compact. In the course of proof, we prove implicitly that G is an algebraic group, of which we became aware only after we had read the letter of Andreotti. In fact, our lemma 3 states that *there exists an imbedding of M into a projective space such that G is isomorphic to the group of projective transformations which send M into itself.* From the compactness of G and from the theorem of Nakano, it follows that G is finite. However, as we shall show in the proof, our method will permit us to obtain the final result without making use of Nakano's result.

We can apply our result to a problem of complex fibre bundles. Let E be a complex analytic fibre bundle over B with fibre F. More precisely,

(1) E, B and F are complex manifolds.

(2) There exists a holomorphic map p (called the projection) of E onto B.

(3) For every point x of B, there exists a neighborhood U of x such that $p^{-1}(U)$ is a trivial bundle; i. e., there exists a holomorphic homeomorphism f_U of $p^{-1}(U)$ onto $U \times F$ such that

$$p(z) = p' \circ f_U(z) \qquad z \in p^{-1}(U),$$

where $p' : U \times F \to U$ is the natural projection.

In this paper, we consider only the case where F is compact. The group G of all holomorphic transformations of F will be the structure group of fibre bundle E, although we do not mention it explicitly. As a direct consequence of our theorem, we shall obtain the following

COROLLARY. *Let (E, B, F, p) be a complex fibre bundle such that F is compact and $c_1(F) < 0$. Then*

(1) *If B is Kaehlerian (compact or not), so is E.*

(2) *If B is algebraic (i. e., projective), so is E.*

Kodaira proved in [11] that E is Kaehlerian (resp. algebraic) if B is compact Kaehlerian (resp. algebraic) and if F is a complex projective space. Borel and Kodaira [3] generalized the result to the case where F is an algebraic manifold with $b_1(F) = 0$ (b_1 is the first Betti number) and the structure group is connected. It is of some interest to note that $c_1(F) > 0$ implies $b_1(F) = 0$, whereas we are considering the case $c_1(F) < 0$. Blanchard proved in his thesis [3] theorems similar to that of Borel-Kodaira without any restriction on the first Betti number of F but with a certain assumption on the topological structure of the fibering. The proof of our corollary does not rely on the result of Blanchard and is very elementary.

Observe the following general fact. Let (E, B, F, p) be a complex fibre bundle with structure group H. (H is necessarily a subgroup of G). Suppose there exists a projective model of F such that H can be represented by a group of projective transformations. Then a similar statement as our corollary holds. (This is an immediate consequence of Kodaira's result). Our corollary can be derived also from this general fact and from Lemma 3. However, the finiteness of G makes the proof much simpler.

2. Proof of Theorem. Let K be the canonical line bundle over M. If $c_1(M) < 0$, then there exists a positive integer k such that the line bundle kK admits a sufficiently many holomorphic cross-sections so that these sections determine an imbedding of M into a complex projective space. More precisely, let $\Gamma(kK)$ be the vector space of all holomorphic sections of kK over M. It is finite dimensional as M is compact. Let $f_0, f_1, \ldots\ldots, f_N$ be a basis for $\Gamma(kK)$. Our conclusion is that

$$\eta : z \to (f_0(z), f_1(z), \ldots\ldots, f_N(z)) \qquad z \in M,$$

defines an imbedding of M into the complex projective space $P_N(C)$ of dimension N. This follows from a result of Kodaira. (Observe how he obtains β in Theorem 3 of [11]). Conversely, if there exists such a positive integer k, then $c_1(M) < 0$. However, we shall not make use of this fact.

Every holomorphic transformation φ of M can be lifted, in a natural way, to a holomorphic automorphism φ of the canonical line bundle K and, consequently, to a holomorphic automorphism φ'' of $kK = K \otimes \ldots\ldots \otimes K$. (Note that the fibre of K over z is $\wedge^n T_z^*$, where T_z^* is the complex co-tangent space to M at z; $n = \dim M = \dim T_z^*$). Therefore, φ induces a linear transformation φ^* of $\Gamma(kK)$. Let $\tilde{\varphi}$ be the projective transformatipn of $P_N(C)$ induced from the linear transformation φ^* of $\Gamma(kK)$. From the definition of η and $\tilde{\varphi}$, it follows that

LEMMA 1. $\tilde{\varphi} \circ \eta = \eta \circ \varphi$.

In other words, the imbedding $\eta : M \to P_N(C)$ allows us to represent G by a group of projective transformations of $P_N(C)$.

Define

$$\rho(\varphi) = \varphi^* \qquad \text{and} \qquad \sigma(\varphi) = \tilde{\varphi}.$$

Then ρ (resp. σ) is a representation of G into the group of linear transformations of $\Gamma(kK)$ (resp. the group of projective transformations of $P_N(C)$).

From the fact that η is injective and from Lemma 1, it follows that σ is faithful. Hence,

LEMMA 2. *Both σ and ρ are faithful.*

We shall prove the following

LEMMA 3. *$\sigma(G)$ consists of exactly those projective transformations of $P_N(C)$ which send $\mu(M)$ into itself.*

Every element of $\sigma(G)$ sends $\eta(M)$ into itself (Lemma 1). Let T be a projective transformation of $P_N(C)$ which sends $\eta(M)$ into itself. Let φ be the restriction of T to $\eta(M)$. As η is an imbedding, φ can be considered as a holomorphic transformation of M. Now, it suffices to show that $T = \sigma(\varphi)$. The restriction of T to $\eta(M)$ (i. e., φ) agrees with $\sigma(\varphi)$ on $\eta(M)$ (Lemma 1). In other words, $\sigma(\varphi)\circ T^{-1}$ is a projective transformation of $P_N(C)$ which induces the identity transformation on $\eta(M)$. The proof of Lemma 3 is reduced to that of the following

LEMMA 4. *If T is a projective transformation of $P_N(C)$ which induces the identity transformation of $\eta(M)$, then T is the identity transformation of $P_N(C)$.*

Let τ be a linear transformation of $\Gamma(kK)$ which induces T. We shall show that $\tau = cI$, where c is a complex number and I is the identity transformation of $\Gamma(kK)$. From $T\circ\eta = \eta$ and from the definition of η, it follows that

$$\tau f((z) = c(z)\cdot f(z) \qquad \text{for all } f \in \Gamma(kK),$$

where $c(z)$ is a non-zero complex number which is independent of f (but may depend on $z \in M$ for the moment). On the other hand, both τf and f are holomorphic sections of the line bundle kK. Hence, $c(z)$ is holomorphic in z. As M is compact, $c(z)$ is a constant. This completes the proof of Lemma 4.

The lemma 3 implies immediately the following

LEMMA 5. *σ (G) is a closed subgroup of the projective transformation group of $P_N(C)$.*

Observe that the lemma 3 implies that $\rho(G)$ is an algebraic group as pointed out by Andreotti.

Our next step is to define a bounded domain in $\Gamma(kK)$ which is invariant by $\rho(G)$. To this end, we introduce a real valued function ν on $\Gamma(kK)$, which behaves like a norm. Every holomorphic section of the canoncal line bunble K is a holomorphic n-form on M, and conversely. Hence, in terms of local coordinate system $z^1, \ldots\ldots z^n$ of M, every element f of $\Gamma(kK)$ can be symbolically written as follows:

$$f = f^*(dz^1 \wedge \cdots\cdots \wedge dz^n)^k,$$

where f^* is a holomorphic function defined in the coordinate neighborhood of $z^1, \ldots\ldots, z^n$. We define

$$\nu(f) = \int_M |f^*|^{2/k} dz^1 \wedge d\bar{z}^1 \wedge \cdots\cdots \wedge dz^n \wedge d\bar{z}^n.$$

Observe that the definition is independent of local coordinate system. We obtain easily the following

LRMMA 6. *(1) $\nu(f) \geqq 0$ and $\nu(f) = 0$ if and only if $f = 0$.*

(2) $\nu(cf) = |c|^{2/k}.\ \nu(f)$ for every complex number c.

(3) ν is a continuous function on the finite dimensional vector space $\Gamma(kK)$.

(4) $\nu(\varphi^* f) = \nu(f)$ for all $\varphi \in G$.

Let D be an open subset of $\Gamma(kK)$ defined by

$$D = \{f;\ \nu(f) < 1\}.$$

LEMMA 7. *D is a bounded domain invariant by $\rho(G)$.*

Every point of D can be joined to the origin by a straight line in D by virtue of (2) of Lemma 6; i.e., D is star like. To see that D is bounded, let $f_0, f_1, \ldots\ldots, f_N$ be a basis for $\Gamma(kK)$. Let s^{2N+1} be the unit sphere in $\Gamma(kK)$ with respect to this basis:

$$S^{2N+1} = \{\Sigma a_j f_j;\ \Sigma |a_j|^2 = 1\}.$$

Let ν_0 be the minimum value of the function ν on S^{2N+1}. Let r be a positive real number such that $r^{2/k}\nu_0 > 1$. Let $S^{2N+1}(r)$ be the sphere of radius r in $\Gamma(kK)$:

$$S^{2N+1}(r) = \{\Sigma a_j f_j;\ \Sigma |a_j|^2 = r^2\}.$$

Then, by virtue of (2) of Lemma 6, D is inside $S^{2N+1}(r)$. The invariance of D by $\rho(G)$ follows from (4) of Lemma 6.

In general, given any point o of a bounded domain D, the group of holomorphic transformations of D which leave o fixed is a *compact* Lie group. (Theorem of H. Cartan [7]; a differential geometric proof can be found in [10]). Let H be the group of linear transformations of $\Gamma(kK)$ which send D onto itself. Obviously, H is a compact Lie group. (Take the origin of $\Gamma(kK)$ as o). Let $\widetilde{H}$ be the subgroup of the group of projective transformations of $P_N(C)$ obtained from H. (Every element of H is a linear transformation of $\Gamma(kK)$, hence induces a projective transformation of $P_N(C)$).

As H is compact, $\widetilde{H}$ is compact. From the lemmas 5 and 7, it follows that

LEMMA 8. *$\sigma(G)$ is compact.*

A well known theorem of Bochner-Montgomery [6] states that the group G of holomorphic transformations of any compact complex manifold M is a complex Lie group and the group action $G \times M \to M$ is a holomorphic map. From this fact, it follows that ρ is a holomorphic map of G into the group of linear transformations of $\Gamma(kK)$. (This can be also derived from the fact that G is an algebraic group). As G is compact, the connected component of the identity of G consists of only the identity transformation. Hence G is a finite group. As we remarked in the introduction, this part can be replaced by the theorem of Nakano.

3. Proof of Corollary. Let G be the group of holomorphic transformations of F. We consider G as the structure group of the bundle E. G is a finite group (Theorem). Let P be the associated principal fibre bundle. As G is finite, P is a covering space (not necessarily connected) of B. Let ds^2_B be a Kaehler metric on the base space B. The projection of P onto B induces a Kaehler metric ds^2_P on P. If B is compact and ds^2_B is a Hodge metric, then P is compact and ds^2_P is a Hodge metric. As P is a principal fibre bundle, G acts on P on the right. From the definition of ds^2_P, it follows that ds^2_P is invariant by G. As F is algebraic and G is finite, there exists a Hodge metric ds^2_F on F invariant by G. Let $ds^2_P + ds^2_F$ be the Kaehler metric on $P \times F$ defined in a natural way. If ds^2_P is a Hodge metric, so is $ds^2_P + ds^2_F$. From the well known relation between E and P (or rather the definition of E), i. e.,

$$P \times_G F = E,$$

it follows that $P \times F$ is a covering space (not necesarily connected) of E since G is finite. The Kaehler metric $ds^2_P + ds^2_F$ induces a Kaehler metric ds^2_E on E. If the former is a Hodge metric, so is the latter. This completes the proof of our corollary.

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