ON ALMOST CONTACT METRIC MANIFOLDS ADMITTING PARALLEL FIELDS OF NULL PLANES

YOSHIHIRO ICHIJYO

(Received October 2, 1963, Revised February 20, 1964)

1. Introduction. Since Prof. S. Sasaki [3], [4][1] introduced the notion of (φ, ξ, η, g) structure in odd dimensional manifolds which is equivalent to the almost contact metric structure, many authors have investigated it vigorously from differential geometric point of view.

In the first place it is seen easily that the almost contact metric manifold admits a certain field of complex null planes (we call it π-plane field) and that the π-plane field is parallel if and only if the tensor $\varphi^i{}_j$ is covariant constant. The main purpose of this paper is to study on the geometry of the almost contact metric manifold M^{*2n+1} of dimension $2n+1$ whose π-plane field is parallel.

The present author wishes to express his hearty thanks to Prof. M. Matsumoto for his kind criticism and encouragement.

2. An almost contact metric manifold admitting certain parallel fields of complex null n-planes. Let M^{2n+1} be an almost contact metric manifold, then there exist tensor fields $\varphi^i{}_j$, ξ^i, η_j[2] and positive definite Riemannian metric tensor g_{ij} over M^{2n+1} satisfying

$$(2.1)\quad \begin{cases} \text{rank } |\varphi^i{}_j| = 2n, & \varphi^i{}_j\xi^j = 0, \quad \varphi^i{}_j\eta_i = 0, \quad \xi^i\eta_i = 1, \\ \varphi^i{}_j\varphi^j{}_k = -\delta^i{}_k + \xi^i\eta_k, & g_{ij}\varphi^i{}_h\varphi^j{}_k = g_{hk} - \eta_h\eta_k, \quad \eta_i = g_{ij}\xi^j. \end{cases}$$

It is obvious that the tensor $\varphi^i{}_j$ satisfies $\varphi^i{}_j\varphi^j{}_k\varphi^k{}_h = -\varphi^i{}_h$, and hence the rank of the matrix $(\gamma^i{}_j) = (\varphi^i{}_j - \sqrt{-1}\ \delta^i{}_j)$ is $n+1$ over M^{2n+1}. Now we consider vectors λ^j satisfying

$$(2.2)\qquad \gamma^i{}_j\lambda^j = 0,$$

and take a field of n-planes π^n over M^{2n+1} spanned by the λ^j's, which is called π-plane field hereafter. From (2.2) we see directly for any vector $\lambda^i{}_{(\alpha)}$ in π-plane field

$$\eta_i\lambda^i{}_{(\alpha)} = 0, \qquad g_{ij}\lambda^i{}_{(\alpha)}\lambda^j{}_{(\beta)} = 0.$$

---

1) Numbers in blackets refer to the references at the end of the paper.

2) In this paper the indices h, i, j, k, l, m run over the range $1, \cdots, 2n+1$; $\alpha, \beta, \gamma, \lambda, \mu$ the range $1, \cdots, n$; a, b, c, d the range $1, \cdots, 2n$; and A, B, C, D the range $1, \cdots, 2n+2$.

Therefore we obtain

THEOREM 2.1. *An almost contact metric manifold M^{2n+1} admits π-plane field, whose vectors are complex null vectors and orthogonal to the vector ξ^i.*

We shall now consider a necessary and sufficient condition for the π-plane field to be parallel. Denoting the covariant differentiation with respect to the Christoffel symbols $\left\{ {i \atop j\ k} \right\}$ by (;), we see directly from (2. 2) $\varphi^i{}_{j;k}\lambda^j + \gamma^i{}_j\lambda^j{}_{;k} = 0$.

Suppose the π-plane field is parallel and $\lambda^i{}_{(\alpha)}$ are basic vectors of π-plane, then λ^i satisfies $\lambda^i{}_{;k} = A^{(\alpha)}{}_k \lambda^i{}_{(\alpha)}$, where $A^{(\alpha)}{}_k$ are components of complex covariant vector fields [7]. Hence we get

$$\varphi^i{}_{j;k}\lambda^j = 0. \tag{2. 3}$$

It is to be noted that $\pi^n \cap \overline{\pi}^n = \{0\}$, by virtue of the definition of λ^i. Then $\pi^n + \overline{\pi}^n (= p^{2n})$ forms a parallel field of complex $2n$-planes orthogonal to ξ^i and contains a parallel field of real $2n$-planes $\widetilde{p}^{2n}$. Therefore in terms of real basic vectors $C^j{}_{(a)}$ of $\widetilde{p}^{2n}$ orthogonal to ξ^i, the relation (2. 3) can be reducible to $\varphi^i{}_{j;k} C^j{}_{(a)} = 0$. Thus we may put $\varphi^i{}_{j;k} = \mu^i{}_k \eta_j$. It follows from this and (2. 1) that

$$\mu^i{}_k = \varphi^i{}_{l;k}\xi^l = -g^{ih}\varphi^l{}_{h;k}\eta_l = -g^{ih}\mu^l{}_k\eta_h\eta_l = g^{ih}\varphi^l{}_m\xi^m{}_{;k}\eta_h\eta_l = 0,$$

and obtain $\varphi^i{}_{j;k} = 0$.

Conversely, if the relation $\varphi^i{}_{j;k} = 0$ holds good, then by differentiating $\gamma^i{}_j\lambda^j{}_{(\alpha)} = 0$ covariantly, we have $\gamma^i{}_j\lambda^j{}_{(\alpha);k} = 0$, that is to say, the π-plane field is a field of parallel planes. Therefore we obtain

THEOREM 2.2. *A necessary and sufficient condition for π-plane field to be parallel is that the tensor $\varphi^i{}_j$ is covariant constant over the almost contact metric manifold M^{2n+1}.*

Next, in an almost contact metric manifold satisfying $\varphi^i{}_{j;k} = 0$, we obtain as well as [4]

COROLLARY 1. *In an almost contact metric manifold, if the π-plane field is parallel, then the tensor $\varphi^i{}_j$, ξ^i and η_j are covariant constant, and the Nijenhuis tensors $N^i{}_{jk}$, N_{jk}, $N^i{}_j$ and N_j vanish.*

When the M^{2n+1} is a contact metric manifold satisfying $\varphi^i{}_{j;k} = 0$, then the M^{2n+1} is a normal contact metric one and the relation

$$2\varphi_{ij;k} = \eta_i g_{jk} - \eta_j g_{ik}$$

holds good [5]. From this we see $\eta_i=0$. This contradicts to (2.1). Thus

COROLLARY 2. *In a contact metric manifold M^{2n+1}, the π-plane field can not be parallel.*

3. Converse theorem. In this paragraph we shall consider the converse of the Theorem 2.2 and obtain

THEOREM 3.1. *If a Riemannian manifold V^{2n+1} of class C^ω admits a field π^n of n-planes which is null and parallel with respect to a given positive definite Riemannian metric g_{ij} on V^{2n+1}, then the V^{2n+1} is an almost contact metric manifold satisfying $\varphi^i{}_{j;k} = 0$.*

PROOF. Since π^n is null and the given metric is positive definite, the real vector of π^n at any point is the zero vector only. Therefore π^n is a field of complex null n-planes. The manifold also admits the complex conjugate field $\bar{\pi}^n$, which is also null and parallel. The planes π^n and $\bar{\pi}^n$ at each point have only the zero vector in common, then their sum $\pi^n+\bar{\pi}^n$ forms a parallel field of complex $2n$-planes and contains a field of real $2n$-planes $\widetilde{p^{2n}}$, whose real basic vectors are $C^i{}_{(a)}$ $(a^i{}_{(\alpha)}, b^i{}_{(\alpha)})$, where $a^i{}_{(\alpha)}$ and $b^i{}_{(\alpha)}$ are respectively real and imaginary part of basic complex vectors $\lambda^i{}_{(\alpha)}$ of π^n and satisfy

$$C^j{}_{(a);k} = B^{(b)}_{(a)k} C^j{}_{(b)}. \tag{3.1}$$

Now consider, at any point P of V^{2n+1}, unit contravariant vector orthogonal to a real vector space $p^{2n}(\mathrm{P})$. Then the relation

$$C^j{}_{(a)}\eta_j = 0 \tag{3.2}$$

must be satisfied over V^{2n+1} by the vector $\eta_i = g_{ij}\xi^j$. Differentiating (3.2) covariantly and using (3.1), we have $\eta_{j;k} = 0$.

Since $\eta_{j;k}=\eta_{k;j}$, there is a scalar field $f = f(x^1, x^2, \cdots, x^{2n+1})$ locally satisfying $\eta_j = \partial_j f$. Then the hypersurfaces K^{2n} defined by $f =$ constant in a local coordinate neighborhood form a family of ∞^1 totally geodesic hypersurfaces. By $K^{2n}(\mathrm{P})$ we mean the hypersurface K^{2n} passing through a point P with local coordinate $x^i{}_0$. Hence if by π^{*n} we denote naturally induced π^n in the $K^{2n}(\mathrm{P})$, then, apparently, $K^{2n}(\mathrm{P})$ admits a parallel field of null n-planes π^{*n}. Therefore, according to Patterson's result [2], the $K^{2n}(\mathrm{P})$ is a Kähler manifold. When we represent by $\varphi^{*a}{}_b$ and $g^*{}_{ab}$ the complex structure tensor and naturally induced metric tensor of $K^{2n}(\mathrm{P})$ respectively, they satisfy

$$\varphi^{*a}{}_b\varphi^{*b}{}_c = -\delta^a{}_c,\ \varphi^{*a}{}_{b;c} = 0,\ g^*{}_{ab} = g^*{}_{cd}\varphi^{*c}{}_a\varphi^{*d}{}_b. \tag{3.3}$$

Since the V^{2n+1} admits a covariant constant vector field η_i, according to the well known theorem [8], there exists a local coordinate system, with respect to which the ds^2 of the manifold is written by

$$ds^2 = g^*_{ab}(u^c)du^a du^b + du^\Delta du^\Delta \qquad (\Delta = 2n+1), \tag{3.4}$$

where $g^*_{ab}(u^c)\, du^a du^b$ is of the Kähler manifold $K^{2n}(\mathrm{P})$. It is clear that the orthogonal trajectories of the family $\{K^{2n}\}$ are geodesics. If we represent the relation between two coordinate systems x^i and (u^c, u^Δ) by $x^i = x^i(u^c, u^\Delta)$, the K^{2n} is given by $u^\Delta =$ constant.

On the other hand, when we put $B^i{}_a = \partial_a x^i$, $B^b{}_j = g^{*ab} g_{jk} B^k{}_a$ and

$$\varphi^i{}_j = \varphi^{*a}{}_b B^i{}_a B^b{}_j, \tag{3.5}$$

the $\varphi^i{}_j$ becomes a $(1-1)$ tensor over M^{2n+1}.

Then, after direct calculation, we can easily see that $\varphi^i{}_j$, ξ^i, η_j and g_{ij} form an almost contact metric structure. In terms of coordinate system (u^a, u^Δ), making use of (3. 3) and (3. 4), we can also verify that

$$\varphi^i{}_{j;k} = 0.$$

Thus we have proved the Theorem 3.1, and at the same time we obtain

THEOREM 3. 2. *The following three conditions are mutually equivalent;*
(I) *A Riemannian manifold V^{2n+1} of class C^∞ admits a parallel field of null n-planes with respect to a given positive definite Riemannian metric.*
(II) *A differentiable manifold V^{2n+1} of class C^∞ is an almost contact metric manifold satisfying $\varphi^i{}_{j;k} = 0$.*
(III) *A Riemannian manifold V^{2n+1} of class C^∞ admits a family of totally geodesic hypersurfaces whose orthogonal trajectories are geodesics, the hypersurfaces being Kähler manifolds whose Kähler metric is naturally induced one from V^{2n+1}.*

4. Some theorems which have resemblance with those in Kähler manifolds. In this paragraph we denote by M^{*2n+1} an almost contact metric manifold satisfying $\varphi^i{}_{j;k} = 0$, and show that M^{*2n+1} has several properties which hold good in Kähler manifolds.

From the Corollary 1 of paragraph 2, the Ricci identities for ξ^i, η_j and $\varphi^i{}_j$ lead us to

$$R^l{}_{kij}\eta_l = 0,\ R^k{}_{lij}\xi^l = 0,\ R^h{}_{lij}\varphi^l{}_k = R^l{}_{kij}\varphi^h{}_l. \tag{4.1}$$

It follows from these that

$$R_{il}\xi^l = 0, \tag{4.2}$$

where $R^i{}_{jkl}$ and R_{jk} are the Riemann curvature tensor and Ricci tensor respectively.

If the manifold is Einstein, (4.2) implies $R_{ij}=0$, because of $\xi^i\eta_i=1$. This result is stated as follows

THEOREM 4.1. *If the almost contact metric manifold satisfying* $\varphi^i{}_{j;k}=0$ *is an Einstein manifold, then the Ricci tensor vanishes.*

If the manifold M^{*2n+1} is of constant curvature, then the M^{*2n+1} is, of course, an Einstein, and from Theorem 4.1 we have $R_{ijkl}=0$. Thus

COROLLARY. *If the almost contact metric manifold satisfying* $\varphi^i{}_{j;k}=0$ *is of constant curvature, then it is of zero curvature.*

This corollary corresponds to Bochner's theorem "*If a Kähler manifold is of constant curvature, then it is of zero curvature.*" [1].

Next, we examine the theorem corresponding to Yano-Mogi's theorem "*If a Kähler manifold is conformally flat, then it is of zero curvature.*" [9].

In the case $n \geqq 2$, the condition for M^{*2n+1} to be conformally flat is represented by

$$R_{ijkl}=\frac{1}{2n-1}\,(R_{jk}g_{il}-R_{jl}g_{ik}+g_{jk}R_{il}-g_{jl}R_{ik}) - \frac{R}{2n(2n-1)}(g_{jk}g_{il}-g_{jl}g_{ik}). \tag{4.3}$$

Proceeding in like manner with the calculation used by Yano-Mogi [9], (4.2) and (4.3) lead us to

$$\frac{R}{2n}\varphi^j{}_i=-(2n-3)R^j{}_a\varphi^a{}_i, \qquad (n-1)R=0. \tag{4.4}$$

From the assumption $n \geqq 2$, (4.4) shows us $R=0$ and $R_{ij}\varphi^j{}_k=0$. Then using (4.2) we obtain $R_{ijkl}=0$.

In the case $n=1$, the condition for M^{*3} to be conformally flat is represented by

$$R_{ij;k}-R_{ik;j}+\frac{1}{4}g_{ik}R_{;j}-\frac{1}{4}g_{ij}R_{;k}=0. \tag{4.5}$$

After direct calculation, (4.2) and (4.4) lead us to $R_{jk}(\varphi^k{}_i+\xi^k\eta_i)=\frac{R}{2}\varphi_{ji}$, from which we have

$$R_{ij}=\frac{R}{2}(g_{ij}-\eta_i\eta_j). \tag{4.6}$$

Substituting (4.6) into (4.5), we have

(4.7) $$R_{ij;k} = 0.$$

Conversely, if (4.6) and (4.7) hold good in M^{*3}, (4.5) also holds good. Hence we obtain

THEOREM 4.2. *In order that an almost contact metric manifold M^{*2n+1} satisfying $\varphi^i{}_{j;k} = 0$ is conformally flat, it is necessary and sufficient that,* (1) *for $n \geqq 2$, the M^{*2n+1} is of zero curvature, and* (2) *for $n=1$, the M^{*3} has a covariant constant Ricci tensor given by* (4.6).

5. Some examples. We shall consider some examples of almost contact metric manifolds satisfying $\varphi^i{}_{j;k} = 0$.

Let K^{2n+2} be a Kähler manifold whose complex structure and Kähler metric tensor are $F^A{}_B$ and g_{AB} respectively. Let us consider an orientable differentiable hypersurface M^{2n+1} of K^{2n+2}, which is locally expressed by the equations $y^A = y^A(x^i)$, and denote the normal unit vector and tangent vectors by N^A and $B^A{}_i = \partial_i y^A$ respectively.

Following Y.Tashiro [6], we put

(5.1) $$\varphi^i{}_j = B^A{}_j F^B{}_A B^i{}_B,\ \xi^i = - N^A F^B{}_A B^i{}_B,\ \eta_j = B^A{}_j F^B{}_A N_B,$$

then the quantities $\varphi^i{}_j$, ξ^i, η_j and induced metric $g_{ij}\,(= g_{AB}B^A{}_iB^B{}_j)$ form an almost contact metric structure of M^{2n+1}. We call this hypersurface almost contact metric hypersurface in K^{2n+2}. The second fundamental tensor H_{ik} of M^{2n+1} imbedded in K^{2n+2} is defined by the equations

(5.2) $$B^A{}_{i;k} = \partial_k B^A{}_i + \left\{ {A \atop BC} \right\} B^B{}_i B^C{}_k - \left\{ {h \atop ik} \right\} B^A{}_h = H_{ik} N^A.$$

Using (5.2), we have

(5.3) $$\varphi^i{}_{j;k} = g^{il}(H_{lk}\eta_j - H_{jk}\eta_l).$$

Hence the condition $\varphi^i{}_{j;k} = 0$ can be written as follows:

(5.4) $$H_{ij} = \alpha\eta_i\eta_j.$$

Thus we have

THEOREM 5.1. *In order that an almost contact metric hypersurface in a Kähler manifold K^{2n+2} satisfies $\varphi^i{}_{j;k} = 0$, it is necessary and sufficient that the second fundamental tensor H_{ij} of M^{2n+1} is given by* (5.4).

COROLLARY. *A totally geodesic hypersurface imbedded in a Kähler manifold K^{2n+2} admits an almost contact metric structure satisfying*

$$\varphi^i{}_{j;k} = 0.$$

Moreover, if an enveloping Kähler manifold K^{2n+2} is flat, the Gauss' equation of a hypersurface and (5.4) implies $R_{hijk} = 0$. Hence

THEOREM 5.2. *If an almost contact metric hypersurface M^{2n+1} in a flat Kähler manifold K^{2n+2} satisfies the condition $\varphi^i{}_{j;k} = 0$, then the M^{2n+1} is also flat.*

REFERENCES

[1] S. BOCHNER, Curvature in Hermitian metric, Bull. Amer. Math. Soc., 53(1947), 179-195.
[2] E. M. PATTERSON, A characterisation of Kähler manifold in terms of parallel fields of planes, Journ. of the London Math. Soc., 28(1953), 260-269.
[3] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Journ., 12(1960), 456-476.
[4] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tôhoku Math. Journ., 13(1961), 281-294.
[5] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, Journ. of the Math. Soc. of Japan, 14(1962), 249-271.
[6] Y. TASHIRO, On contact structure of hypersurfaces in complex manifolds I, Tôhoku Math. Journ., 15(1963), 62-78.
[7] A. G. WALKER, On parallel vector fields of partially null vector space, Quart. Journ. of Math. (Oxford), 20(1949), 135-145.
[8] K. YANO, Sur la parallélism et la concourance dans l'espace de Riemann, Proc. Imp. Acad. Tokyo, 19(1949), 189-197.
[9] K. YANO AND I. MOGI, On real representations of Kähler manifold, Ann. of Math., 61 (1955), 170-189.
[10] K. YANO AND S. BOCHNER, Curvature and Betti Numbers, Ann. of Math., Studies N. 32, Princeton Univ. Press (1953).

FACULTY OF ENGINEERING, TOKUSHIMA UNIVERSITY.