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AN EXTENSION OF AN APPROXIMATION PROBLEM PROPOSED BY K. ITÔ

LOTHAR HOISCHEN

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The following problem, which first was proposed by Itô, is of some interest to the theory of probability:

If $f(n)$ is a sequence with

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |f(n)|^2 < \infty , \tag{1}$$

does there always exist a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |f(n) - P(n)|^2 < \varepsilon$$

for any assigned $\varepsilon > 0$?

Izumi [2] has given an affirmative answer if (1) is strengthened to

$$\sum_{n=0}^{\infty} |f(n)|^2/w^n < \infty \quad \text{for some} \quad w > 0 .$$

A more general existential proof, based upon the Hahn-Banach Theorem is due to Edwards [1].

In this paper we shall obtain a very short and simple proof of the following extension of Itô's problem:

THEOREM. *Suppose that*

$$\int_0^{\infty} H(|f(t)|)\, d\alpha(t) < \infty , \tag{2.1}$$

$f(t)$ continuous for $t \geqslant 0$, $H(t) \geqslant 0$ continuous and not decreasing for $t \geqslant 0$,

$H(0)=0$, *$\alpha(t)$ not decreasing for* $t\geqslant 0$.

$$(2.2)\qquad H(|x+y|)\leqslant K[H(|x|)+H(|y|)]$$

for every x, y with some constant K independent of x, y.

$$(2.3)\qquad \int_0^\infty H(e^{ut})\,d\alpha(t)<\infty \quad \textit{for every} \quad u>0.$$

Then, for every $\varepsilon>0$ there exists a polynomial $P(x)$ such that

$$\int_0^\infty H(|f(t)-P(t)|)\,d\alpha(t)<\varepsilon.$$

PROOF. To prove our theorem we only need the familiar Weierstrass Approximation Theorem:

LEMMA. *Let $h(x)$ be continuous on* $[0, 1]$, *then, for every $\varepsilon>0$ there is a polynomial $P(x)$ such that*

$$|h(x)-P(x)|<\varepsilon \qquad (x\in[0,1]).$$

By (2.1) we now choose a number N, so that

$$(3)\qquad \int_N^\infty H(|f(t)|)\,d\alpha(t)<\varepsilon.$$

(2.3) implies $\int_0^\infty d\alpha(t)<\infty$. Therefore it is easily seen that there are a number $c\geqslant 0$ and a continuous function $g(t)$ for $t\geqslant 0$ such that $g(t)=0$ for $t\geqslant N+c$ and

$$(4)\qquad \int_0^{N+c} H(|f(t)-g(t)|)\,d\alpha(t)<\varepsilon.$$

Thus by (3) and (4)

$$(5)\qquad \int_0^\infty H(|f(t)-g(t)|)\,d\alpha(t)<2\varepsilon.$$

By the above Lemma there is a polynomial $B(x)=\sum_{v=0}^{k} b_v x^v$ such that

$$|g(t)-B(e^{-t})|<\varepsilon \quad \text{for} \quad t\geqslant 0. \tag{6}$$

Hence

$$|B(e^{-t})|<\varepsilon \quad \text{for} \quad t\geqslant N+c. \tag{7}$$

Here

$$B(e^{-x})=\sum_{v=0}^{k} b_v \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(xv)^i.$$

We define the polynomial

$$P_R(x)=\sum_{v=0}^{k} b_v \sum_{i=0}^{R} \frac{(-1)^i}{i!}(xv)^i. \tag{8}$$

It follows immediately from (8) that

$$|P_R(x)|\leqslant \sum_{v=0}^{k} |b_v| \sum_{i=0}^{\infty} \frac{|x|^i v^i}{i!} = \sum_{v=0}^{k} |b_v| e^{|x|v}. \tag{9}$$

Thus we get by (2. 2)

$$\begin{aligned}\int_0^\infty H(|f(t)-P_R(t)|)\,d\alpha(t) &\leqslant K\int_0^\infty H(|f(t)-g(t)|)\,d\alpha(t)\\ &\quad + K^2\int_0^\infty H(|g(t)-B(e^{-t})|)\,d\alpha(t) + K^2\int_0^\infty H(|B(e^{-t})-P_R(t)|)\,d\alpha(t)\\ &= A_1+A_2+A_3.\end{aligned}$$

Here, first $A_1\leqslant 2K\varepsilon$ by (5). Secondly, $A_2\leqslant K^2 H(\varepsilon)\int_0^\infty d\alpha(t)$ by (6). Thirdly,

$$\begin{aligned}A_3 &\leqslant K^2\int_0^M H(|B(e^{-t})-P_R(t)|)\,d\alpha(t) + K^3\int_M^\infty H(|B(e^{-t})|)\,d\alpha(t)\\ &\quad + K^3\int_M^\infty H(|P_R(t)|)\,d\alpha(t) = E_1+E_2+E_3 \quad \text{by} \quad (2.\,2).\end{aligned}$$

We now can choose $M> N+c$ by (9), (2. 2) and (2. 3), so that $E_3<\varepsilon$ uniformly for R. Plainly $P_R(x)$ tends uniformly to $B(e^{-x})$ in every finite interval of x when $R\to\infty$. Therefore we can choose an integer R such that $E_1<\varepsilon$. Finally,

$E_2 \leqslant K^3 H(\varepsilon) \int_0^\infty d\alpha(t)$ by (7). Thus we have proved

$$\int_0^\infty H(|f(t) - P_R(t)|)\, d\alpha(t) < 2(K+1)\,\varepsilon + (K^2+K^3)\, H(\varepsilon) \int_0^\infty d\alpha(t),$$

which completes our theorem.

The conditions (2.2) and (2.3) of our theorem for example are satisfied if

$$H(t) = t^p, \quad p > 0, \qquad \alpha(t) = \sum_{n \leqq t} \frac{\lambda^n}{n!}.$$

Thus we have proved the approximation $\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |f(n) - P(n)|^p < \varepsilon$ for any sequence $f(n)$ with $\sum_{n=0} \frac{\lambda^n}{n!} |f(n)|^p < \infty$. Obviously the special case $p = 2$ is Itô's problem.

REFERENCES

[1] R. E. EDWARDS, An approximation problem proposed by K. Itô, Tôhoku Math. Journ., 11 (1959), 406-408.

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MATH. INSTITUT DER UNIVERSITÄT
GIESSEN, GERMANY.