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TORSIONLESS MODULES

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Throughout this paper each ring R will be a ring with identity element and each module will be unital. We denote the category of right (resp. left) R-modules by $\mathfrak{M}_R$ (resp. ${}_R\mathfrak{M}$).

If A is a right R-module, $A^* = \mathrm{Hom}_R(A, R)$ will be its dual, and we may have the natural R-homomorphism

$$\delta_A : A \to A^{**}.$$

Following H.Bass [3, p. 476], we call A torsionless if δ_A is a monomorphism and reflexive if δ_A is an isomorphism. If X is a subset of A (resp. A^*) we denote its annihilator in A^* (resp. A) by $l(X)$ (resp. $r(X)$). In case X is a subset of R, $l(X)$ (resp. $r(X)$) is just the left (resp. right) annihilator of X in R since $(R_R)^* = {}_RR$.

In this paper we shall be concerned with torsionless modules. In Section 1, we recall elementary basic properties of torsionless modules and discuss the following condition: *Every essential extension of a torsionless module is torsionless.* We end the section with a remark on a theorem of L. E. T.Wu., H. Y. Mochizuki and J. P. Jans [20]. In Section 2 we study S-rings (without any chain condition) and right self-cogenerator rings. We show in Proposition 3 that, if $\mathfrak{M}_R$ has a torsionless cogenerator, then R is a cogenerator in $\mathfrak{M}_R$ (compare this result with C. Faith and E. A. Walker [6, Theorem 4.1]). Our main results are obtained in Section 3. Theorem 1 states the equivalence of the following conditions:

(1) *R is an injective cogenerator in $\mathfrak{M}_R$.*
(2) *The injective hull $E(R_R)$ of R_R is torsionless and R is an S-ring.*
(3) *Each factor module of $R_R \oplus R_R$ is torsionless and R is a right S-ring.*

Note that the implication (1)$\Rightarrow$(2), (3) is due to B. L.Osofsky [16]. Finally, in Section 4, we make some remarks on duality and then give equivalent criteria for the simplicity of U^* for any simple right ideal U.

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1. Preliminaries. To begin with we shall look at characteristic properties of torsionless modules:

(1.1) *The following conditions on a right R-module A are equivalent*:
(1) *A is torsionless.*
(2) $r(A^*)=0$.
(3) *If* $0\neq a\in A$, *then there exists* $b\in A^*$ *such that* $ba\neq 0$.
(4) *A can be imbedded in a direct product of copies of* R_R.
(5) *For any nonzero map* $A_0\to A$ $(A_0\in\mathfrak{M}_R)$, *there is a map* $A\to R_R$ *such that the composition map* $A_0\to A\to R_R$ *is nonzero.*
(6) *A is a submodule of a dual module.*

(1.2) *Let A be a right R-module and* A_0 *a submodule of A. Then the following conditions are equivalent*:
(1) A/A_0 *is torsionless.*
(2) *If* $a\in A$, $a\notin A_0$, *then there exists* $b\in A^*$ *such that* $ba\neq 0$, $bA_0=0$.
(3) $r(l(A_0))=A_0$.

The equivalence of the above conditions are well known, and we shall omit the proofs (see H. Bass [3, p. 477], J. P. Jans [9, p. 68]).

If A is a right R-module, then we set

$$K(A)=\mathrm{Ker}\,\delta_A=\bigcup_{b\in A^*}\mathrm{Ker}\,b.$$

Concerning the properties of $K(A)$, we have the following

(1.3) (1) $A/K(A)$ *is torsionless.*
(2) $K(A)=A$ *if and only if* $A^*=0$.
(3) *If* A_0 *is a submodule of A such that* $K(A)\not\subset A_0$, *then* A/A_0 *is not torsionless.*

PROOF. The proof can be made directly from the definitions.

It is well known that the class of torsionless right R-modules is closed under taking submodules and direct product. In general extensions of torsionless modules may fail to be torsionless. Is it true that essential extension modules of torsionless modules are torsionless? In the following we shall give an answer to the question.

If A is a right R-module then $E(A)$ will denote the injective hull of A, and $A\ '\supset A_0$ will signify that A is an essential extension of a submodule A_0 of A (see B. Eckmann and A. Schopf [5]).

PROPOSITION 1. *The following conditions are equivalent for any ring R:*

(1) *Every essential extension of a torsionless right R-module is torsionless.*

(2) *$\mathfrak{M}_R$ has a faithful, injective, torsionless right R-module.*

(3) *$E(R_R)$ is torsionless.*

PROOF. (1) implies (2). $E(R_R)$ must be torsionless since $E(R_R)\ '\supset R_R$. Thus $E(R_R)$ is faithful, injective, and torsionless.

(2) implies (3). Let M be a faithful, injective, torsionless right R-module. Since M is faithful,

$$R_R \hookrightarrow \prod M \text{ (a direct product of copies of } M).$$

This implies that $E(R_R) \hookrightarrow \prod M$ since $\prod M$ is injective. Thus $E(R_R)$ is torsionless since $\prod M$ is torsionless by assumption.

(3) implies (1). Let A be a torsionless right R-module. We show that $E(A)$ is torsionless, which may imply (1). Since A is torsionless, using (1.1) (4), we have $A \hookrightarrow \prod R_R$. This implies

$$E(A) \hookrightarrow E\left(\prod R_R\right) \hookrightarrow \prod E(R_R).$$

Thus $E(A)$ is torsionless, since $\prod E(R_R)$ is torsionless by (3).

REMARK. L. E. T. Wu, H. Y. Mochizuki and J. P. Jans [20] have proved the following equivalence for a left Artinian ring R:

(4) *$E(R_R)$ is projective.*

(5) *Extensions of torsionless right R-modules are torsionless. Moreover if $A_0 \subset A$ (A_0, $A \in \mathfrak{M}_R$), $A_0^* \neq 0$, then $A^* \neq 0$.*

By a slight modification of their proof, we may establish the equivalence of the statements (3) and (5) above for any ring R, making use of (1.1) and (1.3).

2. S-rings and self-cogenerator rings. Following F. Kasch [11, p. 455] and K. Morita [15], we call a ring R a left S-ring if $l(I) \neq 0$ for each right ideal $I \neq R$. Similarly we may define a right S-ring. An S-ring is a left S-ring, but a right S-ring as well. The following equivalence (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) is well known (see J. P. Jans [8, Theorem 3.2] and K. Morita [15, Theorem 1]).

(2.1) *The following conditions on a ring R are equivalent*:
(1) *R is a left S-ring.*
(2) *R contains a copy of each simple right R-module.*
(3) *$A^* \neq 0$ for every finitely generated right R-module $A \neq 0$.*

PROOF. Since $(R/I)^* \approx l(I)$ for a right ideal I, the equivalence (1)$\Longleftrightarrow$(2) $\Longleftrightarrow$(3) is clear.

For a further discussion of left S-rings, we need a notion of cogenerators in $\mathfrak{M}_R$. A right R-module C is called a cogenerator in $\mathfrak{M}_R$ if, for each $A \in \mathfrak{M}_R$, A can be imbedded in a direct product of copies of C. C is a cogenerator in $\mathfrak{M}_R$ if and only if, for each simple $U \in \mathfrak{M}_R$, C contains a copy of $E(U)$ (see B. L. Osofsky [16, Lemma 1]). R is called a right self-cogenerator ring if R_R is a cogenerator in $\mathfrak{M}_R$.

PROPOSITION 2. *The following conditions are equivalent for any ring R*:
(1) *R is a left S-ring.*
(2) *$E(R_R)$ is a cogenerator in $\mathfrak{M}_R$.*
(3) *Every faithful injective right R-module is a cogenerator in $\mathfrak{M}_R$.*

PROOF. (1) implies (2). By assumption R contains a copy of each simple right R-module, so does $E(R_R)$. Hence $E(R_R)$ is a cogenerator in $\mathfrak{M}_R$ since $E(R_R)$ is injective.

(2) implies (3). Let M be a faithful injective right R-module. Then the same argument as in the proof of Proposition 1 shows that $E(R_R) \subsetneq \prod M$ by the faithfulness and the injectivity of M. Thus, by (2), $\prod M$ is a cogenerator in $\mathfrak{M}_R$, or equivalently, M is a cogenerator in $\mathfrak{M}_R$ (see K. Sugano [17, Lemma 1]).

(3) implies (1). Since $E(R_R)$ is faithful and injective, $E(R_R)$ must be a cogenerator in $\mathfrak{M}_R$ by (3). Let $U \in \mathfrak{M}_R$ be simple. We may assume $U \subset E(R_R)$ since $E(R_R)$ is a cogenerator in $\mathfrak{M}_R$. Then $U \cap R \neq 0$ since $E(R_R)' \supset R$. Thus R contains $U \cap R = U$. It follows from this and (2.1) that R is a left S-ring.

We are now in a position to characterize a ring R with the property that each right R-module is torsionless. Such a ring is indeed a right self-cogenerator ring as is easily seen. First we define a right R-module C: let $\{U_i\}$ be the family of all non-isomorphic simple right R-modules, and let C be the direct sum of the family $\{E(U_i)\}$.

PROPOSITION 3. *The following conditions are equivalent for any ring R:*

(1) *R is a cogenerator in $\mathfrak{M}_R$.*

(2) *C is torsionless.*

(3) *$E(R_R)$ is torsionless and R is a left S-ring.*

(4) *$E(R_R)$ is a torsionless cogenerator in $\mathfrak{M}_R$.*

(5) *$\mathfrak{M}_R$ has a torsionless cogenerator in $\mathfrak{M}_R$.*

PROOF. (1)$\Rightarrow$(2) is clear.

(2) implies (1). Since C is torsionless, $C \hookrightarrow \prod R_R$. Thus $\prod R_R$ is a cogenerator in $\mathfrak{M}_R$, or equivalently, R is a cogenerator in $\mathfrak{M}_R$, since C is a cogenerator in $\mathfrak{M}_R$ by its definition.

(1)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5) is clear by Proposition 2.

(5) implies (1). Let C_0 be a torsionless cogenerator in $\mathfrak{M}_R$. Since C_0 is torsionless, $C_0 \hookrightarrow \prod R_R$. Then the same argument as in the proof (2)$\Rightarrow$(1) above shows that R is a cogenerator in $\mathfrak{M}_R$.

REMARK. K. Sugano [17] has proved the equivalence of the following statements:

(1) *R is a cogenerator in $\mathfrak{M}_R$.*

(2′) *C is projective.*

(6) *Every faithful right R-module is a cogenerator in $\mathfrak{M}_R$.*

We take an interest in the equivalence (5)$\Longleftrightarrow$(6) above.

3. The main results. In this section we study rings R such that R is an injective cogenerator in $\mathfrak{M}_R$. We begin our study with the following lemma which is analogous to H. Bass [3, Theorem 5.4] or K. Morita [15, Theorem 1].

LEMMA 1. *Let R be a right S-ring. Then every finitely generated projective submodule of a torsionless right R-module is always a direct summand.*

PROOF. Let $0 \rightarrow P \rightarrow A$ be an exact sequence of right R-modules with P finitely generated projective and A torsionless. Then we have the dual exact sequence

$$A^* \rightarrow P^* \rightarrow B \rightarrow 0$$

for a suitable choice of a finitely generated left R-module B. Dualizing this sequence again, we get the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & B^* \arrow[r] & P^{**} \arrow[r] & A^{**} \\
 & 0 \arrow[r] & P \arrow[r] \arrow[u, phantom, "\delta_P \wr"'] & A \arrow[u, "\delta_A"'] \\
 & & & 0 . \arrow[u]
\end{tikzcd}$$

Thus $B^*=0$. But this implies, using the fact that B is a finitely generated left R-module and that R is a right S-ring, $B=0$ by (2.1). Hence the epimorphism $A^* \to P^* \to 0$ above splits by virtue of the projectivity of P^*. So does the first exact sequence $0 \to P \to A$, since P is reflexive and A is torsionless.

We are now ready to prove our main result.

THEOREM 1. *The following conditions on a ring R are equivalent*:

(1) *R is an injective cogenerator in $\mathfrak{M}_R$.*

(2) *$E(R_R)$ is torsionless and R is an S-ring.*

(3) *Every factor module of $R_R \oplus R_R$ is torsionless and R is a right S-ring.*

(4) *R is a cogenerator in $\mathfrak{M}_R$ and there are only finitely many non-isomorphic simple right* (*or left*) *ideals.*

PROOF. (1)$\Rightarrow$(2), (3), (4) is found in B. L. Osofsky [16].

(2) implies (1). Let $E(R_R)$ be torsionless and R an S-ring. Then R is a cogenerator in $\mathfrak{M}_R$ by Proposition 3. Next, making use of the above lemma, we conclude that R_R is a direct summand of $E(R_R)$, or equivalently, R is right self-injective since $E(R_R)\,'\supset R$.

(3) implies (1). We first show that R is right self-injective, that is, $E(R_R)=R$. Suppose on the contrary that $a \notin R$ for some $a \in E(R_R)$. Since $R+aR$ is an epimorph of $R_R \oplus R_R$, $R+aR$ is torsionless by (3). In view of the above lemma, R_R is a direct summand of $R+aR$, since $R+aR$ is torsionless and R is a right S-ring by (3). This contradicts the fact that $E(R_R)\,'\supset R+aR\,'\supset R$, $a \notin R$. Thus R is right self-injective. It follows easily that R is a left S-ring, since each cyclic right R-module (which is, in fact, an epimorph of $R_R \oplus R_R$) is torsionless by (3). Thus R is an injective cogenerator in $\mathfrak{M}_R$.

(4) implies (1). Let R be a cogenerator in $\mathfrak{M}_R$. Then R has the same number of simple left ideals as simple right ideals, up to isomorphism (see [12, Theorem 1] or K. Sugano [17, Remark 1]). Now let $\{U_1, \cdots, U_n\}$ be the full set of non-isomorphic simple right R-modules. Note that R is a cogenerator in $\mathfrak{M}_R$ if and only if each $E(U_i)$ is finitely generated projective. Therefore $C=\bigoplus_{i=1}^{n} E(U_i)$ is projective (and injective), and $E(U_1)/E(U_1)J, \cdots,$

$E(U_n)/E(U_n)J$ is a complete set of non-isomorphic simple right R-modules, where and throughout this paper J denotes the Jacobson radical of R (see C. Faith and E. A. Walker [6, p. 214] and B. L. Osofsky [16, p. 375]). Thus each simple right R-module is an epimorph of C and hence C is a generator in $\mathfrak{M}_R$ (see G. Azumaya [2, Theorem 1]). Since $\mathfrak{M}_R$ has an injective generator (indeed C), this implies the injectivity of R_R (see Y. Utumi [19, 3.3]).

REMARK. A ring R is called right PF if every faithful right R-module is a generator in $\mathfrak{M}_R$. Azumaya-Utumi's theorem states that R is right PF if and only if R is right self-injective, R/J is Artinian, and each right ideal $\neq 0$ contains a simple right ideal. The rings characterized in the preceding theorem coincide exactly with right PF-rings.

As an immediate corollary to the preceding theorem we have the following.

COROLLARY. *The following conditions on a ring R are equivalant*:

(1) *R is an injective cogenerator both in ${}_R\mathfrak{M}$ and in $\mathfrak{M}_R$.*

(2) *$E({}_RR)$ and $E(R_R)$ are torsionless and R is an S-ring.*

(3) *Every factor module of ${}_RR\oplus{}_RR$ and of $R_R\oplus R_R$ is torsionless.*

REMARK 1. Let A be a right R-module. Following J. Dieudonné [4] we say that A has perfect duality if

(i) A is reflexive,

(ii) the dual map $A^* \to A_0^*$ is an epimorphism for any submodule A_0 of A, and $A^{**} \to B_0^*$ is also an epimorphism for any submodule B_0 of A^*,

(iii) every factor module of A and A^* is torsionless.

The major unsolved problem concerning perfect duality is whether both conditions (i) and (iii) imply perfect duality. Our theorem may give an affirmative answer to this problem for a free R-module A of a rank $\geqq 2$.

REMARK 2. The author [12] has obtained a further result concerning the above equivalences, namely, as follows:

(3. 1) *The following conditions on a ring R are equivalent*:

(1) *R is an injective cogenerator both in ${}_R\mathfrak{M}$ and in $\mathfrak{M}_R$.*

(2′) *$E({}_RR)$ and $E(R_R)$ are torsionless and R is a right (or left) S-ring.*

(4) *$E(U)$ and $E(V)$ are torsionless for any simple left R-module U and any simple right right ideal V.*

REMARK 3. K. Morita [13, §2] has proved the following equivalences:

(1) *R is an injective cogenerator both in ${}_R\mathfrak{M}$ and in $\mathfrak{M}_R$.*

(5) *The class of reflexive left R-modules is closed under taking submodules and factor modules, and so is the class of reflexive right R-modules.*

(6) *Every finitely generated left, and every finitely generated right, R-module is reflexive.*

(7) *Every cyclic left, and every cyclic right, R-module is reflexive.*

4. Remarks on duality. Let us consider the following condition:

(a) *The dual of any simple right R-module is zero or simple.*

This type of duality has been studied by a number of authors, namely, G. Azumaya [1], J. Dieudonné [4], K. Morita and H. Tachikawa [14] and H. Tachikawa [18] under finiteness assumptions on a ring R. In the following we deal with the condition (a) for general rings R.

LEMMA 2. *The following conditions on a ring R are equivalent*:

(1) *R satisfies* (a).

(2) *If aR, $a\in R$, is simple then $l(r(a))=Ra$.*

(3) $\mathrm{Ext}^1_R(R/U, R)=0$ *for each simple right ideal U.*

PROOF. (1) implies (2). Let R satisfy (a) and aR, $a\in R$, be simple. Then

$$l(r(a))\approx(R/r(a))^*\approx(aR)^*\neq 0$$

is simple by (a). Thus $l(r(a))=Ra$.

(2) implies (1). Assume (2) and let U be a simple right R-module such that $U^*\neq 0$. Then $U\approx aR$ for some $a\in R$. Now $l(r(a))$ is simple. In fact, let $l(r(a))\ni b\neq 0$. Then $r(a)=r(b)$ since $r(a)$ is maximal. Hence $bR\approx R/r(b)$ $=R/r(a)=aR$ is simple, and $l(r(b))=Rb$ by (2). Thus $l(r(a))=l(r(b))=Rb$, and hence $l(r(a))$ is simple. Since $U^*\approx(aR)^*\approx(R/r(a))^*\approx l(r(a))$, U^* is simple.

(2) implies (3). Note that $\mathrm{Ext}^1_R(R/U, R)=0$ if and only if each homomorphism from U to R may be given by the left multiplication of an element of R, where U is a right ideal of R. Now let aR, $a\in R$, be simple and let f be a map from aR into R_R. Clearly $r(a)\subset r(fa)$. Hence, by (2), $Ra=l(r(a))$ $\supset l(r(fa))\ni fa$. Thus f may be given by the left multiplication of an element of R and hence $\mathrm{Ext}^1_R(R/aR, R)=0$.

(3) implies (2). Assume (3) and let aR, $a\in R$, be simple. Let $b\in l(r(a))$. Then $r(b)\supset r(a)$ and hence the mapping $ar\to br$, $r\in R$, from aR into R is well defined. By assumption, this map may be given by the left multiplication of an element of R and hence $b\in Ra$. This implies $l(r(a))=Ra$.

REMARK. If U is a simple right ideal, then either $U^2=0$ or $U=eR$, $e=e^2\in R$ (see N. Jacobson [10, Proposition 1, p. 57]). Hence the condition (3) is equivalent to the following condition:

(3′) $\mathrm{Ext}^1_R(R/U, R)=0$ for each nilpotent simple right ideal U.

By the proof of the above lemma, if R satisfies one of the equivalent conditions in the above lemma, the simplicity of aR, $a\in R$, implies that of Ra, consequently, the left socle of R contains the right socle (see N. Jacobson [10, p.65]).

In what follows, we denote by $[A]$ the composition length of a (left or right) R-module A when A has a composition series. Now, let us assume (a), and let A be a right R-module of finite length. Then, using the induction on the length $[A]$ of A, we see easily that $[A^*]\leqq[A]$. We shall use this fact to show the following which is essentially the same as J. Dieudonné [4, (3.4)] or as K. Morita and H. Tachikawa [14, Theorem 1.1].

PROPOSITION 4. *Let R satisfy the condition* (a) *and its left analogue. Let A be a torsionless right R-module of finite length and let A_0 be any submodule of A. Then A/A_0 is always reflexive.*

PROOF. By the fact mentioned above, $[A^{**}]\leqq[A^*]\leqq[A]$. But, since A is torsionless, $[A]\leqq[A^{**}]$. Thus $[A^{**}]=[A^*]=[A]$, and A is reflexive. Also, A_0 is reflexive since A_0 is a torsionless module of finite length. Moreover the dual sequence

$$0\longrightarrow(A/A_0)^*\longrightarrow A^*\longrightarrow A_0^*\longrightarrow 0$$

is exact by virtue of $[A^*]=[A]$. Dualizing this sequence we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_0^{**} & \longrightarrow & A^{**} & \longrightarrow & (A/A_0)^{**} & \longrightarrow & 0\\
 & & \wr\wr & & \wr\wr & & \uparrow{\scriptstyle \delta_{A/A_0}} & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & A/A_0 & \longrightarrow & 0
\end{array}$$

by the same argument as above. Thus A/A_0 is reflexive.

REMARK. Let the assumptions and the notations be as in the preceding proposition. Then A has perfect duality in the sense of J.Dieudonné [4] by the above proof.

If we combine Lemma 2 with Proposition 4, we can easily deduce, in view of Remark 3, a result of M. Ikeda [7, Proposition 3].

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