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CONVERGENCE AND INVERSION RESULTS FOR A CLASS OF CONVOLUTION TRANSFORMS

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1. Introduction. In former papers [1] and [2] we introduced a generalization of a class of transforms treated mainly by Y. Tanno [4], [5] and [6]. The transforms are related to the meromorphic functions

$$F(s) = \left\{ \prod_{k=1}^{\infty} (1-s/a_k)e^{s/a_k} \Big/ \prod_{k=1}^{\infty} (1-s/c_k)\, e^{s/c_k} \right\} \tag{1.1}$$

where a_k and c_k are real and $0 \leqq a_k/c_k < 1$ and $\Sigma a_k^{-2} < \infty$.

Our investigation of $F(s)$ and $H(t)$ where $H(t)$ is defined by

$$[F(iy)]^{-1} = \int_{-\infty}^{\infty} e^{-iyt}\, dH(t) \tag{1.2}$$

was facilitated by the use of the number $N \equiv N_+ + N_-$ (see [1]).

In this paper we shall investigate the convergence of the transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t) \tag{1.3}$$

where $\alpha(t) \in B.V.(A, B)$ for all A, B satisfying $-\infty < A < B < \infty$, and $G(t) = H'(t)$, or

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt\,, \tag{1.4}$$

where $\varphi(t)$ is locally Lebesgue integrable. We shall use the notations we used in [1] and [2] without introducing them again.

We shall prove

$$\lim_{m\to\infty} R_m(D) f(x) = \varphi(x) \tag{1.5}$$

where

$$R_m(D) = e^{-b_m D} \prod_{k=1}^{m} \{(1-D/a_k)e^{a_k^{-1}D}/(1-D/c_k)e^{c_k^{-1}D}\} ;$$

$\lim_{m\to\infty} b_m = 0$ and

$$(1.7) \qquad \left(1 - \frac{D}{c}\right)^{-1} f(x) = \begin{cases} c\, e^{cx} \displaystyle\int_x^\infty e^{-cy} f(y)\, dy & \text{for } c > 0 \\ -c\, e^{cx} \displaystyle\int_{-\infty}^x e^{-cy} f(y)\, dy & \text{for } c < 0 . \end{cases}$$

We shall find a similar inversion formula for

$$(1.8) \qquad f(x) = \int_{-\infty}^\infty e^{ct}\, G(x-t)\, d\alpha(t) .$$

In section 9 we shall treat the problem when $N_+ + N_- < 3$ (i.e., equal to 1 or 2) in which cases further assumptions have to be made on $\varphi(t)$.

2. Convergence of the transform. In this section we shall treat the convergence of the transform when almost no condition is made on the determining function $\alpha(t)$.

THEOREM 2.1. *Suppose*:

(1) $N_+ + N_- \geqq 3$ *and* $G(t) \in$ *class* I.
(2) $\alpha(t) \in B.V.$ *in any finite interval.*
(3) $\int_{-\infty}^\infty G(x_0-t)\, d\alpha(t)$ *converges conditionally.*

Then $\int_{-\infty}^\infty G(x-t)\, d\alpha(t)$ *converges* (*conditionally*) *uniformly in every finite interval.*

PROOF. By Theorem 6.1 of [2] $H'(t) = G(t) > 0$ and therefore $(G(x-t)/G(x_0-t)) > 0$. Using Theorem 4.1 of [2] we have for $N_+ + N_- \geqq 2$

$$(2.1) \qquad \lim_{t\to\infty} \{G(x-t)/G(x_0-t)\} = e^{\alpha_?(x-x_0)}$$

and

$$\lim_{t\to-\infty} \{G(x-t)/G(x_0-t)\} = e^{\alpha_1(x-x_0)}. \tag{2.2}$$

The estimations (2.1) and (2.2) imply for x in any finite interval

$$0 < G(x-t)/G(x_0-t) < K. \tag{2.3}$$

Using Theorem 4.1 of [2] we have when $N_+ + N_- \geqq 3$

$$\frac{d}{dt}\{G(x-t)/G(x_0-t)\} = O\left(\frac{1}{t^2}\right), \quad |t| \to \infty. \tag{2.4}$$

(In fact if $\mu_1 = \mu_2 = 1$, $\frac{d}{dt}\{G(x-t)/G(x_0-t)\} = O(\exp[-M|t|])$ for $|t| \to \infty$ for some $M > 0$).

We can write now

$$\begin{aligned}\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t) &= \int_{-\infty}^{\infty} (G(x-t)/(G(x_0-t))\,G(x_0-t)\,d\alpha(t) \\ &= -\int_{-\infty}^{\infty} \frac{d}{dt}\{G(x-t)/G(x_0-t)\}\left(\int_{-\infty}^{t} G(x_0-v)\,d\alpha(v)\right)dt \\ &\quad + e^{\alpha_2(x-x_0)}\int_{-\infty}^{\infty} G(x_0-t)\,d\alpha(t).\end{aligned}$$

Since $\int_{-\infty}^{t} G(x_0-t)\,d\alpha(t)$ is bounded (2.4) yields the proof of our theorem.

Q.E.D.

THEOREM 2.2. *Suppose*:

(1) $G(t) \in$ *class* B *and* $G(t) \in$ *class* II.
(2) $\alpha(t) \in B.V.$ *in every finite interval.*
(3) $\int_{-\infty}^{\infty} G(x_0-t)\,d\alpha(t)$ *converges conditionally.*

Then $\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)$ *converges conditionally uniformly in any finite interval* $[x_1, x_2]$ *satisfying* $x_0 < x_1 < x_2 < \infty$.

PROOF. By Theorem 6.8 of [2] $G(t) > 0$ and therefore $0 < G(x-t)/G(x_0-t)$. Arguments similar to that of Theorem 2.1 yield the uniform convergence of

$\int_{-M}^{\infty} G(x-t)\,d\alpha(t)$ for x in any finite interval and any finite M. By Theorem 7.2 of [2] we have

$$\text{(2.5)} \qquad \frac{d}{dt}\log(G(x-t)/G(x_0-t)) = L(x-t+o(1)) - L(x_0-t+o(1)), \qquad t \to -\infty.$$

Since $x > x_0$ and $L(t)$ is monotonic increasing for $t \geqq t_0$ $\log(G(x-t)/G(x_0-t))$ is also monotonic increasing for $t \leqq -M$ (for some M) and therefore so is $G(x-t)/G(x_0-t)$.

By the mean value theorem for integrals

$$\int_{-\infty}^{-M} G(x-t)\,d\alpha(t) = \int_{-\infty}^{-M} (G(x-t)/G(x_0-t))\,d\left\{\int_{-\infty}^{t} G(x_0-v)\,d\alpha(v)\right\}$$

$$= (G(x+M)/G(x_0+M)) \int_{\xi}^{-M} G(x_0-v)\,d\alpha(v) \text{ for some } \xi < -M.$$

Q.E.D.

THEOREM 2.3. *Suppose*:

(1) $G(t) \in$ *class* III *and* $N_+ \geqq 3$.

(2) $\alpha(t) \in B.V.$ *in any finite interval* $[t_1, t_2]$ *satisfying* $T \leqq t_1 < t_2 < \infty$.

(3) $\int_{-\infty}^{\infty} G(x_0-t)\,d\alpha(t)$ *converges conditionally for some* $x_0, x_0 > T + \sum_{k=1}^{\infty}(a_k^{-1} - c_k^{-1})$.

Then $\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)$ *converges uniformly in the interval* $[x_1, x_2]$, *for* $T + \sum_{k=1}^{\infty}(a_k^{-1} - c_k^{-1}) < x_1 < x_2 < \infty$.

PROOF. We may write, since $G(t) = 0$ for $t > \sum_{k=1}^{\infty}(a_k^{-1} - c_k^{-1})$ which is easily deduced by the method of Theorem 6.8 of [2],

$$\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t) = \left\{\int_{x-\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})}^{M} + \int_{M}^{\infty}\right\} G(x-t)\,d\alpha(t) \equiv J_1(x) + J_2(x).$$

For M large enough the estimation of $J_2(x)$ is like that of Theorem 2.1.

Since $G(x-t)$ is bounded and $\alpha(t)$ for $x>T+\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ is of bounded variation in $\left[x-\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1}), M\right]$ $J_1(x)$ convergs. Q.E.D.

3. Convergence of the transform (Continued). In this section we shall discuss transforms for the convergence of which we have to make assumptions on the determining function.

THEOREM 3.1. *Suppose*:

(1) $G(t) \in$ *class* II, $N_+ \geqq 3$.
(2) $\alpha(t) \in B.V.$ *in any finite interval.*
(3) $\int_{-\infty}^{\infty} G(x_0-t)\, d\alpha(t)$ *converges.*
(4) $\alpha(t)=o(e^{kt})$, $t\to-\infty$ *for some negative* k.

Then $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ *converges uniformly in* x *for any finite interval.*

PROOF. Condition (4) is used together with the asymptotic estimate of $G(t)$ to show convergence of $\int_{-\infty}^{0} G(x-t)\, d\alpha(t)$. That $\int_{0}^{\infty} G(x-t)\, d\alpha(t)$ converges follows similarly to the proofs of section 2. Q.E.D.

THEOREM 3.2. *Suppose*:

(1) $N_+ + N_- \geqq 2$.
(2) $\varphi(t) \in L_1$ *in any finite interval if either* $G(t) \in$ *class* I *or* $G(t) \in$ *class* II. $\varphi(t) \in L_1[t_1, t_2]$ *for* t_1, t_2 *satisfying* $T \leqq t_1 < t_2 < \infty$ *if* $G(t) \in$ *class* III.
(3) *For some* $\varepsilon>0$ $|\varphi(t)| \leqq K\exp[(\alpha_2-\varepsilon)t]$ *for* $t>0$; *for* $t<0$ $|\varphi(t)| \leqq K\exp[(\alpha_1+\varepsilon)t]$ *when* $G(t) \in$ *class* I *and* $|\varphi(t)| \leqq K\exp[-Mt]$ *for some* $M>0$ *when* $G(t) \in$ *class* II *or* III.

Then $\int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\, dt$ *converges uniformly in any finite interval when* $G(t) \in$ *class* I *or* II *and in any finite interval* $[x_1, x_2]$, $x_1 > T+\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ *when* $G(t) \in$ *class* III.

PROOF. The proof is an immediate result of Theorem 4.1 of [2] and simple considerations used also in the former section.

4. Asymptotic properties of the determining and generating function. For the convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)\,, \tag{4.1}$$

$f(x)$ and $\alpha(t)$ are called generating and determining functions respectively.

LEMMA 4.1. *If for some $G(t)$, $N_+ + N_- \geqq 3$ and $\alpha_1 \neq \infty$, then there exists a constant $M > 0$ such that for $u > M$ and any $a > 0$*

$$\left| \frac{d}{du} \frac{G(u+a)}{G(u)} \right| \leqq \frac{K}{u^2} \tag{4.2}$$

and

$$\left| \frac{d}{du} \frac{G(u)}{G(u+a)} \right| \leqq \frac{K}{u^2} e^{-\alpha_1 a} \tag{4.3}$$

where K is independent of a.

PROOF. The proof is mainly computational making use of the following estimates for $u > M$

$$|G(u) - (p(u)e^{\alpha_1 u})| \leqq Ke^{ku} \quad \text{and} \quad |G'(u) - (p(u)e^{\alpha_1 u})'| \leqq Ke^{ku} \tag{4.4}$$

where $k < \alpha_1$. Q.E.D.

Using the above result for $G^*(u) = G(-u)$ we obtain:

LEMMA 4.2. *If for some $G(t)$, $N_+ + N_- \geqq 3$ and $\alpha_2 \neq \infty$ then there exists a constant $M > 0$ such that for $u > M$ and any $a > 0$*

$$\left| \frac{d}{du} \frac{G(-u-a)}{G(-u)} \right| \leqq \frac{K}{u^2} \tag{4.5}$$

and

$$\left| \frac{d}{du} \frac{G(-u)}{G(-u-a)} \right| \leqq \frac{K}{u^2} e^{\alpha_2 a} \tag{4.6}$$

where K does not depend on a.

THEOREM 4.3. *If* $G(t) \in$ *class* I, $\alpha(t) \in B.V.$ *in every finite interval and* (4.1) *converges, then*

(a) $\alpha(t) = o(|t|^{-\mu_1} e^{\alpha_1 t})$, $t \to -\infty$. (b) $\alpha(t) = o(t^{-\mu_2} e^{\alpha_2 t})$, $t \to \infty$.

(c) $f(x) = O(e^{\alpha_1 x})$, $x \to -\infty$. (d) $f(x) = O(e^{\alpha_2 x})$, $x \to \infty$.

PROOF. Since $N_+ + N_- \geqq 3$ we have $G(t) \in C^1(-\infty, \infty)$ and

$$G'(t) = [e^{\alpha_1 t} p(t)]' + o(e^{kt}), \quad t \to \infty \quad \text{where } k < \alpha_1$$

and $p(t)$ a polynomial of order μ_1 (see Theorem 4.1 of [2]). The coefficient of t^{μ_1} in $p(t)$ is positive; otherwise $G(t) > 0$ would not be satisfied. Therefore $\alpha_1 p(t) - p'(t) < 0$ for $t > M$ which implies $G'(t) < 0$ there. Having $G'(t) < 0$ for $t > M$ implies that $G(t)$ is monotonically decreasing there. Using Lemma 2.1 c of [3, p. 121] we obtain (a). Similarly we derive (b).

We shall prove (d) now, (c) can be proved similarly. We have

$$f(x) = \left\{ \int_{-\infty}^{0} + \int_{0}^{x+A} + \int_{x+A}^{\infty} \right\} G(x-t)\, d\alpha(t) \equiv I_1(x) + I_2(x, A) + I_3(x, A),$$

$x > 0$ and we shall choose A later ($A > 0$).

$$|I_2(x, A)| \leqq G(x)|\alpha(0)| + G(-A)|\alpha(x+A)| + \left| \int_{0}^{x+A} G'(x-t)\, \alpha(t)\, dt \right|.$$

By Theorem 4.1 of [2] both $G(t)$ and $G'(t)$ are $o(1)$ as $|t| \to \infty$ and therefore $G(t) < K_1$ and $|G'(t)| < K_1$ for some $K_1 > 0$. This implies

$$|I_2(x, A)| \leqq M_1 e^{\alpha_2(x+A)}, \tag{4.7}$$

$$|I_3(x, A)| \leqq \left| \int_{x+A}^{\infty} \frac{d}{dt} (G(x-t)/G(x_0-t)) \cdot \left(\int_{t}^{\infty} G(x_0-u)\, d\alpha(u) \right) dt \right|$$

$$+ (G(-A)/G(x_0-x-A)) \left| \int_{x+A}^{\infty} G(x_0-t)\, d\alpha(t) \right|.$$

Substituting in Lemma 4.2 $x - x_0 = a > 0$ and choosing $A > M$ where M is of Lemma 4.2 we obtain for $x - t \equiv -u \leqq -A$

$$\frac{d}{dt} (G(x-t)/G(x_0-t)) \leqq \frac{K}{(x-t)^2} e^{\alpha_2(x-x_0)}.$$

We have also $(G(-A)/G(x_0-x-A)) \leqq K\exp[\alpha_2(x-x_0)]$ and therefore

$$I_3(x, A)| \leqq M_2 e^{\alpha_2(x-x_0)} . \tag{4.8}$$

$$|I_1(x)| \leqq \frac{G(x)}{G(x_0)} \left| \int_{-\infty}^{0} G(x_0-t)\, d\alpha(t) \right|$$

$$+ \int_{-\infty}^{0} \left| \frac{d}{dt}(G(x-t)/G(x_0-t)) \right| \cdot \left| \int_{-\infty}^{t} G(x_0-v)\, d\alpha(v) \right| dt .$$

We choose now x_0 such that $x > x_0 > M$ (M of Lemma 4.1). Substituting $a=x-x_0$ in estimation (4. 2) of Lemma 4.1 we obtain

$$\left| \frac{d}{dt}(G(x-t)/G(x_0-t)) \right| \leqq K/(x_0-t)^2 \quad \text{and} \quad |G(x)/G(x_0)| \leqq K .$$

Therefore

$$|I_1(x)| \leqq M_3 . \tag{4.9}$$

Combining (4. 7), (4. 8) and (4. 9) we conclude the proof of this theorem.

THEOREM 4.4. *Let* $G(t) \in B$, $\sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1}) = \infty$, $\alpha(t) \in B.V.$ *in any finite interval and* (4. 1) *converges at* $x = x_0$; *then* (b) *and* (d) *of Theorem* 4.3 *are valid.*

THEOREM 4.5. *Let* $G(t) \in$ *class* III, $\alpha(t) \in B.V.$ *in any interval* $[t_1, t_2]$ *satisfying* $T < t_1 \leqq t \leqq t_2 < \infty$ *and let* (4. 1) *converge for* $x > T + \Sigma(a_k^{-1} - c_k^{-1})$ *then* (b) *and* (d) *of Theorem* 4.3 *are valid.*

PROOF OF THEOREMS 4.4 AND 4.5. Similar to that of Theorem 4.3 but using the methods of Theorems 2.2 and 2.3 to estimate $I_1(x)$.

5. Definition and some basic properties of $H_m(t)$ and $G_m(t)$.

Let $\{b_m\}$ be a sequence of real numbers satisfying $\lim_{m\to\infty} b_m = 0$.

Define $R_m(s)$ by

$$R_m(s) = e^{-b_m s} \prod_{k=1}^{m} [(1-s/a_k)\, e^{s/a_k}/(1-s/c_k)\, e^{s/c_k}]. \tag{5.1}$$

Define $F_m(s)$ by

(5.2) $$F_m(s) = F(s)/R_m(s) = e^{b_m s} \prod_{m+1}^{\infty} [(1-s/a_k)\, e^{s/a_k}/(1-s/c_k)\, e^{s/c_k}] .$$

$\exp[-b_m s] F(s) \in$ class F defined in [2].

We can introduce $N_+(m)$ and $N_-(m)$

(5.3) $$N_\pm(m) = \limsup_{\pm x \to \infty} \{N(\{a_k\}_{k=m+1}^{\infty}, x)\} ,$$

that is N_+ and N_- for the sequences $\{a_k\}_{k=m+1}^{\infty}$ and $\{c_k\}_{k=m+1}^{\infty}$.

If $N_+(m)+N_-(m) \geqq 1$ we have

(5.4) $$H_m(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{e^{st}}{sF_m(s)}\, ds$$

and when $N_+(m) + N_-(m) \geqq 2$

(5.5) $$G_m(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{F_m(s)}\, ds .$$

We shall define $M_n(m)$ by

(5.6) $$M_1(m) = \int_{-\infty}^{\infty} t\, dH_m(t)$$

and

(5.7) $$M_n(m) = \int_{-\infty}^{\infty} (t - M_1(m))^n\, dH_m(t)$$

when these integrals converge. Using (2.9) of [2] and the proof of Theorem 2.6 of [2], (5.6) and (5.7) converge when $N_+(m) + N_-(m) \geqq 1$. Obviously

(5.8) $$M_1(m) = b_m .$$

THEOREM 5.1. *Let* $N_+(m) + N_-(m) \geqq 1$ *for* $m > 0$, *then*

(5.9) $$H_m(t_0) \leqq (t_0 - b_m)^{-2} M_{2n}(m) , \qquad t_0 < b_m ,$$

and

(5.10) $$1 - H_m(t_0) \leqq (t_0 - b_m)^{-2} M_{2n}(m) \quad \textit{for} \quad t_0 > b_m .$$

PROOF. $H_m(t)$ is a distribution function and therefore

$$H_m(t_0) = \int_{-\infty}^{t_0} dH_m(t) = \int_{-\infty}^{t_0-b_m} dH_m(t+b_m)$$

$$\leqq \int_{|t|\geqq|t_0-b_m|} dH_m(t+b_m) \leqq (t_0-b_m)^{-2n} \int_{-\infty}^{\infty} t^{2n}\, dH(t+b_m)$$

$$= (t_0-b_m)^{-2m} M_{2n}(m).$$

(5.10) is proved similarly.

LEMMA 5.2. *If* $N_+(m) + N_-(m) \geqq 1$, *then*

$$M_2(m) \equiv S_m^2 = \sum_{k=m+1}^{\infty} (a_k^{-2} - c_k^{-2}). \tag{5.11}$$

PROOF. It is easily seen that

$$M_2(m) = \frac{d^2}{ds^2}[e^{b_m s}(F_m(s))^{-1}]_{s=0} = \sum_{k=m+1}^{\infty} (a_k^{-2} - c_k^{-2}).$$

Q.E.D.

To state and prove the following theorem that establishes the connection between $G(t)$ and $G_m(t)$ we have to introduce the following operator (see Tanno [3] and [4])

$$R_m(D) = e^{b_m D} \prod_{k=1}^{m} (1-D/a_k)\, e^{D/a_k}/(1-D/c_k)\, e^{D/c_k}$$

where $e^{kD}f(x) = f(x+k)$,

$$[(1-D/a)/(1-D/c)]\, f(x) \equiv \frac{c}{a} f(x) + \frac{c(a-c)}{a} e^{cx} \int_x^{\infty} e^{-cy} f(y)\, dy \tag{5.12}$$

for $0 < a < c < \infty$,

$$[(1-D/a)/(1-D/c)]\, f(x) \equiv \frac{c}{a} f(x) - \frac{c(a-c)}{a} e^{cx} \int_{-\infty}^{x} e^{-cy} f(y)\, dy \tag{5.13}$$

for $-\infty < c < a < 0$, and for $c = \pm\infty$

$$[(1-D/a)/(1-D/c)]\, f(x) \equiv (1-D/a) f(x) = f(x) - a^{-1} f'(x).$$

THEOREM 5.3. *If* $N_+(m) + N_-(m) \geqq 2$ *then*

$$R_m(D)\,G(x) = G_m(x)\,. \tag{5.14}$$

PROOF. It is easy to see that $R_m(D)\exp[st] = R_m(s)\exp[sx]$ for $\gamma_1 < \mathrm{Re}\, s < \gamma_2$, (where $\gamma_1 = \max_{c_k<0}(c_k, -\infty)$ and $\gamma_2 = \min_{c_k>0}(c_k, \infty)$). Formally we have

$$R_m(D)\,G(x) = R_m(D)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{sx}ds}{F(s)}$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{R_m(D)e^{sx}}{F(s)}\,ds = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{sx}}{F_m(s)} = G_m(x)\,.$$

We have only to justify interchanging $R_m(D)$ and the integral sign. We have to justify that for $c > 0$

$$\int_x^\infty e^{-cy}dy\int_{-i\infty}^{i\infty}\frac{e^{sy}}{F(s)}\,ds = \int_{-i\infty}^{i\infty}\frac{ds}{F(s)}\int_x^\infty e^{-cy}e^{sy}\,dy \tag{5.15}$$

which is true by Fubini's Theorem since $N_+ + N_- \geqq 2$.

Similarly we can show when $c < 0$ the analogous formula to (5.15) holds. We may repeat the procedure since $N_+(m) + N_-(m) \geqq 2$. Terms like $\exp[kD]$ and $(1-D/a)$ can enter under the integral sign (the latter using $N_+(m)+N_-(m) \geqq 2$) as was done for convolution of variation diminishing kernels. Q.E.D.

REMARK 5.3. The operators $[(1-D/a)/(1-D/c)]$ defined in (5.12) and (5.13) can be obtained from (1.7) on which we operate with $(1-a^{-1}D)$ where $D \equiv d/dx$.

REMARK 5.4. The ordering in the sequences $\{a_k\}$ and $\{c_k\}$ make no difference to $F(s)$ but different orderings, even though keeping $0 \leqq a_k/c_k < 1$, yield different sequences of $F_m(s)$ and $G_m(t)$ and for different orderings $N_+(m) + N_-(m)$ can be different numbers. It is not hard to show that if the number of positive (negative) a_k's is either zero or infinity we can arrange $\{a_k\}$ and $\{c_k\}$ so that $N_+(m) = N_+$ ($N_-(m) = N_-$). In fact, any order for which the subsequences of positive (negative) a_k's or c_k's are ordered by their magnitude yields the above mensioned result. When there are positive (negative) a_k's in finite number which is the same as the number of positive (negative) finite c_k's, again we can have $N_+(m) = N_+$ ($N_-(m) = N_-$). We can also maintain $N_+(m) + N_-(m) = N_+ + N_-$ if $N_+ + N_- = \infty$. This leaves the case $N_+ + N_- = n$ and either $N_+ > 0$ and the number of positive a_k's is finite or $N_- > 0$ and the number of negative a_k's is finite.

REMARK 5.5. In section 9 we shall overcome the difficulty of having $N_+ + N_- \geqq 3$ but $N_+(m) + N_-(m) < 3$.

6 .Operating with $R_m(D)$ on $f(x)$. In this section we shall operate with $R_m(D)$ on $f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ when $G(t)$, the kernel function, satisfies various conditions.

THEOREM 6.1. *Suppose*:

(1) $G(t) \in$ *class* I, $N_+ + N_- \geqq 3$.
(2) $\alpha(t) \in B.V.$ *in any finite interval.*
(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ *converges.*
(4) $N_+(m) + N_-(m) \geqq 3$.

Then

$$R_m(D) f(x) = \int_{-\infty}^{\infty} G_m(x-t)\, d\alpha(t)$$

converges.

For the proof we shall need the following Lemma.

LEMMA 6.2. *Let assumptions* (1), (2) *and* (3) *of Theorem* 6.1 *be satisfied*, $c \notin [\alpha_1, \alpha_2]$ *and* $G_1(t) \equiv (1 - c^{-1}D)^{-1} G(t)$, *then* $\int_{-\infty}^{\infty} G_1(x-t) d\alpha(t)$ *converges.*

PROOF. The meromorphic function $F_1(s)$ corresponding to $G_1(t)$ has the same zeros as $F(s)$ (corresponding to $G(t)$) and it may have, at most, one more zero which is $c, c \notin [\alpha_1, \alpha_2]$. By Theorem 4.1 of [2] we can obtain

$$\frac{d}{dt}(G_1(x-t)/G(x-t)) = O\left(\frac{1}{|t|^2}\right), \quad |t| \to \infty .$$

Therefore the proof of the convergence of $\int_{-\infty}^{\infty} G_1(x-t) d\alpha(t)$ is immediate.

Q.E.D.

PROOF OF THEOREM 6.1. Formally we write

$$R_m(D) f(x) = R_m(D) \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$$

$$= \int_{-\infty}^{\infty} R_m(D)\, G(x-t)\, d\alpha(t) = \int_{-\infty}^{\infty} G_m(x-t)\, d\alpha(t).$$

To justify the interchance of $R_m(D)$ and the integral it is enough to justify it for one term. For $0 < a < c < \infty$ $(c > \alpha_2)$

(6.1) $$[(1-a^{-1}D)/(1-c^{-1}D)] f(x) = \frac{c}{a} f(x) + \frac{c(a-c)}{a} e^{cx} \int_x^{\infty} f(y)\, e^{-cy}\, dy$$

$$= \frac{c}{a} \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t) + \frac{c(a-c)}{a} e^{cx} \int_x^{\infty} \left\{ \int_{-\infty}^{\infty} G(y-t)\, d\alpha(t) \right\} e^{-cy} dy.$$

We have to show that

(6.2) $$\left(1-\frac{D}{a}\right)\left(1-\frac{D}{c}\right)^{-1} f(x)$$

$$= \int_{-\infty}^{\infty} \left[\frac{c}{a} G(x-t) + \frac{c(a-c)}{a} e^{cx} \int_x^{\infty} G(y-t)\, e^{-cy}\, dy \right] d\alpha(t).$$

After simplification we have to show

$$ce^{cx} \int_x^{\infty} \left\{ \int_{-\infty}^{\infty} G(y-t)\, d\alpha(t) \right\} e^{-cy}\, dy = ce^{cx} \int_{-\infty}^{\infty} \left\{ \int_x^{\infty} G(y-t)\, e^{-cy}\, dy \right\} d\alpha(t)$$

(both exist and are equal). We shall show that for an arbitrary positive number ε there exists $A, A > x$ such that

(a) $$I(A) \equiv ce^{cx} \int_A^{\infty} \left\{ \int_{-\infty}^{\infty} G(y-t)\, d\alpha(t) \right\} e^{-cy}\, dy < \varepsilon,$$

(b) $$J(A) \equiv ce^{cx} \int_{-\infty}^{\infty} \left\{ \int_A^{\infty} G(y-t)\, e^{-cy}\, dy \right\} d\alpha(t) < \varepsilon.$$

We recall by Theorem 4.3 that

$$\left| \int_{-\infty}^{\infty} G(y-t)\, d\alpha(t) \right| \leq Ke^{\alpha_2 y}$$

for $y > 0$ and therefore for big enough A,

$$I(A) \leqq ce^{cx} \int_A^\infty K e^{\alpha_2 y} e^{-cy} \, dy = K c e^{cx} e^{(\alpha_2 - c)A} (c-\alpha_2)^{-1} < \varepsilon .$$

We have also

$$ce^{cx} \int_x^\infty G(y-t) \, e^{-cy} \, dy \equiv G_1(x-t)$$

where

$$F_1(s) = F(s)(1-s/c) ,$$

substituting this we obtain

$$J(A) = e^{c(x-A)} \int_{-\infty}^\infty G_1(A-t) \, d\alpha(t)$$

which converges by Lemma 6.2. By Theorem 4.3

$$\int_{-\infty}^\infty G_1(A-t) \, d\alpha(t) \leqq K e^{\alpha_2 A}$$

and therefore for big enough A, $J(A) \leqq \varepsilon$.

It is enough to show now that for finite A

$$ce^{cx} \int_x^A \left\{ \int_{-\infty}^\infty G(y-t) \, d\alpha(t) \right\} e^{-cy} \, dy = ce^{cx} \int_{-\infty}^\infty \left\{ \int_x^A G(y-t) \, e^{-cy} \, dy \right\} d\alpha(t) .$$

One can choose B and C large enough such that

$$\left| \left\{ \int_{-\infty}^{-B} + \int_C^\infty \right\} G(y-t) \, d\alpha(t) \right| < \varepsilon \qquad \text{uniformly for} \quad y \in (x, A)$$

and therefore

$$ce^{cx} \left| \int_x^A \left[\left\{ \int_{-\infty}^{-B} + \int_C^\infty \right\} G(y-t) \, d\alpha(t) \right] e^{-cy} \, dy \right| < \varepsilon .$$

By Theorem 2.1 we can choose B and C such that

$$\left|\left\{\int_{-\infty}^{-B} + \int_{C}^{\infty}\right\} G_1(\xi-t)\, d\alpha(t)\right| < \varepsilon/2 \qquad \text{uniformly for} \quad \xi \in [x, A]$$

and therefore

$$\left| ce^{cx}\left\{\int_{-\infty}^{-B} + \int_{C}^{\infty}\right\}\left[\int_{x}^{A} G(y-t)\, e^{-cy}\, dy\right] d\alpha(t)\right|$$

$$= \left|\left\{\int_{-\infty}^{-B} + \int_{C}^{\infty}\right\}\left[G_1(x-t) - e^{c(x-A)}\, G_1(A-t)\right] d\alpha(t)\right|$$

$$\leqq \frac{\varepsilon}{2} + e^{c(x-A)}\,\frac{\varepsilon}{2} < \varepsilon\,.$$

By Fubini's Theorem

$$ce^{cx}\int_{x}^{A}\left\{\int_{-B}^{C} G(y-t)\, d\alpha(t)\right\} e^{-cy}\, dy = ce^{cx}\int_{-B}^{C}\left\{\int_{x}^{A} G(y-t)\, e^{-cy}\, dy\right\} d\alpha(t)$$

and therefore (6. 2) is proved. Q.E.D.

THEOREM 6.3. *Suppose*:

(1) $G(t) \in$ *class* B *and also* $G(t) \in$ *class* II.
(2) $\alpha(t) \in B.V.$ *in any finite interval.*
(3) $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ *converges for* $x > \gamma$.

Then

$$R_m(D) f(x) = \int_{-\infty}^{\infty} G_m(x-t)\, d\alpha(t)\,.$$

For the proof we shall need the following Lemma.

LEMMA 6.4. *Let assumptions* (1), (2) *and* (3) *of Theorem* 6.3 *be satisfied then*

$$\int_{-\infty}^{\infty} G_1^*(x-t)\, d\alpha(t) \quad \textit{converges for } x > \gamma$$

where

$$G_1^*(t) \equiv \left(1-\frac{D}{c}\right)^{-1} G(t) \quad \textit{and} \quad c \in \{c_k\}$$

PROOF. We can show that $\int_A^\infty G_1^*(x-t)\,d\alpha(t)$ converges by a method similar to that used in the proof of Lemma 6.2. Choose B so large that $t > B + x$, $G(t)$ is monotonically decreasing and therefore $G(t-y)$ is monotonically decreasing in y for $t < -B$ and $y \geqq x$. We can write now

$$0 < \frac{G_1^*(x-t)}{G(x-t)} = \frac{ce^{cx}}{G(x-t)} \int_x^\infty e^{-cy} G(y-t)\,dy < 1\,. \tag{6.3}$$

The proof will be completed by Lemma 2.1a of [3, p. 120] if we prove for $x > x_0 > \gamma$ that $G_1^*(x-t)/G(x_0-t)$ has no changes of trend for $t < -B$. We define $G_1^{**}(x-t) = \exp[c^{-1}D_x]G_1^*(x-t)$, obviously $G_1^{**}(t) \in$ class B and also to class II. Therefore

$$\frac{d}{dt}(G_1^{**}(x-t)/G(x_0-t)) = L_1^{**}(x-t+o(1)) - L(x_0-t+o(1)), \quad t \to -\infty, \tag{6.4}$$

where $L(t)$ and $L_1^{**}(t)$ satisfy

$$\begin{aligned} t &= \sum_{k=1}^\infty L(t)\,[(a_k(a_k+L(t)))^{-1} - (c_k(c_k+L(t)))^{-1}] \\ &= \sum_{k=1}^\infty L_1^{**}[(a_k(a_k+L_1^{**}(t)))^{-1} - (c_k(c_k+L_1^{**}(t)))^{-1}] \\ &\quad + L_1^{**}(t)\cdot(c(c+L_1^{**}(t)))^{-1}\,. \end{aligned} \tag{6.5}$$

(6.5) implies $L_1^{**}(t) = L(t-c^{-1}+o(1))$, $t \to -\infty$ or when we choose B large enough

$$\frac{d}{dt}(G_1^{**}(x-t)/G(x_0-t))$$

is positive for $t < -B$ and for $x - c^{-1} > x_0 > \gamma$ and therefore so is $\frac{d}{dt}(G_1^*(x-t)/G(x_0-t))$ for $x > x_0 > \gamma$. Q.E.D.

PROOF OF THEOREM 6.3. One easily see that

$$e^{AD} f(x) = \int_{-\infty}^\infty e^{AD} G(x-t)\,d\alpha(t) = \int_{-\infty}^\infty G(x+A-t)\,d\alpha(t)$$

and that the last integral converges for $x > \delta - A$.

Since $G_m(t)$ satisfy the same conditions as $G(t)$ it is enough to show

$$(6.6) \qquad ce^{cx}\int_x^\infty e^{-cy}\,dy\int_{-\infty}^\infty G(y-t)\,d\alpha(t) = ce^{cx}\int_{-\infty}^\infty\left\{\int_x^\infty G(y-t)\,e^{-cy}\,dy\right\}d\alpha(t)$$

and

$$(6.7) \qquad \left(1-\frac{D}{a}\right)\int_{-\infty}^\infty G(x-t)\,d\alpha(t) = \int_{-\infty}^\infty\left[\left(1-\frac{D}{a}\right)G(x-t)\right]d\alpha(t),$$

where $a \in \{a_k\}$ and $c \in \{c_k\}$. The equation (6.7) which is needed when the c_k corresponding to a is ∞ is proved as in Theorem 5.2a of [3, p. 129]. To prove (6.6) we see that both double integrals converge, that on the right because of Lemma 6.4 and that on the left because of Theorem 4.4. The proof of (6.6) follows now similarly to that of Theorem 6.1. Q.E.D.

THEOREM 6.5. *Suppose*:

(1) $G(t) \in$ *class* III *and* $N_+ \geqq 3$.
(2) $\alpha(t) \in B.V.$ *for* t *in any finite interval to the right of* T.
(3) $\int_{-\infty}^\infty G(x-t)\,d\alpha(t)$ *converges for* $x > T + \sum_{k=1}^\infty (a_k^{-1} - c_k^{-1})$.

Then

$$R_m(D)f(x) = \int_{-\infty}^\infty G_m(x-t)\,d\alpha(t)$$

the integral converges uniformly for x *in any finite interval right of*

$$T + b_m + \sum_{k=m+1}^\infty (a_k^{-1} - c_k^{-1}).$$

LEMMA 6.6. *Let assumptions* (1), (2) *and* (3) *of Theorem* 6.5 *be satisfied, then* $\int_{-\infty}^\infty G_1(x-t)\,d\alpha(t)$ *converges for* $x > T + \sum_{k=1}^\infty (a_k^{-1} - c_k^{-1})$ *where* $G_1(x) = \left(1-\frac{D}{c}\right)^{-1} G(x)$ *and* $c > \alpha_2$.

PROOF OF THEOREM 6.5 AND LEMMA 6.6. Similar to that of former Theorems and Lemmas of this section using here $G(t) \in$ class III and $G(t) = 0$ for $t > \sum_{k=1}^\infty (a_k^{-1} - c_k^{-1})$.

THEOREM 6.7. *Suppose*:

(1) $G(t) \in$ *class* II.
(2) $\alpha(t) \in B.V.$ *in any finite interval and for some* $K>0$ *and* $p>0$ $|\alpha(t)| \leqq Ke^{pt}$ *for* $t<0$.
(3) $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ *converges.*

Then

$$R_m(D) f(x) = \int_{-\infty}^{\infty} G_m(x-t)\, d\alpha(t)\,.$$

PROOF. By Theorem 3.1 $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ converges uniformly in any finite interval. Following the proof of Theorem 6.1 we can complete the proof of this theorem.

7. The basic inversion Theorems. We shall invert in this section the transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt \tag{7.1}$$

where $N_+ + N_- \geqq 3$. Similar results will be achieved for $N_+ + N_- \geqq 1$ in section 9 but then asymptotic restrictions have to be given on $\varphi(t)$ which will imply that if $N_+ + N_- \geqq 3$ the results of section 9 are a special case of those in this section.

For the proof we shall need the following Lemma, which will be helpful also in the following sections.

LEMMA 7.1. *Suppose* $N_+(m) + N_-(m) \geqq 2$, $\delta > 0$ *and* c *is finite, then*

$$\lim_{m\to\infty} \int_{|t|>\delta} G_m(t)\, e^{-ct}\, dt = 0 \tag{7.2}$$

and

$$\lim_{m\to\infty} \int_{|t|>\delta} t\, G_m'(t)\, e^{-ct}\, dt = 0\,. \tag{7.3}$$

PROOF. Choose m so large that $\alpha_1(m) < c < \alpha_2(m)$ (that is α_1 and α_2

corresponding to the sequence $\{a_k\}_{m+1}^{\infty}$) and choose also $\eta > 0$ such that $\alpha_1(m) < c-2\eta < c+2\eta < \alpha_2(m)$.

Since for $|t| \geqq \delta$ we have

$$e^{-ct} \leqq e^{-ct} \sinh^2 \eta t/\sinh^2 \eta\delta$$

we obtain

$$\int_{|t|\geqq\delta} G_m(t)\, e^{-ct}\, dt \leqq \int_{-\infty}^{\infty} G_m(t)\, e^{-ct} \sinh^2 \eta t\, (\sinh \eta\delta)^{-2}\, dt$$

$$\leqq \frac{1}{4(\sinh \eta\delta)^2}\left[\frac{1}{F_m(c+2\eta)} + \frac{1}{F_m(c-2\eta)} - \frac{2}{F_m(c)}\right]$$

and since for every finite a $\lim_{m\to\infty} F_m(a) = 1$ we conclude the proof of (7.2); the proof of (7.3) is similar. Q.E.D.

THEOREM 7.2. *Suppose*:

(1) $G(t) \in$ *class* I, $N_+ + N_- \geqq 3$.
(2) $\varphi(t) \in L_1(A, B)$ *for any* A, B *satisfying* $-\infty < A < B < \infty$.
(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt$ *converges.*
(4) $\varphi(t)$ *is continuous at* $t=x$ *and for some* p *and* k $|\varphi(t)| \leqq K \cosh pt$.
(5) $N_+(m) + N_-(m) \geqq 3$.

Then

$$\lim_{m\to\infty} R_m(D) f(x) = \varphi(x)\,.$$

THEOREM 7.3. *Suppose*:

(1) $G(t) \in$ *class* II.
(2) *Assumptions* (2), (3) *and* (4) *of Theorem* 7.2 *are satisfied.*

Then

$$\lim_{m\to\infty} R_m(D) f(x) = \varphi(x)\,.$$

THEOREM 7.4. *Suppose*:

(1) $G(t) \in$ *class* III, $N_+ + N_- \geqq 3$.
(2) $\varphi(t) \in L_1(A, B)$ *for any* A, B *satisfying* $T < A < B < \infty$.
(3) $f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt$ *converges at* $x > T + \sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1})$.

(4) *Assumption* (4) *(for* $t>0$) *and* (5) *of Theorem* 7.2 *are satisfied.*

Then

$$\lim_{m\to\infty} R_m(D) f(x) = \varphi(x) .$$

REMARK 7.5. For Theorem 7.4 we can always have such an arrangement of a_k's so that $N_+(m) \geqq 3$ if $N_+ \geqq 3$. See also Remark 5.4.

PROOF OF THEOREMS 7.2, 7.3 AND 7.4. By the corresponding theorems of section 6 we have

$$R_m(D) f(x) = \int_{-\infty}^{\infty} G_m(x-t)\,\varphi(t)\,dt . \tag{7.4}$$

Using $\int_{-\infty}^{\infty} G_m(t)\,dt = 1$ and (7.2) for $c=0$ we have for a point of continuity x of $\varphi(t)$

$$\left|\int_{x-\delta}^{x+\delta} G_m(x-t)[\varphi(t)-\varphi(x)]\,dt\right| \leqq \varepsilon \int_{x-\delta}^{x+\delta} G_m(x-t)\,dt \leqq \varepsilon . \tag{7.5}$$

Also the above arguments imply

$$\begin{aligned} \varphi(x) &= \int_{-\infty}^{\infty} G_m(x-t)\,\varphi(x)\,dt \\ &= \int_{x-\delta}^{x+\delta} G_m(x-t)\,\varphi(x)\,dt + o(1) , \qquad m\to\infty . \end{aligned} \tag{7.6}$$

To complete the proof we have only to show that for all δ

$$\int_{|x-t|>\delta} G_m(x-t)\,\varphi(t)\,dt = o(1) , \qquad m\to\infty , \tag{7.7}$$

which is an easy consequence of (7.2).

For Theorem 7.4 we have to use the proof of Theorem 2.3 which implies $G_m(-\delta) = 0$ for m big enough. Q.E.D.

8. Inversion Theorems. In this section inversion theorems will be established for

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t). \tag{8.1}$$

THEOREM 8.1. *Suppose:*

(1) $G(t) \in$ *class* I, $N_+ + N_- \geqq 3$.
(2) $\alpha(t) \in B.V.$ *in any finite interval.*
(3) (8.1) *converges.*
(4) $N_+(m) + N_-(m) \geqq 3$.
(5) x_1 *and* x_2 *are points of continuity of* $\alpha(t)$.

Then for m big enough we have

A. $\alpha_1 < c < \alpha_2$ *implies*

$$\begin{aligned}\int_{x_1}^{x_2} & e^{-cx} R_m(D) f(x)\, dx \\ &= \int_{-\infty}^{\infty} G_m(x_2-t)\, e^{-c(x_2-t)}\, \alpha(t)\, dt - \int_{-\infty}^{\infty} G_m(x_1-t)\, e^{-c(x_1-t)}\, \alpha(t)\, dt \\ &= \alpha(x_2) - \alpha(x_1) + o(1), \qquad m \to \infty .\end{aligned}$$

B. $\alpha_2 \leqq c$ *implies the existence of* $\alpha(+\infty)$ *and*

$$\begin{aligned}\int_{x_1}^{\infty} & e^{-cx} R_m(D) f(x)\, dx \\ &= \int_{-\infty}^{\infty} G_m(x_1-t)\, e^{-c(x_1-t)}(\alpha(\infty) - \alpha(t))\, dt \\ &= \alpha(\infty) - \alpha(x_1) + o(1), \qquad m \to \infty .\end{aligned}$$

C. $c \leqq \alpha_1$ *implies the existence of* $\alpha(-\infty)$ *and*

$$\begin{aligned}\int_{-\infty}^{x_1} & e^{-cx} R_m(D) f(x)\, dx \\ &= \int_{-\infty}^{\infty} G_m(x_1-t)\, e^{-c(x_1-t)}(\alpha(t) - \alpha(-\infty))\, dt \\ &= \alpha(x_1) - \alpha(-\infty) + o(1), \qquad m \to \infty .\end{aligned}$$

PROOF. First, we have to note that it is enough to assume that (8.1) converges for some x_0, since

$$\int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t) = \int_{-\infty}^{\infty} G(x-t)\, d\beta(t)$$

where $\beta(t) = \int_0^t e^{cu}\, d\alpha(u)$ is of bounded variation in any finite interval which permits the use of Theorem 2.1 as well as Theorem 6.1. As a result of Theorem 4.1 of [2] the function $G(x-t)e^{ct}$ is monotonic for $t \in (M, \infty)$ and $(-\infty, -M)$ for some large M and for every real c. (In the proof one has to distinguish five cases (1) $c < \alpha_1$, (2) $c = \alpha_1$, (3) $\alpha_1 < c < \alpha_2$, (4) $\alpha_2 = c$ and (5) $\alpha_2 > c$.) The monotonicity of $\exp[ct]\, G(x-t)$ implies by Lemma 2.1c of [3, pp. 121-122]:

(a) For $\alpha_1 < c < \alpha_2$

$$\alpha(t) = o(G(x_0-t)\, e^{ct})^{-1}, \qquad |t| \to \infty .$$

(b) For $c \geqq \alpha_2$ $\alpha(\infty) = \lim_{t\to\infty} \alpha(t)$ exists and

$$[\alpha(\infty) - \alpha(t)] = o(G(x_0-t)\, e^{ct})^{-1}, \qquad |t| \to \infty .$$

(c) For $c \leqq \alpha_1$, $\alpha(-\infty) = \lim_{t\to\infty} \alpha(t)$ exists and

$$[\alpha(t) - \alpha(-\infty)] = o(G(x_0-t)\, e^{ct})^{-1}, \qquad |t| \to \infty .$$

By Theorem 6.1

$$\begin{aligned}\int_{x_1}^{x_2} e^{-cx} R_m(D) f(x)\, dx &= \int_{x_1}^{x_2} \int_{-\infty}^{\infty} G_m(x-t)\, e^{-c(x-t)}\, d\alpha(t) \\ &= \int_{x_1}^{x_2} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} [G_m(x-t)\, e^{-c(x-t)}]\, \alpha(t)\, dt \\ &= \int_{-\infty}^{\infty} G_m(x_2-t)\, e^{-c(x_2-t)}\, \alpha(t) dt - \int_{-\infty}^{\infty} G_m(x_1-t)\, e^{-c(x_1-t)} \alpha(t)\, dt.\end{aligned}$$

The change in the order of integration is clearly permissible since one can see easily that the inner integral converges uniformly in the finite interval $x \in [x_1, x_2]$. The rest of the proof of A is by using Theorem 7.2. The proof of B and C follows steps similar to those of the proof of Theorem 6.1 a of [3, pp. 132-134] which are justified by lemmas and theorems of this paper.

Q.E.D.

Similarly one can prove the following theorems using, of course, the corresponding Theorems for classes II and III instead of class I.

THEOREM 8.2. *Suppose*:

(1) $G(t) \in$ *class* II.
(2) *Assumptions* (2), (3), (4) *and* (5) *of Theorem* 8.1 *are satisfied.*
(3) $|\alpha(t)| \leqq Ke^{pt}$ *for some* $p < 0$ *and* $t < 0$.

Then (A) *and* (B) *of Theorem* 8.1 *are valid.*

THEOREM 8.3. *Suppose*:

(1) $G(t) \in$ *class* III.
(2) $\alpha(t) \in B.V.$ $t \in (A, B)$ *for any* A, B *satisfying* $T < A < B < \infty$.
(3) (8.1) *converges for* $x > T + \sum_{k=1}^{\infty}(a_k^{-1} - c_k^{-1})$.
(4) $N_+(m) + N_-(m) \geqq 3$.
(5) $\alpha(t)$ *is continuous at* x_1 *and* x_2.

Then (A) *and* (B) *of Theorem* 8.1 *are valid for* $x_1, x_2, > T$.

9. Some generalizations and remarks.

(a) In Remark 5.4 we noticed that we may arrange the sequences $\{a_k\}$ and $\{c_k\}$ so that, except in one case, $N_+(m) + N_-(m) = N_+ + N_-$. We shall show here how to invert the transform even in this case.

Suppose that the negative a_k's are finite and that $n = N_- > 0$ (N_- is of course finite here). Take the n smallest positive c_k's $c_{k_1}, \cdots, c_{k_n}$ and write

$$f_*(t) = \left(1 - \frac{D}{c_{k_n}}\right)^{-1} \cdots \left(1 - \frac{D}{c_{k_1}}\right)^{-1} f(x) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(1 - \frac{D}{c_{k_i}}\right)^{-1} G(x-t)\, d\alpha(t)$$

which is permitted by the method of Theorem 6.1. Now

$$\prod_{i=1}^{n} \left(1 - \frac{D}{c_{k_i}}\right)^{-1} G(x-t) = G_*(x-t)$$

is a kernel satisfying $N_+ = N_+ + n = N_+ + N_-$ and now we can use Theorems 6.1 and 7.2 to find $\varphi(t)$ and $\alpha(t)$ from $f_*(t)$ instead of $f(t)$. When there are only finite positive a_k's and $N_+ > 0$ a similar technique is employed.

(b) When $N_+ + N_- = 2$ the inversion theory of section 7 still holds if $\varphi(t)$ satisfies the following:

(i) For $G(t) \in$ class I $|\varphi(t)| \leqq K \exp[(\alpha_1+\varepsilon)t]$, $t<0$ and $|\varphi(t)| \leqq K \exp[(\alpha_2-\varepsilon)t]$, $t>0$ for some K and $\varepsilon>0$.

(ii) For $G(t) \in$ class II $|\varphi(t)| \leqq K \exp[(\alpha_2-\varepsilon)t]$, $t>0$ and $|\varphi(t)| \leqq K e^{-pt}$, $t<0$ for some $K>0$, $p>0$ and $\varepsilon>0$.

(iii) For $G(t) \in$ class III

$$|\varphi(t)| \leqq K \exp[(\alpha_2-\varepsilon)t], \quad t>T.$$

Under these assumptions we obtain for $x \to \infty$

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt \\ &\leqq K \int_0^{\infty} G(x-t)\,e^{(\alpha_2-\varepsilon)t}\,dt + \left| \int_{-\infty}^{0} G(x-t)\,\varphi(t)\,dt \right| \\ &\equiv I_1(x) + I_2(x) \end{aligned}$$

$$I_2(x) = o(1), \quad x \to \infty \text{ for all } G(t)$$

$$I_1(x) \leqq K e^{(\alpha_2-\varepsilon)x} \int_{-\infty}^{\infty} G(v)\,e^{-(\alpha_2-\varepsilon)v}\,dv \leqq K_2 e^{(\alpha_2-\varepsilon)x}.$$

For $G(t) \in$ class I we can prove $|f(x)| \leqq K_3 \exp[(\alpha_1+\varepsilon)x]$, $x \to -\infty$. Let $c \in \{c_k\}$

$$\begin{aligned} f_*(x) &= \left(1-\frac{D}{c}\right)^{-1} f(x) \\ &= \int_{-\infty}^{\infty} \left(1-\frac{D}{c}\right)^{-1} G(x-t)\,\varphi(t)\,dt \\ &= \int_{-\infty}^{\infty} G_*(x-t)\,\varphi(t)\,dt. \end{aligned}$$

Then $G_*(x-t)$ satisfies $N_+ + N_- \geqq 3$, and the inversion operator is applicable to $f_*(x)$.

(c) When $N_+ + N_- = 1$ we can obtain a result similar to those of section 7 if $\varphi(t)$ is continuous and satisfies I, II or III of (b) when $G(t) \in$ class I, II or III respectively.

$$f(x) = \int_{-\infty}^{\infty} \varphi(x-t)\, dH(t)$$

$$\leqq K \int_{-\infty}^{x} e^{(\alpha_2-\varepsilon)(x-t)}\, dH(t) + K \int_{x}^{\infty} e^{(\alpha_1+\varepsilon)(x-t)}\, dH(t).$$

Using (2.9) of [2] which implies $H(t) \leqq K \exp[(\alpha_2-\varepsilon)t]$ and $1 - H(t) \leqq K \exp[(\alpha_1 + \varepsilon)t]$ and by the method of (a) and (b) find a transform $f_*(x) = \int_{-\infty}^{\infty} G_*(x-t)\,\varphi(t)\, dt$ for which G_* satisfies $N_+ + N_- \geqq 3$.

We should note here that the restriction that $\varphi(t)$ is continuous is not necessary. It will suffice that the Lebesgue-Stieltjes integral $f(x) = \int_{-\infty}^{\infty} \varphi(x-t) dH(t)$ converge for all x when $G(t) \in$ class I and for $x > A$ for some $A > 0$ otherwise.

(d) It should be noted that improvement of some Theorems here would be possible if we knew that $G_m(t)$ were monotonic in $(-\infty, -\delta)$ and (δ, ∞) and not only for $(-\infty, -A_m)$ (A_m, ∞). That refers to the theorems of section 7. Theorems 8.1 and 8.3 are the best possible in the sense that whenever the convergence occured the inversion was valid.

(e) The class of transforms

$$F(s) = \prod_{k=1}^{\infty} (1-s^2/a_k^2)/(1-s^2/c_k^2) \tag{9.1}$$

where $\sum a_k^{-2} < \infty$, $0 \leqq a_k/c_k < 1$, $N_+ + N_- > 2$ (and therefore $N_+ + N_- \geqq 4$), which includes the class treated by Y. Tanno in [4] and [5], satisfies $tG'(t) < 0$ for $t \neq 0$ and $tG'_{2m}(t) < 0$ for $t \neq 0$. The proof follows the one used for a subclass by Hirschman and Widder [3, p. 221]. The above fact enables us to replace assumption 4 of Theorem 7.2 by

$$\frac{1}{h} \int_{0}^{h} [\varphi(x+y) - \varphi(x)]\, dy = o(1), \qquad h \to o,$$

when our class is defined by (9.1).

The proof is as follows:

$$I_1 = \int_{x-\delta}^{x+\delta} G_{2m}(x-t)[\varphi(t) - \varphi(x)]\, dt$$

$$= G_{2m}(\delta)\, \alpha(\delta) - G_{2m}(\delta)\, \alpha(-\delta) + \int_{|x-t|<\delta} G'_{2m}(x-t)\, \alpha(t-x)\, dt$$

where $\alpha(t-x)=\int_x^t [\varphi(\tau)-\varphi(x)]\,d\tau$ and $|\alpha(t-x)| = o(t-x)$, $t\to x$,

$$I_1 \leqq \varepsilon \int_{-\infty}^{\infty} G'_{2m}(x-t)(x-t)\,dt + o(1) \leqq \varepsilon + o(1)\,, \qquad m\to\infty\,.$$

By (7.6) we see that it is enough to prove (7.7).

$$\int_{|x-t|>\delta} G_m(x-t)\,\varphi(t)\,dt$$

$$= \int_{|x-t|>\delta} G'_m(x-t)\,\alpha(t)\,dt + o(1)\,, \qquad m\to\infty\,,$$

where

$$\alpha(t) = \int_0^t \varphi(v)\,dv\,.$$

Using Theorems 4.3(a) and (b) we obtain (7.7) by integration by parts.

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