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THE HAUSDORFF DIMENSION OF THE SINGULAR SETS OF COMBINATION GROUPS

Dedicated to Professor Y. Tōki on his sixtieth birthday

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Introduction. It is well known that the Hausdorff dimension of the singular set of a Schottky group is positive and smaller than 2 ([5]). Moreover, it is also shown that there exists a Schottky group with a fundamental domain bounded by four circles whose singular set has positive 1-dimensional Hausdorff measure ([2]). Recently one of the present authors proved the existence of Kleinian groups whose singular sets have positive (3/2)-dimensional measure ([4]). The fundamental domains of groups mentioned above are domains bounded by a finite number of mutually disjoint circles. From these facts, it is natural for us to set up the following problem: Does the Hausdorff dimension of the singular sets of finitely generated Kleinian groups with fundamental domains bounded by mutually disjoint circles climb up, when the number of the boundary circles increases? What is the supremum of the Hausdorff dimensions determined by all such groups?

In this paper we shall give the result that the Hausdorff dimension increases strictly according to increment of the number of boundary circles.

1. Statement of theorem. Let us denote by B the unbounded domain in the complex plane whose boundary consists of $N(\geqq 1)$ mutually disjoint circles $\{K_i\}_{i=1}^N$. We shall form a discontinuous group of linear transformations with the fundamental domain B in the following. Take p pairs of boundary circles from $\{K_i\}_{i=1}^N$ and denote them by $\{H_i, H_i'\}_{i=1}^p$. Let S_i $(1 \leqq i \leqq p)$ be a hyperbolic or loxodromic transformation which transforms the outside of H_i onto the inside of H_i'. We denote by S_i^{-1} the inverse transformation of S_i. Consider the $N-2p(\geqq 0)$ remaining boundary circles among $\{K_i\}_{i=1}^N$ and denote them by $\{K_j^*\}_{j=1}^q$, where $N = 2p + q$. Let S_j^* $(1 \leqq j \leqq q)$ be an elliptic transformation with period 2 which transforms the outside of K_j^* onto the inside of K_j^*. A group G, generated by $\{S_i\}_{i=1}^p$ and $\{S_j^*\}_{j=1}^q$, is a discontinuous group with a fundamental domain B. In the special case of $N = 2p$, G is a Schottky group, which contains the elementary group of the case $p = 1$. If N is odd, there exists necessarily at least one elliptic transformation with period 2 and G is a Kleinian group. If $p = 0$ especially, G is generated by the only elliptic transformations $\{S_j^*\}_{j=1}^N$ with period 2. From now on G will always denote such a group unless otherwise is stated.

Let $G_i (i = 1, 2)$ be two discontinuous groups defined in the above with fundamental domain $B_i (i = 1, 2)$. Assume that all of the boundary circles of B_1 and B_2 are mutually disjoint. Then it is easily seen that the free product of G_1 and G_2, denoted by G, form a discontinuous group with a fundamental domain $B_1 \cap B_2$ ([6], [7]). Following Ford ([6]), we call G the combination group of G_1 and G_2.

Let us denote by E and $E_i (i = 1, 2)$ the singular sets of G and $G_i (i = 1, 2)$, that is, the totality of limit points of G and $G_i (i = 1, 2)$, respectively. It is evident that $E_1 \cup E_2$ is a proper subset of E. Hence it holds

$$d(E) \geqq d(E_i),\ i = 1, 2\ ,$$

where $d(E)$ and $d(E_i)$ denote the Hausdorff dimension of E and E_i, respectively. Here the Hausdorff dimension of a point set F in the z-plane is defined as the unique non-negative number $d(F)$ satisfying

$$M_d(F) = 0,\ \text{if}\ d > d(F)$$

and

$$M_d(F) = +\infty,\ \text{if}\ 0 \leqq d < d(F)\ ,$$

where $M_d(F)$ denotes the d-dimensional Hausdorff measure of F.

The purpose of this paper is to prove the following theorem.

THEOREM. *Suppose that the number of the boundary circles of either B_1 or B_2 is at least two. Then it holds*

$$d(E) > \max\,(d(E_1), d(E_2))\ .$$

Before proving our theorem, we shall give the well-known results as corollaries of our theorem.

COROLLARY 1. ([5], [9]) *Let E be the singular set of a non-cyclic Schottky group G. Then it holds*

$$d(E) > 0$$

and hence the capacity of E is positive.

PROOF. Since G is non-cyclic, there are at least two hyperbolic or loxodromic generators, say T_1 and T_2, of G. Let G_1 and G_2 denote the cyclic groups generated by T_1 and T_2, respectively, and let E_1 and E_2 denote their singular sets, respectively. Denoting E_3 the singular set of the combination group of G_1 and G_2, we have from our theorem

$$d(E_3) > d(E_1) = 0 .$$

Noting that $E \supset E_3$, we have immediately that $d(E) > 0$. q.e.d.

A discontinuous group of linear transformations is called a Kleinian group if its singular set has at least three points. Myrberg ([9]) proved that a Kleinian group contains always a Schottky group as a subgroup. Consequently we have

COROLLARY 2. ([5], [9]) *Let E be the singular set of any Kleinian group. Then $d(E) > 0$ and hence the capacity of E is positive.*

2. Propositions and Lemma for the proof of theorem. Let us denote by U a generator of G_1 or its inverse and by $\mathscr{U}_1 = \{U\}$ the totality of such U, that is, the generator system. Similarly V and $\mathscr{U}_2 = \{V\}$ are defined with respect to G_2. We denote by $T \circ S$ the composition of two transformations T and S, that is, $T \circ S(z) = T(S(z))$. Then any element $S (\neq$ identity) of G has the form

$$S = T_1 \circ T_2 \circ \cdots \circ T_m , \tag{1}$$

where $T_k \in \mathscr{U}_1 \cup \mathscr{U}_2$ for $k = 1, 2, \cdots, m$ and $(T_k)^{-1} \neq T_{k+1}$ for $k = 1, 2, \cdots, m-1$. If S has the form (1), S is called an element of grade m in G and is denoted by $S = S_{(m)}$. Let us denote by $R_{S_{(m)}}$ the radius of the isometric circle of the linear transformation $S_{(m)}$, that is,

$$R_{S_{(m)}} = \frac{1}{|c_m|} ,$$

for

$$S_{(m)}(z) = \frac{a_m z + b_m}{c_m z + d_m}, \quad a_m d_m - b_m c_m = 1 .$$

Next we put for any $\mu \geqq 0$

$$L_m(G, \mu) = \sum_{S_{(m)} \in G} [R_{S_{(m)}}]^\mu , \tag{2}$$

where the sum is taken over all elements of grade m in G. Further we shall define for any $\mu \geqq 0$

$$L(G, \mu) = \sum_{m=1}^{\infty} L_m(G, \mu) . \tag{3}$$

Quite similarly, we can define $L_m(G_i, \mu)$ and $L(G_i, \mu)$ for $i = 1, 2$.

Following two propositions play the important role in the proof of our theorem.

PROPOSITION 1. ([1], [3]) *Let G be a discontinuous group with a fundamental domain B whose boundary circles consist of at least two circles, and let E be its singular set. Let μ be a non-negative number. Then the following two statements are equivalent to each other*:

(i) $L(G, 2\mu) < +\infty$,

(ii) $M_\mu(E) = 0$,

where $M_\mu(E)$ denotes the μ-dimensional Hausdorff measure of E.

PROPOSITION 2. ([3]) *Under the same assumption as in Proposition 1, it holds*

$$0 < M_d(E) < +\infty,$$

where d is the Hausdorff dimension of E.

Using the above propositions, we obtain the following lemma.

LEMMA. *Denote by $d_1 = d(E_1)$ the Hausdorff dimension of E_1 and put $\mu' = 2d_1$. Then it holds*

$$\lim_{\delta \to +0} L(G_1, \mu' + \delta) = +\infty. \tag{4}$$

PROOF. Assume that this lemma is not true. Then there exist a decreasing sequence $\{\mu_n\}_{n=1}^{\infty}$ of positive numbers and a constant $M > 0$ such that

$$\mu_n \to \mu' \text{ as } n \to \infty$$

and

$$L(G_1, \mu_n) \leqq M \text{ for } n = 1, 2, \cdots.$$

From (3), we have

$$\sum_{m=1}^{l} L_m(G_1, \mu_n) \leqq M$$

for any l and any n. When l is fixed, the left side of the above inequality is a continuous function of μ. Hence, tending μ_n to μ', we have

$$\sum_{m=1}^{l} L_m(G_1, \mu') \leqq M.$$

Since l is any fixed positive integer, we have

$$L(G_1, \mu') \leqq M.$$

Hence from Proposition 1 we obtain

$$M_{\frac{\mu'}{2}}(E_1) = M_{d_1}(E_1) = 0,$$

which contradicts Proposition 2. Thus our lemma is proved. q.e.d.

3. Proof of theorem. Now let us begin the proof of our theorem. Without loss of generality we may assume that the number of the boundary circles of B_1 is more than one and $d(E_1) \geqq d(E_2)$. We put

$$S_{(m)} = (T_1 \circ T_2 \circ \cdots \circ T_\nu) \circ (T_{\nu+1} \circ \cdots \circ T_n) = S_{(\nu)} \circ S_{(m-\nu)} ,$$

and consider the radii $R_{S_{(m)}}$, $R_{S_{(\nu)}}$ and $R_{S_{(m-\nu)}}$ of the isometric circles of $S_{(m)}$, $S_{(\nu)}$ and $S_{(m-\nu)}$, respectively. As to these values, the following relation holds ([6]):

$$R_{S_{(m)}} = \frac{R_{S_{(\nu)}} \cdot R_{S_{(m-\nu)}}}{|S_{(m-\nu)}(\infty) - S_{(\nu)}^{-1}(\infty)|} , \tag{5}$$

where $S_{(m-\nu)}(\infty)$ and $S_{(\nu)}^{-1}(\infty)$ denote the images of ∞ by $S_{(m-\nu)}$ and $S_{(\nu)}^{-1}$, the inverse of $S_{(\nu)}$, respectively. Now we shall divide the sum $L_m(G, \mu)$ into the following partial sums:

$$\begin{aligned} L_m(G, \mu) &= \sum_{S_{(m)} \in G} [R_{S_{(m)}}]^\mu \\ &= \sum\nolimits^{(0)} [R_{S_{(m)}}]^\mu + \sum\nolimits^{(1)} [R_{S_{(m)}}]^\mu + \cdots + \sum\nolimits^{(m)} [R_{S_{(m)}}]^\mu \\ &= \sum_{\nu=0}^{m} \left(\sum\nolimits^{(\nu)} [R_{S_{(m)}}]^\mu\right) . \end{aligned} \tag{6}$$

Here $\sum^{(\nu)}$ $(\nu = 0, 1, \cdots, m)$ denotes the sum taken over all elements of the following form:

$$\begin{aligned} S_{(m)} &= (U_1 \circ \cdots \circ U_\nu) \circ V_{\nu+1} \circ (T_{\nu+2} \circ \cdots \circ T_m) \\ &= U_{(\nu)} \circ V_{\nu+1} \circ S_{(m-\nu-1)} , \end{aligned} \tag{7}$$

where $U_k \in \mathscr{Y}_1$ $(1 \leqq k \leqq \nu)$, $V_{\nu+1} \in \mathscr{Y}_2$ and $T_k \in \mathscr{Y}_1 \cup \mathscr{Y}_2$ $(\nu + 2 \leqq k \leqq m)$ and hence $U_{(\nu)}$ and $S_{(m-\nu-1)}$ are elements of grade ν and $m - \nu - 1$ in G_1 and G, respectively.

Now take a sufficiently large number $\rho > 1$ such that all of the boundary circles of B are contained in the inside of a circle with the radius $\rho/2$ and the center at the origin. Further take a positive constant $\lambda < 1$ such that $0 < \lambda < R_V$ for any $V \in \mathscr{Y}_2$. Then using the relations (5) and (7), we obtain

$$R_{S_{(m)}} \geqq \frac{\lambda}{\rho^2} R_{U_{(\nu)}} \cdot R_{S_{(m-\nu-1)}} \tag{8}$$

for $\nu = 1, 2, \cdots, m - 2$. Substituting (8) into (6), we have the following inequality:

$$\begin{aligned} L_m(G, \mu) > \left(\frac{\lambda}{\rho^2}\right)^\mu \{ & L_1(G_1, \mu) L_{m-2}(G, \mu) + L_2(G_1, \mu) L_{m-3}(G, \mu) \\ & + \cdots + L_{m-3}(G_1, \mu) L_2(G, \mu) + L_{m-2}(G_1, \mu) L_1(G, \mu)\} \end{aligned} \tag{9}$$

for $m \geqq 3$. The verification of (9) is easy if the number of the boundary circles of B_2 is at least two. In the case that its number is only one, we consider the generator system

$$\hat{\mathscr{Y}}_2 = \{\hat{V};\ \hat{V} = S_1^* \circ U \circ S_1^{*-1},\ U \in \mathscr{Y}_1\}$$

instead of $\mathscr{Y}_2 = \{S_1^*\}$. Then the fundamental domain $\hat{B}_2$ of $\hat{G}_2$ generated by $\hat{\mathscr{Y}}_2$ is bounded by the same number of boundary circles as the one of B_1. Since $G_1 * G_2 \supset G_1 * \hat{G}_2$, the following discussion holds also for $G_1 * \hat{G}_2$, where $G_1 * G_2$ denotes the combination group of G_1 and G_2. Summing up the term $L_m(G, \mu)$ from 1 to n with respect to m, we obtain from (9) the following inequality:

$$\begin{aligned}(10)\quad \sum_{m=1}^{n} L_m(G, \mu) &> \sum_{m=3}^{n} L_m(G, \mu) \\ &> \left(\frac{\lambda}{\rho^2}\right)^{\mu}\left[L_1(G, \mu)\left\{\sum_{m=1}^{n-2} L_m(G_1, \mu)\right\} + L_2(G, \mu)\left\{\sum_{m=1}^{n-3} (G_1, \mu)\right\}\right. \\ &\quad \left. + \cdots + L_{n-2}(G, \mu) L_1(G_1, \mu)\right] .\end{aligned}$$

Setting $P_n(\mu) = \sum_{m=1}^{n} L_m(G, \mu)$ and $Q_n(\mu) = \sum_{m=1}^{n} L_m(G_1, \mu)$ for $n = 1, 2, \cdots$ and using $n + 2$ instead of n in (10), we have

$$\begin{aligned}(11)\qquad P_{n+2}(\mu) &> \left(\frac{\lambda}{\rho^2}\right)^{\mu}\Big\{L_1(G, \mu)Q_n(\mu) + L_2(G, \mu)Q_{n-1}(\mu) + \cdots \\ &\quad + L_n(G, \mu)Q_1(\mu)\Big\} .\end{aligned}$$

On the other hand we have from Lemma that for any large number M there exist a small number δ depending only on M and a large positive integer n_0 depending on M and δ such that

$$(12)\qquad Q_n(2d_1 + \delta) > M \qquad \text{for any } n > n_0 .$$

Putting $\mu = 2d_1 + \delta$ in (11), we obtain from (12)

$$\begin{aligned}(13)\qquad P_{n+2}(2d_1 + \delta) &> \left(\frac{\lambda}{\rho^2}\right)^{2d_1+\delta} M\{L_1(G, 2d_1 + \delta) + \cdots \\ &\quad + L_{n-n_0}(G, 2d_1 + \delta)\}\end{aligned}$$

for any $n > n_0$. If we put $a = (\lambda/\rho^2)^{2d_1+\delta} \cdot M$ and $n_1 = n_0 + 2$ in (13), we have for any $n \geqq n_1$

$$(14)\qquad P_n(2d_1 + \delta) > aP_{n-n_1}(2d_1 + \delta) .$$

After k times repetitions of (14) for $n = kn_1 + l$ $(0 \leqq l < n_1)$, we reach to the following inequality:

(15) $$P_n(2d_1+\delta) > a^k P_l(2d_1+\delta) \geqq a^k L_1(2d_1+\delta) .$$

We may consider that $a > 1$, for we can take a sufficiently large number M for fixed λ and ρ. Since $L_1(2d_1+\delta) > 0$, it holds from (3) and (15)

$$L(G, 2d_1+\delta) = \lim_{n\to\infty} P_n(2d_1+\delta) = +\infty .$$

Hence it holds for some $\delta > 0$

(16) $$L(G, 2d_1+\delta) = +\infty .$$

We obtain from Proposition 1 and (16)

$$M_{d_1+\frac{\delta}{2}}(E) > 0 ,$$

which implies that $d(E) \geqq d_1 + \delta/2$. Since $\delta > 0$, it holds

$$d(E) > d_1 .$$

Thus our theorem is completely proved. q.e.d.

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