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K-THEORY OF LENS-LIKE SPACES AND S^1-ACTIONS ON $S^{2m+1} \times S^{2n+1}$

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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0. Introduction. A smooth S^1-action $\psi: S^1 \times X \to X$ on a smooth manifold X is called *principal* if the isotropy subgroup

$$I(x) = \{z \in S^1 \mid \psi(z, x) = x\}$$

consists of the identity element alone for each point $x \in X$. A principal smooth S^1-action (ψ, X) on a closed (oriented) smooth manifold X is called to *bord* (with orientation) if there is a principal smooth S^1-action (Φ, W) on a compact (oriented) smooth manifold W and there is an equivariant (orientation preserving) diffeomorphism of (ψ, X) onto $(\Phi, \partial W)$, the boundary of W. It is well known that any principal smooth S^1-action on a homotopy sphere does not bord (cf. [3], Theorem 23.2). On the contrary, any principal smooth S^1-action on a closed oriented smooth manifold which is cohomologically a product $S^{2m+1} \times S^{2n+1}$ of odd-dimensional spheres bords with orientation ([7], Theorem 7.3; [5], Theorem 1). Moreover let $(\psi_1, \Sigma_1^{2n+1})$, $(\psi_2, \Sigma_2^{2n+1})$ be any principal smooth S^1-actions on homotopy spheres, then there is an equivariant continuous mapping $f: \Sigma_1 \to \Sigma_2$ and f induces a homotopy equivalence of the orbit manifold Σ_1/ψ_1 to the orbit manifold Σ_2/ψ_2 (cf. [2], Proposition 3.1). On the contrary, there are infinitely many cohomologically distinct principal smooth S^1-actions on $S^{2m+1} \times S^{2n+1}$ $(m \neq n)$ ([5], Corollary of Lemma 2.2).

In this paper we consider the equivariant K-theory of certain S^1-manifolds $S^{2m+1} \times S^{2n+1}$ and we show that there are topologically distinct principal smooth S^1-actions on $S^{2m+1} \times S^{2n+1}$ which can not be distinguished by the cohomology ring structure of the orbit spaces. To state our results precisely, we introduce some notations. Let

$$a = (a_0, a_1, \cdots, a_m)\,, \qquad b = (b_0, b_1, \cdots, b_n)$$

be sequences of positive integers and denote by

$$S^{2m+1}(a_0, a_1, \cdots, a_m) \times S^{2n+1}(b_0, b_1, \cdots, b_n)$$

the product of spheres $S^{2m+1} \times S^{2n+1}$ with the smooth S^1-action $\psi_{a,b}$ defined by

$$\psi_{a,b}(\lambda, (u_0, \cdots, u_m), (v_0, \cdots, v_n)) = ((\lambda^{a_0}u_0, \cdots, \lambda^{a_m}u_m), (\lambda^{b_0}v_0, \cdots, \lambda^{b_n}v_n))$$

in complex coordinates. Then the S^1-action $\psi_{a,b}$ is principal if and only if each a_i is coprime to each b_j. When the S^1-action $\psi_{a,b}$ is principal, the orbit manifold is denoted by

$$M(a_0, a_1, \cdots, a_m; b_0, b_1, \cdots, b_n) .$$

In particular, $M(a_0; b_0, \cdots, b_n)$ is naturally diffeomorphic to the lens space $L(a_0; b_0, \cdots, b_n)$ obtained from S^{2n+1} by the identification

$$(v_0, v_1, \cdots, v_n) = (\lambda^{b_0}v_0, \lambda^{b_1}v_1, \cdots, \lambda^{b_n}v_n)$$

for all $\lambda \in C$, $\lambda^{a_0} = 1$. Let $Z[x]$ be the polynomial ring with integer coefficients and let $p_1(x), \cdots, p_n(x)$ be elements of $Z[x]$. And let

$$\{p_1(x), \cdots, p_n(x)\}$$

be the ideal in $Z[x]$ generated by the polynomials $p_1(x), \cdots, p_n(x)$. Now we can state our result as follows.

THEOREM 1.3. *If there is an equivariant continuous mapping from* $S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n)$ *to* $S^{2m+1}(c_0, \cdots, c_m) \times S^{2n+1}(d_0, \cdots, d_n)$, *then the ideal* $\{\prod_{i=0}^m (1 - x^{a_i}), \prod_{j=0}^n (1 - x^{b_j})\}$ *contains the ideal* $\{\prod_{i=0}^m (1 - x^{c_i}), \prod_{j=0}^n (1 - x^{d_j})\}$, *where* a_i, b_j, c_i, d_j *are positive integers.*

As an application of Theorem 1.3, we have the following result.

THEOREM 2.1. *Under the hypotheses of Theorem* 1.3,

(1) *if* $n \geqq m + 1$, *then* $\prod_{i=0}^m c_i \equiv 0 \pmod{\prod_{i=0}^m a_i}$,

(2) *if* $n \geqq m + 2$ *and* $\prod_{i=0}^m a_i = \prod_{i=0}^m c_i$, *then* $\sum_{i=0}^m a_i \equiv \sum_{i=0}^m c_i \pmod 2$.

Next, let $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ be sequences of positive integers with $0 \leqq m \leqq n$, and assume that each a_i is coprime to each b_j. Then we have the following result about the orbit manifold.

THEOREM 3.6. *There is a canonical ring isomorphism*

$$K(M(a_0, \cdots, a_m; b_0, \cdots, b_n)) \cong K(M(a_0, \cdots, a_m; \underbrace{1, \cdots, 1}_{n+1})) .$$

The result for $m = 0$ has been obtained by Mahammed ([4], Theorem 2.1.)

Finally we consider the conditions of the existence of equivariant diffemorphisms.

1. Equivariant *K*-theory of $S^{2m+1} \times S^{2n+1}$. Let G be a compact Lie group. Let $E = L_1 \oplus \cdots \oplus L_n$ be a sum of complex G-line bundles on a compact G-space X and let

$$\pi: S(E) \to X$$

be the sphere bundle. Then, since the Thom isomorphism theorem is true for E ([1], Proposition 2.7.2; [6], p. 140), we have an exact sequence

$$(*) \quad K_G^1(X) \xrightarrow{\pi^*} K_G^1(S(E)) \xrightarrow{\pi_*} K_G(X) \xrightarrow{\varphi} K_G(X) \xrightarrow{\pi^*} K_G(S(E)) \xrightarrow{\pi_*} K_G^1(X) ,$$

where φ is the multiplication by $\lambda_{-1}[E] = \prod_{i=1}^n (1 - [L_i])$.

Let $(a_0, \cdots, a_m)$ be the sequence of positive integers. Denote by $C^{m+1}(a_0, \cdots, a_m)$ the complex vector space C^{m+1} with the S^1-action ψ given by

$$\psi(\lambda, (z_0, \cdots, z_m)) = (\lambda^{a_0} z_0, \cdots, \lambda^{a_m} z_m) .$$

LEMMA 1.1. *The ring structure of* $K_{S^1}^*(S^{2m+1}(a_0, \cdots, a_m))$ *is as follows.*

$$K_{S^1}(S^{2m+1}(a_0, \cdots, a_m)) = Z[x]\Big/\Big\{\prod_i (1 - x^{a_i})\Big\} ,$$

$$K_{S^1}^1(S^{2m+1}(a_0, \cdots, a_m)) = 0 ,$$

where x *is the class of the* S^1*-line bundle* $S^{2m+1}(a_0, \cdots, a_m) \times C^1(1)$.

PROOF. Since $K_G(\text{one-point}) = R(G)$, the representation ring of G and $K_G^1(\text{one-point}) = 0$, there is an exact sequence

$$0 \longrightarrow K_{S^1}^1(S^{2m+1}(a_0, \cdots, a_m)) \longrightarrow R(S^1) \xrightarrow{\varphi} R(S^1)$$
$$\xrightarrow{\pi^*} K_{S^1}(S^{2m+1}(a_0, \cdots, a_m)) \longrightarrow 0 .$$

Here $R(S^1) = Z[t, t^{-1}]$, $t = [C(1)]$ and φ is the multiplication by $\prod_{i=0}^m (1 - t^{a_i})$. Therefore φ is injective and

$$0 = \prod_{i=0}^m (1 - x^{a_i}) = 1 - x \cdot p(x)$$

in $K_{S^1}(S^{2m+1}(a_0, \cdots, a_m))$ for some $p(x)$ in $Z[x]$, if $x = \pi^*(t)$. Therefore $\pi^*(t^{-1}) = p(x)$. q.e.d.

LEMMA 1.2. *Let* $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ *be sequences of positive integers. Then the ring*

$$K_{S^1}(S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n))$$

is isomorphic to the ring

$$Z[x]\Big/\Big\{\prod_{i=0}^m (1 - x^{a_i}), \prod_{j=0}^n (1 - x^{b_j})\Big\} ,$$

where x *is the class of the* S^1*-line bundle*

$$S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n) \times C^1(1) .$$

PROOF. This follows similarly from the exact sequence (*) for S^1-vector bundle

$$E = S^{2m+1}(a_0, \cdots, a_m) \times C^{n+1}(b_0, \cdots, b_n)$$

and Lemma 1.1, so we leave it to the reader. q.e.d.

Now we can prove the main result stated in the introduction.

THEOREM 1.3. *If there is an equivariant continuous mapping from* $S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n)$ *to* $S^{2m+1}(c_0, \cdots, c_m) \times S^{2n+1}(d_0, \cdots, d_n)$, *then the ideal* $\{\prod_{i=0}^m (1 - x^{a_i}), \prod_{j=0}^n (1 - x^{b_j})\}$ *contains the ideal* $\{\prod_{i=0}^m (1 - x^{c_i}), \prod_{j=0}^n (1 - x^{d_j})\}$, *where* a_i, b_j, c_i, d_j *are positive integers.*

PROOF. By Lemma 1.2,

$$K_{S^1}(S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n)) = Z[x]\Big/\Big\{\prod_{i=0}^m (1 - x^{a_i}), \prod_{j=0}^n (1 - x^{b_j})\Big\} ,$$

$$K_{S^1}(S^{2m+1}(c_0, \cdots, c_m) \times S^{2n+1}(d_0, \cdots, d_n)) = Z[y]\Big/\Big\{\prod_{i=0}^m (1 - y^{c_i}), \prod_{j=0}^n (1 - y^{d_j})\Big\} ,$$

where x is the class of the S^1-line bundle

$$S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n) \times C^1(1)$$

and y is the class of the S^1-line bundle

$$S^{2m+1}(c_0, \cdots, c_m) \times S^{2n+1}(d_0, \cdots, d_n) \times C^1(1) .$$

Let

$$f\colon S^{2m+1}(a) \times S^{2n+1}(b) \to S^{2m+1}(c) \times S^{2n+1}(d)$$

be an equivariant continuous mapping. Then f induces a ring homomorphism

$$f^*\colon K_{S^1}(S^{2m+1}(c) \times S^{2n+1}(d)) \to K_{S^1}(S^{2m+1}(a) \times S^{2n+1}(b))$$

with $f^*(y) = x$. Then the statement is clear. q.e.d.

2. Existence of equivariant continuous mappings. As an application of Theorem 1.3, we have the following result.

THEOREM 2.1. *If there is an equivariant continuous mapping from* $S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n)$ *to* $S^{2m+1}(c_0, \cdots, c_m) \times S^{2n+1}(d_0, \cdots, d_n)$, *where* a_i, b_j, c_i, d_j *are positive integers, and*

(1) *if* $n \geqq m + 1$, *then* $\prod_{i=0}^m c_i \equiv 0 \pmod{\prod_{i=0}^m a_i}$,

(2) *if* $n \geqq m + 2$ *and* $\prod_{i=0}^m a_i = \prod_{i=0}^m c_i$, *then* $\sum_{i=0}^m a_i \equiv \sum_{i=0}^m c_i \pmod 2$.

PROOF. By the hypotheses and Theorem 1.3, there are polynomials

$p(x)$, $q(x)$ in $Z[x]$ satisfying

$$\prod_{i=0}^{m}(x^{c_i}-1)=p(x)\cdot\prod_{i=0}^{m}(x^{a_i}-1)+q(x)\cdot\prod_{j=0}^{n}(x^{b_j}-1)\,.$$

Put $y=x-1$, then we have

$$(\ast)\quad \prod_{i=0}^{m}\left(\sum_{k=1}^{c_i}\binom{c_i}{k}y^{k-1}\right)=p(y+1)\prod_{i=0}^{m}\left(\sum_{k=1}^{a_i}\binom{a_i}{k}y^{k-1}\right)+y^{n-m}q(y+1)\prod_{j=0}^{n}\left(\sum_{k=1}^{b_j}\binom{b_j}{k}y^{k-1}\right).$$

Since $n>m$, if we put $y=0$, we have

$$\prod_{i=0}^{m}c_i=p(1)\cdot\prod_{i=0}^{m}a_i\,,$$

and this shows (1). Next, if $\prod_{i=0}^{m}a_i=\prod_{i=0}^{m}c_i$, then $p(1)=1$ in the equation $(\ast)$. Therefore there is a polynomial $f(y)$ in $Z[y]$ such that

$$p(y+1)=1+y\cdot f(y)\,.$$

Then we have the following equation from $(\ast)$ and the hypotheses $n\geqq m+2$,

$$\left(\prod_{i=0}^{m}c_i\right)\cdot\left(1+\left(\sum_{i=0}^{m}\frac{c_i-1}{2}\right)y\right)=\left(\prod_{i=0}^{m}a_i\right)\cdot\left(1+\left(\sum_{i=0}^{m}\frac{a_i-1}{2}\right)y+yf(y)\right)+y^2g(y)$$

for some polynomial $g(y)$ in $Z[y]$. Since $\prod_{i=0}^{m}a_i=\prod_{i=0}^{m}c_i\neq 0$, we have the equation

$$\left(\prod_{i=0}^{m}a_i\right)\left(\sum_{i=0}^{m}\frac{c_i-a_i}{2}\right)=\left(\prod_{i=0}^{m}a_i\right)f(y)+yg(y)\,.$$

And, since $f(0)$ is an integer, we have

$$\sum_{i=0}^{m}a_i\equiv\sum_{i=0}^{m}c_i \pmod 2\,. \qquad \text{q.e.d.}$$

REMARK 2.2. Considering the coefficients of higher terms y^k of $(\ast)$, one may have more and more necessary conditions for the existence of an equivariant mappings, if $n-m$ is large.

REMARK 2.3. If $n>m$, the integral cohomology ring of the orbit manifold $M(a_0,\cdots,a_m;b_0,\cdots,b_n)$ defined in the introduction is isomorphic to the ring

$$Z[c,y]\Big/\left\{y^2,\,c^{n+1},\,\left(\prod_{i=0}^{m}a_i\right)\cdot c^{m+1},\,yc^{m+1}\right\}\,,$$

where $\deg c=2$, $\deg y=2n+1$ and c is the Euler class of the principal S^1-bundle

$$\pi\colon S^{2m+1}(a)\times S^{2n+1}(b)\to M(a_0,\cdots,a_m;b_0,\cdots,b_n)$$

([5], Theorem 2, (ii)). Then the cohomology ring of the orbit manifold is determined by the integer $\prod_{i=0}^{m} a_i$. Therefore Theorem 2.1 shows that there are topologically distinct principal smooth S^1-actions on $S^{2m+1} \times S^{2n+1}$ for $n \geqq m+2$ which can not be distinguished by the cohomology ring structure of the orbit spaces.

3. *K*-theory of lens-like spaces. Let $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ be sequences of positive integers and assume that each a_i is coprime to each b_j. Then there is a natural ring isomorphism

$$K(M(a_0, \cdots, a_m; b_0, \cdots, b_n)) \cong K_{S^1}(S^{2m+1}(a_0, \cdots, a_m) \times S^{2n+1}(b_0, \cdots, b_n)) .$$

In this section, we consider the ideal

$$\left\{\prod_{i=0}^{m} (1 - x^{a_i}), \prod_{j=0}^{n} (1 - x^{b_j})\right\}$$

of $Z[x]$, and we have a generalization of the theorem of Mahammed ([4], Theorem 2.1).

LEMMA 3.1. *Let a, b be positive integers and assume that a is coprime to b. Then there are polynomials $p(x)$, $q(x)$ in $Z[x]$ such that*

$$1 - x = (1 - x^a)p(x) + (1 - x^b)q(x) .$$

PROOF. Suppose that $a = bc + d$, $c > 0$ and $d \geqq 0$. Then

$$1 - x^d = (1 - x^a) - (1 - x^b)x^d\left(\sum_{i=0}^{c-1} x^{ib}\right) .$$

Repeating this process, we have the result. q.e.d.

LEMMA 3.2. *Let $a_0, a_1, \cdots, a_m, b$ be positive integers and assume that each a_i is coprime to b. Then there are polynomials $p(x)$, $q(x)$ in $Z[x]$ such that*

$$(1 - x)^{m+1} = p(x) \prod_{i=0}^{m} (1 - x^{a_i}) + q(x)(1 - x^b) .$$

PROOF. From Lemma 3.1, there are polynomials $p_i(x)$, $q_i(x)$ in $Z[x]$ such that

$$1 - x = p_i(x)(1 - x^{a_i}) + q_i(x)(1 - x^b) .$$

Then there is a polynomial $q(x)$ in $Z[x]$ such that

$$(1 - x)^{m+1} = \left(\prod_{i=0}^{m} p_i(x)\right)\left(\prod_{i=0}^{m} (1 - x^{a_i})\right) + q(x)(1 - x^b) . \qquad \text{q.e.d.}$$

LEMMA 3.3. *Let $(a_0, \cdots, a_m)$, $(b_0, \cdots b_n)$ be sequences of positive integers and assume that each a_i is coprime to each b_j. Then there are polynomials $p(x)$, $q(x)$ in $Z[x]$ such that*

$$(1-x)^{(m+1)(n+1)} = p(x)\prod_{i=0}^{m}(1-x^{a_i}) + q(x)\prod_{j=0}^{n}(1-x^{b_j}) .$$

PROOF. This follows from Lemma 3.2 by the same way as in Lemma 3.2. q.e.d.

LEMMA 3.4. *Let $a_0, a_1, \cdots, a_m$ be positive integers. Then there are polynomials $p(x)$, $q(x)$ in $Z[x]$ such that*

$$\left(\prod_{i=0}^{m} a_i\right)^{n(m+1)}(1-x)^{m+1} = p(x)\prod_{i=0}^{m}(1-x^{a_i}) + q(x)(1-x)^{(m+1)(n+1)} .$$

PROOF. There is a polynomial $f(x)$ in $Z[x]$ such that

$$\prod_{i=0}^{m}(1-x^{a_i}) = (1-x)^{m+1}\left(\prod_{i=0}^{m} a_i + (1-x)f(x)\right) .$$

By induction on k, there are polynomials $p_k(x)$, $q_k(x)$ in $Z[x]$ for $0 \leqq k \leqq n(m+1)$ such that

$$\left(\prod_{i=0}^{m} a_i\right)^{k}(1-x)^{(m+1)(n+1)-k} = p_k(x)\prod_{i=0}^{m}(1-x^{a_i}) + q_k(x)(1-x)^{(m+1)(n+1)} .$$

We leave it to the reader. q.e.d.

LEMMA 3.5. *Let $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ be sequences of positive integers with $0 \leqq m \leqq n$, and assume that each a_i is coprime to each b_j. Then there are polynomials $p(x)$, $q(x)$ in $Z[x]$ such that*

$$(1-x)^{n+1} = p(x)\prod_{i=0}^{m}(1-x^{a_i}) + q(x)\prod_{j=0}^{n}(1-x^{b_j}) .$$

PROOF. From Lemma 3.3 and Lemma 3.4, there are polynomials $a(x)$, $b(x)$, $c(x)$, $d(x)$ in $Z[x]$ such that

$$\left(\prod_{i=0}^{m} a_i\right)^{n(m+1)}(1-x)^{m+1} = a(x)\prod_{i=0}^{m}(1-x^{a_i}) + b(x)\prod_{j=0}^{n}(1-x^{b_j}) ,$$

$$\left(\prod_{j=0}^{n} b_j\right)^{m(n+1)}(1-x)^{n+1} = c(x)\prod_{i=0}^{m}(1-x^{a_i}) + d(x)\prod_{j=0}^{n}(1-x^{b_j}) .$$

By assumption, there are integers M, N such that

$$M\left(\prod_{i=0}^{m} a_i\right)^{m(n+1)} + N\left(\prod_{j=0}^{n} b_j\right)^{m(n+1)} = 1 .$$

Then

$$p(x) = M(1-x)^{n-m}a(x) + Nc(x) ,$$

$$q(x) = M(1-x)^{n-m}b(x) + Nd(x)$$

are desired polynomials. q.e.d.

THEOREM 3.6. *Let* $(a_0, \cdots, a_m)$, $(b_0, \cdots, b_n)$ *be sequences of positive integers with* $0 \leqq m \leqq n$, *and assume that each* a_i *is coprime to each* b_j. *Then there is a canonical ring isomorphism*

$$K(M(a_0, \cdots, a_m; b_0, \cdots, b_n)) \cong K(M(a_0, \cdots, a_m; \underbrace{1, \cdots, 1}_{n+1}))\ .$$

PROOF. By Lemma 3.5,

$$\left\{\prod_{i=0}^{m}(1 - x^{a_i}), \prod_{j=0}^{n}(1 - x^{b_j})\right\} = \left\{\prod_{i=0}^{m}(1 - x^{a_i}), (1 - x)^{n+1}\right\}$$

as the ideals of $Z[x]$. Then the statement is clear from Lemma 1.2.

q.e.d.

4. **Existence of equivariant diffeomorphisms.** In this section, we will prove the following result.

THEOREM 4.1. *Let* a_i, b_j, c_i, d_j *be positive integers. If*

$$S^{2m+1}(a_0, a_1, \cdots, a_m) \times S^{2n+1}(b_0, b_1, \cdots, b_n)$$

is equivariantly diffeomorphic to

$$S^{2m+1}(c_0, c_1, \cdots, c_m) \times S^{2n+1}(d_0, d_1, \cdots, d_n)\ ,$$

then

$$\sum_{i=0}^{m} a_i^{2p} + \sum_{j=0}^{n} b_j^{2p} = \sum_{i=0}^{m} c_i^{2p} + \sum_{j=0}^{n} d_j^{2p}$$

for $0 < 2p \leqq \min(m, n)$.

REMARK 4.2. For principal S^1-actions, the above result is equivalent to Corollary (1) of Theorem 3 in [5] which is obtained by the Pontrjagin classes of the orbit manifolds.

Let G be a compact Lie group and X a compact G-space. Let us denote by

$$r \colon K_G(X) \to KO_G(X)$$
$$c \colon KO_G(X) \to K_G(X)$$

the real restriction and the complexification respectively.

LEMMA 4.3. *Let* E *be a complex* G*-vector bundle over* X. *Then*

$$cr[E] = [E] + [E^*]\ ,$$

where E^* *is the conjugate vector bundle of* E.

PROOF. The result follows easily from the following fact. Let V be a complex G-vector space with a complex structure J. Denote by $V \otimes_R C$, the complexification of underlying real G-vector space V, whose complex structure is the multiplication by $1 \otimes \sqrt{-1}$. Let V_+, V_- be complex G-vector spaces in $V \otimes_R C$ defined by

$$V_+ = \{v \otimes 1 - J(v) \otimes \sqrt{-1} \mid v \in V\} ,$$
$$V_- = \{v \otimes 1 + J(v) \otimes \sqrt{-1} \mid v \in V\} .$$

Then V_+ is G-isomorphic to V, V_- is conjugate G-isomorphic to V, and $V \otimes_R C$ is G-isomorphic to $V_+ \oplus V_-$. q.e.d.

If X is a smooth G-manifold, then its tangent vector bundle TX is a G-vector bundle over X by the induced G-action, and its class $[TX]$ is an element of $KO_G(X)$.

LEMMA 4.4. *Let $a = (a_0, a_1, \cdots, a_m)$, $b = (b_0, b_1, \cdots, b_n)$ be sequences of positive integers. Then*

$$[T(S^{2m+1}(a) \times S^{2n+1}(b))] = r(x^{a_0} + \cdots + x^{a_m} + x^{b_0} + \cdots + x^{b_n} - 1)$$

in $KO_{S^1}(S^{2m+1}(a) \times S^{2n+1}(b))$. Here the element x is the class of the complex S^1-line bundle

$$S^{2m+1}(a) \times S^{2n+1}(b) \times C^1(1) .$$

PROOF. This follows from the fact that the tangent S^1-vector bundle of $C^{m+1}(a) \times C^{n+1}(b)$ restricted to $S^{2m+1}(a) \times S^{2n+1}(b)$ is the real restriction of the complex S^1-vector bundle

$$S^{2m+1}(a) \times S^{2n+1}(b) \times C^{m+1}(a) \times C^{n+1}(b)$$

and the normal S^1-vector bundle of the embedding of $S^{2m+1}(a) \times S^{2n+1}(b)$ in $C^{m+1}(a) \times C^{n+1}(b)$ is trivial 2-plane bundle. q.e.d.

LEMMA 4.5. *Let a_i, b_j, c_i, d_j be positive integers. Assume $0 \leqq m \leqq n$. If*

$$S^{2m+1}(a_0, a_1, \cdots, a_m) \times S^{2n+1}(b_0, b_1, \cdots, b_n)$$

is equivariantly diffeomorphic to

$$S^{2m+1}(c_0, c_1, \cdots, c_m) \times S^{2n+1}(d_0, d_1, \cdots, d_n) ,$$

then

$$\sum_{i=0}^{m} (x^{a_i} + x^{-a_i}) + \sum_{j=0}^{n} (x^{b_j} + x^{-b_j}) = \sum_{i=0}^{m} (x^{c_i} + x^{-c_i}) + \sum_{j=0}^{n} (x^{d_j} + x^{-d_j})$$

in the ring $Z[x]/\{(1-x)^{m+1}\}$.

PROOF. Let

$$f: S^{2m+1}(a) \times S^{2n+1}(b) \longrightarrow S^{2m+1}(c) \times S^{2n+1}(d)$$

be an equivariant diffeomorphism. Then we have a commutative diagram:

$$\begin{tikzcd} KO_{S^1}(S^{2m+1}(c) \times S^{2n+1}(d)) \arrow[r, "c"] \arrow[d, "\cong"', "f^*"] & K_{S^1}(S^{2m+1}(c) \times S^{2n+1}(d)) \arrow[d, "\cong"', "f^*"] \\ KO_{S^1}(S^{2m+1}(a) \times S^{2n+1}(b)) \arrow[r, "c"] & K_{S^1}(S^{2m+1}(a) \times S^{2n+1}(b)) . \end{tikzcd}$$

The similar argument as in the proof of Theorem 1.3 shows that

$$\sum_{i=0}^{m} (x^{a_i} + x^{-a_i}) + \sum_{j=0}^{n} (x^{b_j} + x^{-b_j}) = \sum_{i=0}^{m} (x^{c_i} + x^{-c_i}) + \sum_{j=0}^{n} (x^{d_j} + x^{-d_j})$$

in the ring $Z[x]/\{\prod_{i=0}^{m} (1 - x^{a_i}), \prod_{j=0}^{n} (1 - x^{b_j})\}$, from (1.2), (4.3) and (4.4). Since $0 \leqq m \leqq n$, the ideal $\{(1 - x)^{m+1}\}$ contains the ideal

$$\left\{\prod_{i=0}^{m} (1 - x^{a_i}), \prod_{j=0}^{n} (1 - x^{b_j})\right\} .$$

Thus we have the equation in the ring $Z[x]/\{(1 - x)^{m+1}\}$. q.e.d.

PROOF OF THEOREM 4.1. If we define

$$\binom{a}{k} = \frac{a(a-1)(a-2) \cdots (a-k+1)}{k!}$$

for positive integer k, and $\binom{a}{0} = 1$, then we get the equation for positive integer a

$$(1) \qquad (1 + y)^a + (1 + y)^{-a} = \sum_{k \geqq 0} \left\{\binom{a}{k} + \binom{-a}{k}\right\} y^k$$

in $Z[[y]]$, the ring of formal power series with integer coefficients. And there are rational numbers $n(k, p)$ such that

$$(2) \qquad \binom{a}{k} + \binom{-a}{k} = \sum_{p=0}^{k} n(k, p) a^p ,$$

$$n(k, p) = 0 \text{ for odd } p, \ n(k, k) \neq 0 \text{ for even } k .$$

On the other hand, the ring $Z[x]/\{(1 - x)^{m+1}\}$ is a free abelian group with the generators

$$1, x - 1, (x - 1)^2, \cdots, (x - 1)^m .$$

So, if we assume $0 \leqq m \leqq n$, we have the following equations

$$(3) \qquad \sum_{p=0}^{k} n(k, p) \left\{\sum_{i=0}^{m} a_i^p + \sum_{j=0}^{n} b_j^p - \sum_{i=0}^{m} c_i^p - \sum_{j=0}^{n} d_j^p\right\} = 0$$

for $0 \leqq k \leqq m$ by Lemma 4.5 and (1). By induction on p, we have

$$\sum_{i=0}^{m} a_i^{2p} + \sum_{j=0}^{n} b_j^{2p} = \sum_{i=0}^{m} c_i^{2p} + \sum_{j=0}^{n} d_j^{2p}$$

for $0 < 2p \leqq m$, from (2) and (3). This completes the proof of Theorem 4.1.

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