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ON THE COMMUTING EXTENSIONS OF NEARLY NORMAL OPERATORS

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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1. We say that a bounded linear operator A on a Hilbert space H is nearly normal if A commutes with A^*A. Recall that if B is a normal operator on K and H is an invariant subspace for B, then the operator $A = B|H$ is said to be subnormal. If the smallest reducing subspace for B containing H is K, then B is said to be the minimal normal extension of A. This is unique to an isomorphism (cf. [3]). It is well-known that every nearly normal operator is subnormal. The purpose of this paper is to give certain necessary and sufficient conditions under which two nearly normal operators on H have the mutually commuting normal extensions on K.

2. For our purpose, we shall consider the following problem: Given a nearly normal operator A on H, and B, its minimal normal extension on K, when can an operator T on H be extended to an operator T^e on K in such a manner that T^e commutes with B? This problem for general subnormal operators was first solved by Bram [1]. We state here it without proof.

PROPOSITION. *Let A on H be a subnormal operator with the minimal normal extension B on K. Then the necessary and sufficient condition that an operator T on H has an extension T^e on K such that T^e commutes with B is that* (a) *T commutes with A, and* (b) *there exists a positive constant c such that for every finite set $x_0, x_1, \cdots, x_r$ in H we have*

$$\sum_{m,n=0}^{r} \langle A^m Tx_n, A^n Tx_m\rangle \leqq c\cdot \sum_{m,n=0}^{r} \langle A^m x_n, A^n x_m\rangle \ .$$

If the extension T^e exists, it is unique.

Now we state the following lemmas given in [4] without proof.

LEMMA 1. *If A is a nearly normal operator on H and if E is the projection from H on $\mathscr{N}_A = \{x \in H; Ax = 0\}$, then $E \in R(A) \cap R(A)'$ where*

$R(A)$ denotes the smallest von Neumann algebra containing A and $R(A)'$ its commutant.

LEMMA 2. *If A is a nearly normal operator on H such that $\mathcal{N}_A = \{0\}$, then, in the polar decomposition $A = V|A|$ of A, V is an isometry and commutes with $|A|$.*

To prove the following theorem except the norm condition $\|T^e\|_K = \|T\|_H$, we have only to show that the operator T satisfies the condition (b) of proposition.

THEOREM 1. *Let A be a nearly normal operator on H, with the polar decomposition $A = V|A|$, such that $\mathcal{N}_A = \{0\}$, and let B on K be the minimal normal extension of A. Then, if an operator T on H commutes with V and $|A|$, then there exists an extension T^e on K of T such that T^e commutes with B and $\|T^e\|_K = \|T\|_H$. Moreover, the existence of the extension T^e is unique.*

PROOF. By Lemma 2, V is an isometry and commutes with $|A|$. Let U be the minimal unitary extension on K' of V, then, for any finite set $x_0, x_1, \cdots, x_r$ in H, we have

$$\begin{aligned}\sum_{m,n=0}^{r} \langle A^m T x_n, A^n T x_m\rangle_H &= \sum_{m,n=0}^{r} \langle V^m |A|^m T x_n, V^n |A|^n T x_m\rangle_H \\ &= \sum_{m,n=0}^{r} \langle V^m T |A|^n x_n, V^n T |A|^m x_m\rangle_H \\ &= \sum_{m,n=0}^{r} \langle U^m T |A|^n x_n, U^n T |A|^m x_m\rangle_{K'} = \left\| \sum_{n=0}^{r} U^{*n} T |A|^n x_n \right\|_{K'}^2 .\end{aligned}$$

Since, for any fixed non-negative integer k,

$$\begin{aligned}U^{*n} T |A|^n x_n &= U^{*n+k} U^k T |A|^n x_n = U^{*n+k} V^k T |A|^n x_n \\ &= U^{*n+k} T V^k |A|^n x_n = U^{*n+k} T U^k |A|^n x_n = U^{*n+k} T U^{n+k} U^{*n} |A|^n x_n\end{aligned}$$

for all $n = 0, 1, \cdots, r$, we have, by choosing k such as $n + k = r$ for each n,

$$\begin{aligned}\left\| \sum_{n=0}^{r} U^{*n} T |A|^n x_n \right\|_{K'}^2 &= \left\| U^{*r} T U^r \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{K'}^2 \\ &= \left\| T U^r \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{K'}^2 = \left\| T U^r \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{H}^2 \\ &\leqq \|T\|_H^2 \left\| U^r \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{H}^2 = \|T\|_H^2 \left\| U^r \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{K'}^2 \\ &= \|T\|_H^2 \left\| \sum_{n=0}^{r} U^{*n} |A|^n x_n \right\|_{K'}^2 = \|T\|_H^2 \sum_{m,n=0}^{r} \langle A^m x_n, A^n x_m\rangle_H .\end{aligned}$$

Hence, by Proposition, there exists uniquely an extension T^e on K of T such that T^e commutes with B (and hence with B^*). For any finite set $x_0, x_1, \cdots, x_r$ in H, we have

$$T^e \sum_{n=0}^{r} B^{*n} x_n = \sum_{n=0}^{r} B^{*n} T^e x_n = \sum_{n=0}^{r} B^{*n} T x_n$$

and

$$\begin{aligned} \left\| T^e \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 &= \left\| \sum_{n=0}^{r} B^{*n} T x_n \right\|_K^2 = \sum_{m,n=0}^{r} \langle B^m T x_n, B^n T x_m \rangle_K \\ &= \sum_{m,n=0}^{r} \langle A^m T x_n, A^n T x_m \rangle_H \leqq \| T \|_H^2 \sum_{m,n=0}^{r} \langle A^m x_n, A^n x_m \rangle_H \\ &= \| T \|_H^2 \left\| \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 . \end{aligned}$$

Since the linear set $\mathscr{D}$ of vectors of the form

$$\sum_{n=0}^{r} B^{*n} x_n, \quad x_n \in H$$

is dense in K by the minimality of B ($\mathscr{D}$ contains H and reduces B), we have $\| T^e \|_K \leqq \| T \|_H$ and hence $\| T^e \|_K = \| T \|_H$.

REMARK 1. In above theorem, if T is an isometry, then we have

$$\begin{aligned} \left\| T^e \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 &= \sum_{m,n=0}^{r} \langle A^m T x_n, A^n T x_m \rangle_H \\ &= \sum_{m,n=0}^{r} \langle T^* T A^m x_n, A^n x_m \rangle_H = \sum_{m,n=0}^{r} \langle A^m x_n, A^n x_m \rangle_H \\ &= \left\| \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 \end{aligned}$$

and since $\mathscr{D}$ is dense in K, T^e is also an isometry on K. If T is normal, then T^{*e} exists and

$$T^{*e} \sum_{n=0}^{r} B^{*n} x_n = \sum_{n=0}^{r} B^{*n} T^* x_n$$

and hence we have

$$\begin{aligned} \left\langle T^{*e} \sum_{n=0}^{r} B^{*n} x_n, \sum_{m=0}^{s} B^{*m} y_m \right\rangle_K &= \left\langle \sum_{n=0}^{r} B^{*n} T^* x_n, \sum_{m=0}^{s} B^{*m} y_m \right\rangle_K \\ &= \sum_{n=0}^{r} \sum_{m=0}^{s} \langle A^m T^* x_n, A^n y_m \rangle_H = \sum_{n=0}^{r} \sum_{m=0}^{s} \langle A^m x_n, A^n T y_m \rangle_H \\ &= \left\langle \sum_{n=0}^{r} B^{*n} x_n, \sum_{m=0}^{s} B^{*m} T y_m \right\rangle_K = \left\langle \sum_{n=0}^{r} B^{*n} x_n, T^e \sum_{m=0}^{s} B^{*m} y_m \right\rangle_K \\ &= \left\langle T^{e*} \sum_{n=0}^{r} B^{*n} x_n, \sum_{m=0}^{s} B^{*m} y_m \right\rangle_K . \end{aligned}$$

Since $\mathscr{D}$ is dense in K, $T^{e*} = T^{*e}$ and hence we have

$$\left\| T^{e*} \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 = \left\| \sum_{n=0}^{r} B^{*n} T^* x_n \right\|_K^2 = \sum_{m,n=0}^{r} \langle A^m T^* x_n, A^n T^* x_m \rangle_H$$
$$= \sum_{m,n=0}^{r} \langle A^m T x_n, A^n T x_m \rangle_H = \left\| \sum_{n=0}^{r} B^{*n} T x_n \right\|_K^2 = \left\| T^e \sum_{n=0}^{r} B^{*n} x_n \right\|_K^2 .$$

Therefore T^e is also normal. If T is a non-negative self-adjoint operator, then we have

$$\left\langle T^e \sum_{n=0}^{r} B^{*n} x_n, \sum_{n=0}^{r} B^{*n} x_n \right\rangle_K = \left\langle \sum_{n=0}^{r} B^{*n} T x_n, \sum_{n=0}^{r} B^{*n} x_n \right\rangle_K$$
$$= \sum_{m,n=0}^{r} \langle A^m T x_n, A^n x_m \rangle_H = \sum_{m,n=0}^{r} \langle A^m T^{1/2} x_n, A^n T^{1/2} x_m \rangle_H$$
$$= \left\| \sum_{n=0}^{r} B^{*n} T^{1/2} x_n \right\|_K^2 \geqq 0$$

and hence T^e is also a non-negative self-adjoint operator.

As an application of Theorem 1, we have the following.

THEOREM 2. *Let A be a nearly normal operator on H with the polar decomposition $A = V|A|$ such that $\mathscr{N}_A = \{0\}$ and let B on K be the minimal normal extension of A with the polar decomposition $B = U|B|$. Then U is unitary and $V = U|H$ and $|A| = |B| \mid H$.*

PROOF. Since, by Lemma 2, A, V and $|A|$ satisfy the condition of Theorem 1, A, V and $|A|$ have extensions A^e, V^e and $|A|^e$ respectively such that they commute with B. Moreover, by Remark 1, V^e is isometric and $|A|^e$ is non-negative self-adjoint. For any $x \in H$, we have

$$V^e |A|^e x = V^e |A| x = V|A| x = Ax = A^e x = Bx = |A| V x$$
$$= |A|^e V x = |A|^e V^e x .$$

Therefore each of $V^e|A|^e$, A^e, B and $|A|^e V^e$ is an extension on K of A and commutes with B. Hence, by the uniqueness of the extension of Theorem 1, we have

$$B = A^e = V^e |A|^e = |A|^e V^e .$$

From this, we have easily

$$|B| = |A^e| = |A|^e .$$

If $Bx = 0$, $x = x_1 \oplus x_2$, $x_1 \in H$, $x_2 \in K \ominus H$, then

$$0 = \|Bx\|_K^2 = \|V^e |A|^e x\|_K^2 = \| |A|^e x\|_K^2 = \| |A|^e x_1 \oplus |A|^e x_2\|_K^2$$
$$= \| |A|^e x_1\|_K^2 + \| |A|^e x_2\|_K^2 = \| |A| x_1\|_H^2 + \| |A|^e x_2\|_K^2$$
$$= \|V|A| x_1\|_H^2 + \| |A|^e x_2\|_K^2 = \|Ax_1\|_H^2 + \| |A|^e x_2\|_K^2$$

and hence $Ax_1 = 0$. Since $\mathcal{N}_A = \{0\}$, $x_1 = 0$. This implies $\mathcal{N}_B \subset K \ominus H$. Clearly $\mathcal{N}_B$ is a reducing subspace for B and, by the minimality of B, we have $\mathcal{N}_B = \{0\}$. Therefore $U = V^e$ and U is unitary.

By Theorem 2, we can prove the following converse assertion of Theorem 1.

THEOREM 3. *Let A be a nearly normal operator on H with the polar decomposition $A = V|A|$ such that $\mathcal{N}_A = \{0\}$ and let B on K be the minimal normal extension of A. Then, if an operator T on H has an extension T^e on K such that T^e commutes with B, then T commutes with V and $|A|$.*

PROOF. Let $B = U|B|$ be the polar decomposition of B. Then $T^eB = BT^e$ implies that T^e commutes with U and $|B|$ because B is normal. Since, by Theorem 2, $V = U|H$ and $|A| = |B|\,|H$, for any $x \in H$, we have

$$TVx = T^eVx = T^eUx = UT^ex = VTx$$

and

$$T|A|x = T^e|A|x = T^e|B|x = |B|T^ex = |A|Tx\,.$$

Therefore T commutes with V and $|A|$.

By Lemma 1 and by the above theorems, we have the following.

THEOREM 4. *Let A be a nearly normal operator on H with the polar decomposition $A = V|A|$ and let B on K be the minimal normal extension of A. Then a necessary and sufficient condition that an operator T on H has an extension T^e on K such that T^e commutes with B and $\|T^e\|_K = \|T\|_H$ is that T commutes with V and $|A|$. If the extension T^e exists, it is unique.*

PROOF. Let E be the projection from H on $\mathcal{N}_A$, then, by Lemma 1, $E \in R(A) \cap R(A)'$. Since $\mathcal{N}_A = \mathcal{N}_{|A|}$, we have

$$A = A_1 \oplus 0, \quad |A| = |A_1| \oplus 0 \quad \text{and} \quad V = V_1 \oplus 0$$

on

$$H = (I - E)H \oplus EH$$

and V_1 is an isometry, $\mathcal{N}_{|A_1|} = \{0\}$, $E \in R(|A|)$ and hence $A_1 = V_1|A_1|$ is the polar decomposition of A_1. Since, clearly, A_1 is nearly normal and since $\mathcal{N}_{A_1} = \mathcal{N}_{|A_1|} = \{0\}$, if B_1 is the minimal normal extension on K_1 of A_1, then, by the same reason as in the proof of Theorem 2, $\mathcal{N}_{B_1} = \{0\}$, and hence, by the minimality of B, $\mathcal{N}_B = \mathcal{N}_A$, that is,

$$B = B_1 \oplus 0 \quad \text{on} \quad K = K_1 \oplus EH\,.$$

Let F be the projection from K on $\mathscr{N}_B$, then clearly $F \in R(B)$.

Sufficiency: If T commutes with V and $|A|$, then we have

$$T = T_1 \oplus T_2 \quad \text{on} \quad H = (I - E)H \oplus EH$$

because $E \in R(|A|)$ and hence T_1 commutes with V_1 and $|A_1|$. By Theorem 1, T_1 has the unique extension T_1^e on K_1 such that T_1^e commutes with B_1 and $\|T_1^e\|_{K_1} = \|T_1\|_{(I-E)H}$. Therefore

$$T^e = T_1^e \oplus T_2 \quad \text{on} \quad K = K_1 \oplus EH$$

is the required unique extension of T which commutes with B and $\|T^e\|_K = \|T\|_H$.

Necessity: If T^e on K is the extension of T on H which commutes with B, then we have

$$T^e = T^{e\prime} \oplus T^{e\prime\prime} \quad \text{on} \quad K = K_1 \oplus EH$$

because $FK = EH$ and $F \in R(B)$ and hence $T^{e\prime}$ commutes with B_1. Hence, by Theorem 3, $T' = T^{e\prime} | (I - E)H$ commutes with V_1 and $|A_1|$ and hence

$$T' \oplus T^{e\prime\prime} \quad \text{on} \quad H = (I - E)H \oplus EH$$

commutes with V and $|A|$. Easily we have $T = T' \oplus T^{e\prime\prime}$ which completes the proof.

As a special case of Theorem 4, we have the following.

COROLLARY 1. *Let V be an isometry on H with the minimal unitary extension U on K. Then a necessary and sufficient condition that an operator T on H has an extension T^e on K such that T^e commutes with U and $\|T^e\|_K = \|T\|_H$ is that T commutes with V. If the extension T^e exists, it is unique.*

REMARK 2. In [2], Douglas also proved this corollary as an application of the result of Sz.-Nagy and Foias concerning the co-isometric extensions of contractions.

As an application of Theorem 4, we have the following.

COROLLARY 2. *Let A be a nearly normal operator on H with the polar decomposition $A = V|A|$ and let W be an isometry on H. Then, if W commutes with A, then W commutes with V and $|A|$ also.*

PROOF. Let B be the minimal normal extension on K of A. Then, since W is an isometry and since W commutes with A, we can easily show that W satisfies the conditions of proposition. And hence W has the unique extension W^e on K which commutes with B. Therefore, by Theorem 4, W commutes with V and $|A|$.

3. Our main theorem in this paper is the following.

THEOREM 5. *Let A_i, $i = 1, 2$ be nearly normal operators on H with the polar decompositions $A_i = V_i |A_i|$, $i = 1, 2$, respectively. Then the necessary and sufficient condition under which A_i, $i = 1, 2$ have the mutually commuting normal extensions C_i, $i = 1, 2$, respectively, acting on the minimal extension space K in the sense that K is the smallest space which contains H and reduces C_i, $i = 1, 2$, respectively, is that* (a) *V_2 commutes with V_1 and $|A_1|$, and* (b) *$|A_2|$ commutes with V_1 and $|A_1|$.*

PROOF. Necessity: For each $i = 1, 2$, let $C_i = W_i |C_i|$ be the polar decomposition of C_i and let B_i on K_i be the minimal normal extension of A_i with the polar decomposition $B_i = U_i |B_i|$. Then we have

$$C_i = B_i \oplus B_i', \quad W_i = U_i \oplus U_i' \quad \text{and} \quad |C_i| = |B_i| \oplus |B_i'|$$

on

$$K = K_i \oplus (K \ominus K_i) .$$

Since, by the proof of Theorem 4,

$$A_i | H \ominus \mathcal{N}_{A_i} = (V_i | H \ominus \mathcal{N}_{A_i})(|A_i| \ | H \ominus \mathcal{N}_{A_i})$$

is the polar decomposition of $A_i | H \ominus \mathcal{N}_{A_i}$ and $B_i | K_i \ominus \mathcal{N}_{A_i}$ is the minimal normal extension of $A_i | H \ominus \mathcal{N}_{A_i}$, by Theorem 2, we have

$$V_i | H \ominus \mathcal{N}_{A_i} = (U_i | K_i \ominus \mathcal{N}_{A_i}) | H \ominus \mathcal{N}_{A_i}$$

and

$$|A_i| \ | H \ominus \mathcal{N}_{A_i} = (|B_i| \ | K_i \ominus \mathcal{N}_{A_i}) | H \ominus \mathcal{N}_{A_i} .$$

Hence we have

$$V_i = U_i | H = W_i | H \quad \text{and} \quad |A_i| = |B_i| \ | H = |C_i| \ | H .$$

Since $C_1 C_2 = C_2 C_1$, we have

$$W_2 W_1 = W_1 W_2, \quad W_2 |C_1| = |C_1| W_2, \quad |C_2| W_1 = W_1 |C_2|$$

and

$$|C_2| |C_1| = |C_1| |C_2| .$$

Therefore, for any $x \in H$, we have

$$V_2 V_1 x = W_2 V_1 x = W_2 W_1 x = W_1 W_2 x = V_1 V_2 x ,$$
$$V_2 |A_1| x = W_2 |C_1| x = |C_1| W_2 x = |A_1| V_2 x ,$$
$$|A_2| V_1 x = |C_2| W_1 x = W_1 |C_2| x = V_1 |A_2| x$$

and

$$|A_2| |A_1| x = |C_2| |C_1| x = |C_1| |C_2| x = |A_1| |A_2| x .$$

Sufficiency: Let B_1 on K_1 be the minimal normal extension of A_1. Then, by Theorem 4, A_2, V_2 and $|A_2|$ have the unique extensions A_2^e, V_2^e and $|A_2|^e$ on K_1, respectively, such that they commute with B_1. For each $i = 1, 2$, let E_i be the projection from H on $\mathscr{N}_{A_i}$, then we have easily $\mathscr{N}_{A_i} = \mathscr{N}_{|A_i|}$ and $E_i \in R(|A_i|)$. Since $|A_1||A_2| = |A_2||A_1|$, we have $E_1E_2 = E_2E_1$ and hence, by the hypothesis and by Lemmas 1 and 2, we have

$$A_1 = A_1' \oplus A_1'' \oplus 0 \oplus 0, \quad V_1 = V_1' \oplus V_1'' \oplus 0 \oplus 0\,,$$
$$|A_1| = |A_1'| \oplus |A_1''| \oplus 0 \oplus 0, \quad A_2 = A_2' \oplus 0 \oplus A_2''' \oplus 0\,,$$
$$V_2 = V_2' \oplus 0 \oplus V_2''' \oplus 0 \quad \text{and} \quad |A_2| = |A_2'| \oplus 0 \oplus |A_2'''| \oplus 0$$

on

$$H = (I - E_1)(I - E_2)H \oplus (I - E_1)E_2H \oplus E_1(I - E_2)H \oplus E_1E_2H\,.$$

And, by the proof of Theorem 4, we have

$$B_1 = B_1^{(1)} \oplus 0 \quad \text{on} \quad K_1 = K_1^{(1)} \oplus E_1H\,,$$

where $B_1^{(1)}$ is the minimal normal extension of $A_1' \oplus A_1''$. Therefore $(I - E_1)(I - E_2)H$ and $(I - E_1)E_2H$ are invariant subspaces of $B_1^{(1)}$. These imply that

$$K_1' = \vee\{B_1^{(1)*n}x;\quad x \in (I - E_1)(I - E_2)H, \quad n \geqq 0\}$$

is a reducing subspace of $B_1^{(1)}$ and $(I - E_1)E_2H \subset K_1^{(1)} \ominus K_1'$. And hence, we have

$$B_1^{(1)} = B_1' \oplus B_1'' \quad \text{on} \quad K_1^{(1)} = K_1' \oplus K_1''$$

and, by the construction of K_1' and by the minimality of $B_1^{(1)}$, B_1' and B_1'' are the minimal normal extensions of A_1' and A_1'', respectively. Therefore, by Theorem 1 and by the uniqueness of the extension of Theorem 4, we have

$$A_2^e = A_2'^e \oplus 0 \oplus (A_2''' \oplus 0), \quad V_2^e = V_2'^e \oplus 0 \oplus (V_2''' \oplus 0)$$

and

$$|A_2|^e = |A_2'|^e \oplus 0 \oplus (|A_2'''| \oplus 0) \quad \text{on} \quad K_1 = K_1' \oplus K_1'' \oplus E_1H$$

and, by Remark 1, $V_2'^e$ is isometric and $|A_2'|^e$ is non-negative self-adjoint. For any $x \in H$, we have

$$V_2^e|A_2|^e x = V_2^e|A_2|x = V_2|A_2|x = A_2x = |A_2|V_2x = |A_2|^e V_2^e x$$

because, by Lemma 2, $V_2' \oplus V_2'''$ commutes with $|A_2'| \oplus |A_2'''|$. Hence $V_2^e|A_2|^e$, $|A_2|^e V_2^e$ and A_2^e are extensions on K_1 of A_2 and clearly they commute with B_1. Therefore, by the uniqueness of the extension of Theorem 4, we have

$$A_2^e = V_2^e |A_2|^e = |A_2|^e V_2^e$$

and hence A_2^e is a nearly normal operator on K_1 with the polar decomposition $A_2^e = V_2^e |A_2|^e$, that is, $|A_2|^e = |A_2^e|$. Let C_2 be the minimal normal extension of A_2^e, acting on K. Then, by Theorem 4, B_1 has the unique extension C_1 on K which commutes with C_2. By the same reason as above, we have

$$C_2 = C_2' \oplus 0 \oplus C_2''' \oplus 0 \quad \text{on} \quad K = K' \oplus K_1'' \oplus K''' \oplus E_1E_2H ,$$

where C_2' and C_2''' are the minimal normal extensions of $A_2'^e$ and A_2''', respectively, and hence we have

$$C_1 = B_1'^e \oplus B_1'' \oplus 0 \oplus 0 \quad \text{on} \quad K = K' \oplus K_1'' \oplus K''' \oplus E_1E_2H .$$

By Remark 1, $B_1'^e$ is normal and hence C_1 is normal. Now we have only to prove the minimality of K. Let L be the smallest subspace containing H which reduces C_1 and C_2. Then we have $K_1 \subset L \subset K$ because K_1 is the minimal normal extension space of A_1 and hence we have

$$C_2 = (C_2 | L) \oplus (C_2 | K \ominus L) \quad \text{on} \quad K = L \oplus (K \ominus L) ,$$

where $C_2 | L$ and $C_2 | K \ominus L$ are normal. For any $x \in K_1$,

$$A_2^e x = C_2 x = (C_2 | L)x .$$

Since C_2 is the minimal normal extension of A_2^e, $L = K$. This completes the proof.

As a special case of Theorem 5, we have the following.

COROLLARY 3. (cf. [2]) *A necessary and sufficient condition under which isometries V_i, $i = 1, 2$ on H have the mutually commuting unitary extensions U_i, $i = 1, 2$, respectively, acting on the minimal extension space K in the sense that K is the smallest space which contains H and reduces U_1 and U_2 is that V_1 commutes with V_2.*

By Corollary 2 and by Theorem 5, we have the following.

COROLLARY 4. *Let A be a nearly normal operator on H and let V be an isometry on H. Then a necessary and sufficient condition under which A and V have the mutually commuting normal extensions B and U (in fact, U is unitary), respectively, acting on the minimal extension space K in the sense that K is the smallest space which contains H and reduces B and U is that A commutes with V.*

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