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# ON THE BORDISM GROUPS OF SEMI-FREE $S^a$-ACTIONS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. **Introduction.** Let $G$ be a compact Lie group and $(X, A)$ a topological $G$-space pair. Let $\tau$ be a fixed $G$-action on $(X, A)$. Then we can define the singular oriented $G$-bordism group $\Omega_*^G(X, A; \tau)$ which forms an equivariant generalized homology theory. In this note we let $G = S^a$ $(a = 1, 3)$ and $\Omega_*^{S^a}(X, A; \tau)$, $\hat{\Omega}_*^{S^a}(X, A; \tau)$ be $S^a$-semi-free, free bordism groups respectively. Then we can obtain the next theorem by applying the result of F. Uchida [3].

THEOREM. *The following triangles are exact*

(a)
$$\begin{tikzcd}[column sep=small] \hat{\Omega}_*^{S^1}(X, A; \tau) \arrow[rr, "i_*"] & & \Omega_*^{S^1}(X, A; \tau) \arrow[dl, "\alpha_*"] \\ & \sum_{k\geq 0} \Omega_{*-2k}(F_\tau \times BU(k), (F_\tau \cap A) \times BU(k)) \arrow[ul, "\partial"] & \end{tikzcd}$$

(b)
$$\begin{tikzcd}[column sep=small] \hat{\Omega}_*^{S^3}(X, A; \tau) \arrow[rr, "i_*"] & & \Omega_*^{S^3}(X, A; \tau) \arrow[dl, "\alpha_*"] \\ & \sum_{k\geq 0} \Omega_{*-4k}(F_\tau \times BSp(k), (F_\tau \cap A) \times BSp(k)) \arrow[ul, "\partial"] & \end{tikzcd},$$

*where $F_\tau$ is the fixed-point set of $\tau$.*

2. **Preliminaries.** Let $G$ be a compact Lie group, and let $(X, A; \tau)$ be a topological space pair with a fixed $G$-action $\tau$. Let $M$ be an oriented differentiable $G$-manifold with boundary. We consider a triple $(M, \phi, f)$, where $\phi: G \times M \to M$ is an orientation preserving differentiable $G$-action and $f: (M, \partial M) \to (X, A)$ a continuous $G$-equivariant map. We call this triple the singular oriented $G$-manifold in a $G$-pair $(X, A; \tau)$. Two such triples $(M_1^n, \phi_1, f_1)$ and $(M_2^n, \phi_2, f_2)$ are $G$-bordant to each other if there is an oriented $G$-manifold $N^{n+1}$ with a $G$-action $\Phi$ such that $\Phi | M_i = \phi_i$ $(i = 1, 2)$ and an equivariant $G$-map $F: N \to X$ such that (1) $\partial N^{n+1}$ contains $M_1^n \cup (-M_2^n)$ as a regular $G$-submanifold with the induced orientation, and (2) $F | M_i^n = f_i$ $(i = 1, 2)$ and $F(\partial N^{n+1} - (M_1^n \cup M_2^n)) \subset A$. This relation is an

equivalence relation. We denote $G$-bordant class of $(M^n, \phi, f)$ by $[M^n, \phi, f]$. The collection of such classes is denoted by $\Omega_n^G(X, A; \tau)$. Then $\Omega_n^G(X, A; \tau)$ is an abelian group by disjoint union as usual. We call this group the bordism group of $G$-action of dimension $n$. We can naturally define a right $\Omega_*$ module structure on $\Omega_*^G(X, A; \tau) = \sum_{n\geq 0} \Omega_n^G(X, A; \tau)$. Next for an equivariant map $g: (X, A; \tau) \to (X', A'; \tau')$ we define a homomorphism $g_*: \Omega_*^G(X, A; \tau) \to \Omega_*^G(X', A'; \tau')$ by $g_*[M, \phi, f] = [M, \phi, gf]$. And also we define the boundary operator $\partial_n: \Omega_n^G(X, A; \tau) \to \Omega_{n-1}^G(A; \tau)$ by restriction, i.e. $\partial_n[M, \phi, f] = [\partial M, \phi, f]$. Then $\{\Omega_*^G(X, A; \tau), g_*, \partial_*\}$ is an equivariant generalized homology theory.

Now we consider the special case of $G = S^a$ $(a = 1, 3)$ and their actions are free and semi-free. Here a $G$-action $\phi: G \times M \to M$ is called semi-free if the isotropy group $\{g \in G \mid \phi(g, x) = x\}$ consists of the identity element alone for each $x \in M - F$, where $F$ is the fixed-point set of $\phi$. Then we denote the semi-free and free bordism groups by $\Omega_*^{S^a}(X, A; \tau)$ and $\hat{\Omega}_*^{S^a}(X, A; \tau)$ respectively, $(a = 1, 3)$.

**3. A study of $\Omega_*^{S^a}(X, A; \tau)$.** We use essentially the following lemma.

LEMMA (F. Uchida [3]). *Let $\phi: S^a \times M \to M$ $(a = 1, 3)$ be a semi-free differentiable action. Let $F^k$ denote the union of the $k$-dimensional components of the set of all stationary points of $\phi$. Then the normal bundle $\nu_k$ of an embedding $F^k \subset M$ has naturally a complex structure for $a = 1$ and a quaternionic structure for $a = 3$, such that the induced $S^a$-action on $\nu_k$ is a scalar multiplication.*

We now define a homomorphism

$$\alpha_*: \Omega_n^{S^1}(X, A; \tau) \to \sum_{k\geq 0}^{[n/2]} \Omega_{n-2k}(F_\tau \times BU(k),\ (F_\tau \cap A) \times BU(k)) .$$

For given $[M, \phi, f] \in \Omega_n^{S^1}(X, A; \tau)$, let $F_\phi$ be the fixed-point set of $\phi$, and $F_\phi^{n-2k}$ be the union of the $(n-2k)$-dimensional components of $F_\phi$ which is an orientable submanifold of $M$. According to the above lemma, the normal bundle of $F_\phi^{n-2k}$ has a complex structure, so it is a complex $k$-dimensional vector bundle classified by a map $\nu^k: F_\phi^{n-2k} \to BU(k)$.

Then we have a map

$$(f \mid_{F_\phi^{n-2k}}) \times \nu^k: (F_\phi^{n-2k}, \partial F_\phi^{n-2k}) \to (F_\tau \times BU(k),\ (F_\tau \cap A) \times BU(k)) .$$

Define $\alpha_*[M, \phi, f] = \sum_{k=0}^{[n/2]} [F_\phi^{n-2k}, (f\mid_{F_\phi^{n-2k}}) \times \nu^k]$, this is a well defined homomorphism, where $F_\tau$ is the fixed-point set of $\tau$. Next we shall define

$$\partial: \sum_{k\geq 0} \Omega_{n-2k}(F_\tau \times BU(k),\ (F_\tau \cap A) \times BU(k)) \to \hat{\Omega}_{n-1}^{S^1}(X, A; \tau) .$$

Let $[M^{n-2k}, f_k] \in \Omega_{n-2k}(F_\tau \times BU(k), (F_\tau \cap A) \times BU(k))$ and let $\pi_2$ be a projection of $F \times BU(k)$ to second factor. Let $\xi^k$ be the complex vector bundle over $M$ induced by $\pi_2 f_k$ from the universal bundle over $BU(k)$. Let $D(\xi^k)$ and $S(\xi^k)$ denote the associated disk bundle and sphere bundle respectively and let $\pi^k \colon E(\xi^k) \to M$ be the projection, where $E(\xi^k)$ is the total space of $\xi^k$. Let $\phi_k \colon E(\xi^k) \times S^1 \to E(\xi^k)$ be the scalar multiplication. Then it acts freely on $S(\xi^k)$. Therefore $[S(\xi^k), \phi_k |_{S(\xi^k)}, \pi_1 f_k \pi^k |_{S(\xi^k)}] \in \hat{\Omega}^{S1}_{n-1}(X, A; \tau)$, where $\pi_1$ is a projection of $F_\tau \times BU(k)$ to first factor. We may then define $\partial(\sum_{k \geqq 0} [M^{n-2k}, f_k]) = \sum_{k \geqq 0} [S(\xi^k), \phi_k |_{S(\xi^k)}, \pi_1 f_k \pi^k |_{S(\xi^k)}]$, this is also a well-defined homomorphism. Let $i_* \colon \hat{\Omega}^{S1}_n(X, A; \tau) \to \Omega^{S1}_n(X, A; \tau)$ be the canonical forgetting homomorphism. We can also define the same type homomorphisms for $S^3$-actions as above, replacing the complex structure of the normal bundle of fixed point set by the quaternionic structure. Then we have the following theorem.

THEOREM. *The following triangles are exact*

(a)
$$\begin{tikzcd}[column sep=small]
\hat{\Omega}^{S1}_*(X, A; \tau) \arrow[rr, "i_*"] & & \Omega^{S1}_*(X, A; \tau) \arrow[dl, "\alpha_*"] \\
& \sum_{k \geqq 0} \Omega_{*-2k}(F_\tau \times BU(k), (F_\tau \cap A) \times BU(k)) \arrow[ul, "\partial"] &
\end{tikzcd}$$

(b)
$$\begin{tikzcd}[column sep=small]
\hat{\Omega}^{S3}_*(X, A; \tau) \arrow[rr, "i_*"] & & \Omega^{S3}_*(X, A; \tau) \arrow[dl, "\alpha_*"] \\
& \sum_{k \geqq 0} \Omega_{*-4k}(F_\tau \times BSp(k), (F_\tau \cap A) \times BSp(k)) \arrow[ul, "\partial"] &
\end{tikzcd}$$
,

*where $F_\tau$ is the fixed-point set of $\tau$.*

PROOF. (a) We can easy to see that $i_* \partial = 0$, $\alpha_* i_* = 0$ and $\partial \alpha_* = 0$. To prove $\operatorname{Ker} \alpha_* \subset \operatorname{Im} i_*$. Let $[M, \phi, f] \in \Omega^{S1}_n(X, A; \tau)$ with $\alpha_*[M, \phi, f] = 0$. Then $\sum_{k \geqq 0} [F^{n-2k}_\phi, (f |_{F^{n-2k}_\phi}) \times \nu^k] = 0$. So there exists $(V^{n-2k+1}, f')$ such that $\partial V \supset F^{n-2k}_\phi$, $f' \colon (V, \partial V \backslash F^{n-2k}_\phi) \to (F_\tau, F_\tau \cap A)$, $f' \mid F^{n-2k}_\phi = f \mid F^{n-2k}_\phi$, and exists a complex $k$-vector bundle $\xi^k$ over $V$ such that $\xi \mid F^{n-2k}_\phi = \nu^k$. Let $W = M \times I \cup \bigcup_{k \geqq 0} D(\xi^k)$, where we identify each $D(\nu^k)$ to $D(\xi^k) \mid F^{n-2k}_\phi$ on $M \times 1$. $S^1$ acts on $W$ by $\phi \times 1$ on $M \times I$ and fibre-wise multiplication on $D(\xi^k)$. Let $\pi(\nu^k) \colon D(\nu^k) \to F^{n-2k}_\phi$ be a projection. We define an equivariant map $h \colon W \to X$ by $h \mid M \times I = f'' \pi_1$ and $h \mid D(\xi^k) = f' \pi^k$, where $f''$ is an equivariant homotopic map to $f$ such that $f'' \mid F_\phi = f \mid F_\phi$, $f'' \mid D(\nu^k) = f |_{F^{n-2k}_\phi} \pi(\nu^k)$. Let $M' = (M \backslash \bigcup \operatorname{Int}(D(\nu^k))) \cup (\bigcup S(\xi^k)) / \bigcup \partial D(\nu^k)$. Then $[M', \mu \mid M', h \mid M'] \in \hat{\Omega}^{S1}_n(X, A; \tau)$, and we can easy to see $i_*[M', \mu \mid M', h \mid M'] = [M, \phi, f]$, where $\mu$ is the above $S^1$ action on $W$. To prove $\operatorname{Ker} i_* \subset \operatorname{Im} \partial$.

Let $[M, \phi, f] \in \hat{\Omega}_n^{S^1}(X, A; \tau)$ with $i_*[M, \phi, f] = 0$. Then there exists $(V^{n+1}, \mu, g)$ such that $\partial V \supset M$, $g: V \to X$, $g \mid M = f$, $g(\partial V \backslash M) \subset A$ and $\mu: S^1 \times V \to V$ is semi-free $S^1$ action with $\mu \mid M = \phi$. So $\mu$ is free on $M$. Therefore $M$ is disjoint to the fixed-point set of $\mu$, which denotes $F = \bigcup F^{n+1-2k}$. Then $\partial[\bigcup (F^{n+1-2k}, (g \mid F^{n+1-2k}) \times \nu^k)] = [M, \phi, f]$. To prove $\operatorname{Ker} \partial \subset \operatorname{Im} \alpha_*$. Let $[\bigcup (M^{n-2k}, g_k)] \in \sum \Omega_{n-2k}(F_\tau \times BU(k), (F_\tau \cap A) \times BU(k))$ with $\partial[\bigcup (M^{n-2k}, g_k)] = 0$. Then $[\bigcup S(\xi^k), \bigcup \phi_k, \bigcup \pi_1 g_k \pi^k \mid S(\xi^k)] = 0$. So there exists $(N, \mu, h)$ such that $\partial N \supset \bigcup S(\xi^k)$, $h \mid \bigcup S(\xi^k) = \bigcup \pi_1 g_k \pi^k \mid S(\xi^k)$, and $\mu$ is free $S^1$ action on $N$ with $\mu \mid \bigcup S(\xi^k) = \bigcup \phi_k$. Let $W = N \cup (\bigcup D(\xi^k))/ \bigcup S(\xi^k)$. We define $S^1$ action $\phi$ on $W$ by $\phi \mid N = \mu$, $\phi \mid D(\xi^k) = \phi_k$, and let $f: (W, \partial W) \to (X, A)$ be $f \mid N = h$, $f \mid \bigcup D(\xi^k) = \bigcup \pi_1 g_k \pi^k \mid D(\xi^k)$. Then fixed-point set of $\phi$ is $\bigcup M^{n-2k}$, so $\alpha_*[M, \phi, f] = [\bigcup M^{n-2k}, \bigcup g_k]$. We conclude the proof of (a). Case (b) can be proved the same way as (a). q.e.d.

PROPOSITION 1.

$$\Omega_*^{S^a}(X, A; 1) \cong \Omega_*^{S^a}(pt; 1) \,{}_{\Omega_*}\!\otimes \Omega_*(X, A), \quad for \quad a = 1, 3\,.$$

PROPOSITION 2. *The sequence*

$$0 \longrightarrow \Omega_n^{S^a}(F_\tau; \tau) \xrightarrow{i_*} \Omega_n^{S^a}(X; \tau) \xrightarrow{j_*} \Omega_n^{S^a}(X, F_\tau; \tau) \longrightarrow 0$$

*is split exact sequence for $a = 1$ or 3.*

These Propositions are proved by the same way as [2], replacing involutions and unorientedness by $S^a$ action and orientedness, so we omit the proofs.

4. **The Smith homomorphism.** Let $[M^n, \phi, f] \in \hat{\Omega}_n^{S^1}(X, A; \tau)$ and $2N + 1 > n$, then there exists an equivariant differentiable map $g: (M^n, \phi) \to (S^{2N+1}, \rho_1)$ which is transverse regular on $S^{2N-1} \subset S^{2N+1}$, where $\rho_1$ is $S^1$ action. Let $V^{n-2} = g^{-1}(S^{2N-1})$. The Smith homomorphism $\Delta: \hat{\Omega}_n^{S^1}(X, A; \tau) \to \hat{\Omega}_{n-2}^{S^1}(X, A; \tau)$ is defined by $\Delta[M^n, \phi, f] = [V^{n-2}, \phi \mid V, f \mid V]$ (cf. [1], § 26). Similarly let $[M^n, \phi, f] \in \hat{\Omega}_n^{S^3}(X, A; \tau)$ and $4N + 3 > n$, then there exists an equivariant differentiable map $g: (M^n, \phi) \to (S^{4N+3}, \rho_3)$ which is transverse regular on $S^{4N-1} \subset S^{4N+1}$, where $\rho_3$ is $S^3$ action. Let $V^{n-4} = g^{-1}(S^{4N-1})$. The Smith homomorphism $\Delta: \hat{\Omega}_n^{S^3}(X, A; \tau) \to \hat{\Omega}_{n-4}^{S^3}(X, A; \tau)$ is defined by $\Delta[M^n, \phi, f] = [V^{n-4}, \phi \mid V, f \mid V]$. Then we can obtain the following proposition.

PROPOSITION 3. *The sequence*

$$\cdots \longrightarrow \hat{\Omega}_{n+1}^{S^a}(X \times S^a, A \times S^a; \tau \times \rho_a) \xrightarrow{\pi_*} \hat{\Omega}_{n+1}^{S^a}(X, A; \tau) \longrightarrow \hat{\Omega}_{n-a}^{S^a}(X, A; \tau)$$
$$\xrightarrow{1 \times \rho_{a*}} \hat{\Omega}_n^{S^a}(X \times S^a, A \times S^a; \tau \times \rho_a) \xrightarrow{\pi_*} \cdots$$

*is exact, where* $\pi_*$ *is induced by the projection* $\pi: X \times S^a \to X$, *and* $(1 \times \rho_{a*})[M^{n-a}, \phi, f] = [M \times S^a, \phi \times \rho_a, f \times 1]$, *and* $a = 1$ *or* 3.

This exact sequence is same type as Wu's [4]. So the proof is an obvious repetition of the proof in [4], replacing $Z_p$ action by $S^a$ action and taking care of dimensions of spheres, disks and so on.

## REFERENCES

[1] P. E. CONNER AND E. E. FLOYD, Differentiable Periodic Maps, Springer-Verlag, Berlin 1964.

[2] R. E. STONG, Bordism and involutions, Ann. of Math. 90 (1969), 47–74.

[3] F. UCHIDA, Cobordism groups of semi-free $S^1$-and $S^3$-actions, Osaka J. of Math. 7 (1970), 345–351.

[4] C.-M. WU, Bordism and maps of odd prime period, Osaka J. of Math. 8 (1971), 405–424.

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