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3-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH DENSE ORBITS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Let M be an n-dimensional connected differentiable Riemannian manifold $(n > 1)$ admitting an intransitive effective connected Lie group H of isometries on M. (The word "differentiable" means "of class C^∞".) For each $p \in M$, the differentiable submanifold $H(p) = \{h(p) \mid h \in H\}$ is usually called an H-orbit. Let $I(M)$ denote the Lie group of all isometries on M and $I_0(M)$ its identity component. The group H can be regarded as an analytic subgroup of $I_0(M)$ and the closure $\bar{H}$ (in $(I_0(M))$ of H forms a subgroup which is a connected Lie group. The closure of an H-orbit consists of one point or has the structure of a regularly imbedded connected differentiable submanifold (cf. [2]). This follows from the fact that the closure $\overline{H(p)}$, $p \in M$, coincides with the $\bar{H}$-orbit through p, i.e., $\overline{H(p)} = \bar{H}(p)$. We call such a manifold $\overline{H(p)}$ the *closure manifold* of $H(p)$. For any $q \in \overline{H(p)}$, we can see $\overline{H(p)} = \overline{H(q)}$.

In the following, *suppose an H-orbit regarded as a subset of M is dense in M.* Then $I_0(M)$ acts on M transitively and hence it is shown that M is complete. The following two theorems have been already proved (see [3]):

THEOREM 1. 1) *Every H-orbit is dense in M,*

2) *any element of $\bar{H}$ carries every H-orbit onto an H-orbit, and*

3) *M has the structure of a foliated manifold* (cf. [4]) *with the H-orbits as its leaves.*

THEOREM 2. *The group $\bar{H}$ has a 1-parameter subgroup γ with the following properties: for any $x \in M$,*

1) *$\gamma(x) \subset H(x)$, but the closure manifold $\overline{\gamma(x)}$ (in M) is not included in $H(x)$,*

2) *$\overline{\gamma(x)}$ is homeomorphic to a torus of dimension > 1 and a Euclidean metric is induced from M, and*

3) *$H(x)$ has a structure of product bundle with $\gamma(y)$, $y \in H(x)$, as its fibers.*

The purpose of this note is, on the foundation of the theorems above, to establish the following theorem. From this theorem we may see an intuitive structure of M in connection with γ-orbits.

THEOREM. *Suppose further M is 3-dimensional, compact and orientable. Then,*

1) *M is homeomorphic to a torus,*
2) *the metric on M is locally Euclidean,*
3) *the group $\bar{H}$ is the transitive group of parallel translations on M, and more precisely, M is expressed as one of the Types* I–III.

To interpret the Types above, we shall first define some notations. Let T^m denote an m-dimensional torus with Euclidean metric and σ a 1-parameter group of isometries on T^m such that a σ-orbit is dense in T^m. Then, the group σ is a 1-parameter group of parallel translations on T^m and the σ-orbits are parallel to each other. Let I denote the segment $\{t \mid 0 \leqq t \leqq c\}$, $c > 0$, of straight line. For 3-dimensional Riemannian manifolds with the same structure as M, we define Types as follows:

Type I: Riemannian manifold T^3 with the σ-orbits as its γ-orbits, such that its H-orbits coincide with the γ-orbits.

Type II: Riemannian manifold T^3 with the σ-orbits as its γ-orbits. The H-orbits are defined by 2-dimensional planes (totally geodesic submanifolds), parallel to each other, which contain γ-orbits.

Type III: Riemannian manifold constructed from the metric product $T^2 \times I$ by identifying (x, c) with $(\psi(x), 0)$ for all $x \in T^2$, where ψ denotes a parallel translation of T^2. (This manifold is homeomorphic to a torus.) Here, for each $(x, t) \in T^2 \times I$, the γ-orbit through the point is defined by $(\sigma(x), t)$ and similarly the H-orbit by a plane consisting of the set of γ-orbits intersecting the geodesic through (x, t), parallel to a fixed geodesic which is not contained in a closure manifold of γ-orbit.

Before proving Theorem, we prepare two lemmas. First take up an n-dimensional connected foliated manifold N with complete bundle-like metric, which is also a fiber bundle over a circle C with the leaves as its fibers. Let $L(p)$ denote the leaf through $p \in N$ and L_p the subspace of the tangent space N_p at p, tangent to $L(p)$. Let Γ_p denote the geodesic through $p \in N$, orthogonal to $L(p)$, with the orientation concordant with a fixed one of C, by the canonical projection of N onto C. The geodesic Γ_p intersects orthogonally all the leaves. Let $\Gamma_p(s)$ denote the geodesic Γ_p parametrized by arc-length s, where $\Gamma_p(0) = p$. There exists the smallest positive number c such that $\Gamma_p(c) \in L(p)$, and it is independent of p. For any real number a, let φ_a denote the map of N onto itself defined by

$\varphi_a(x) = \Gamma_x(a)$ for any $x \in N$. The map φ_a carries every leaf onto a leaf. We call such a map a *leaf map*. Particularly, put $\varphi = \varphi_c$.

LEMMA 1. *In N suppose the point set $\{\varphi^\lambda(x)\}$ $(\lambda = 1, 2, \cdots)$, for any fixed $x \in N$, has x as one of its accumulation points if the set is infinite. Then, every isometry in $I_0(N)$ carries every leaf of N onto a leaf.*

PROOF. Suppose Lemma 1 does not hold true. Then there exists an isometry $f \in I_0(N)$ near enough to the identity, which does not carry some leaf onto a leaf. So we have a point $p \in N$ such that $f_* \cdot L_p \neq L_{f(p)}$. Put $\Gamma' = f \cdot \Gamma_p$. The geodesic Γ' passes through $f(p)$ and is orthogonal to $f_* \cdot L_p$, but intersects every leaf obliquely.

1) The case where Γ_p is closed. There is the smallest positive integer m such that $\Gamma_p(mc) = p$. Then the length of Γ_p is equal to mc and so is also that of Γ'. However, it is seen that the length of Γ' is greater than mc, the metric on N being bundle-like. This is obviously a contradiction.

2) The case where Γ_p is non-closed. Put $p_\lambda = \Gamma_p(\lambda c)$, then $p_\lambda \in L(p)$ and $p_\lambda = \varphi^\lambda(p)$. The set $\{p_\lambda\}$ has p as one of its accumulation points by the assumption. Let $\Gamma_{p,\lambda}$ denote the geodesic arc $\Gamma_p(s)$, $0 \leqq s \leqq \lambda c$. Put $\Gamma'_\lambda = f \cdot \Gamma_{p,\lambda}$. We may show that, if we take some integer $\tau > 0$ such that p_τ is near enough to p, then Γ'_τ has longer length than $\Gamma_{p,\tau}$. This contradicts the fact that Γ'_τ and $\Gamma_{p,\tau}$ have the same length.

Accordingly, every isometry in $I_0(N)$ near enough to the identity carries every leaf onto a leaf. We may thus see that Lemma 1 is true.

LEMMA 2. *Suppose N satisfies the following conditions:*

1) *every leaf is homeomorphic to a torus and the induced metric is Euclidean,*

2) *$I_0(N)$ is transitive, and*

3) *$I_0(N)$ has a subgroup G which leaves each leaf invariant and which is there the transitive group of parallel translations.*

Then, N is regarded as a Riemannian manifold constructed from the metric product $T^{n-1} \times I$ by identifying (x, c) with $(\psi(x), 0)$ for all $x \in T^{n-1}$ and for some $\psi \in I_0(T^{n-1})$.

PROOF. For any $p \in N$, we have $g \in G$ by the assumption 3) such that $g(p) = \varphi(p)$. Generally, $g^\lambda(p) = \varphi^\lambda(p)$. This shows that, if the point set $\{\varphi^\lambda(p)\}$ is infinite, the set has p as one of its accumulation points. Accordingly, by Lemma 1 every isometry in $I_0(N)$ carries every leaf onto a leaf.

Next, it is easy to see that, for a 1-parameter group of G, its orbits are geodesics in each leaf and are preserved by any leaf map. We may hence conclude that any leaf map is a projective motion of every leaf onto a leaf. Take up the closed geodesics C_i $(i = 1, 2, \cdots, n-1)$ in $L(p)$ generating the fundamental group $\pi_1(L(p), p)$, then these images by a leaf map φ_a are also closed geodesics in the leaf $L(\varphi_a(p))$. Since, further, there is an isometry in $I_0(N)$ carrying $L(p)$ onto $L(\varphi_a(p))$, we may easily see that a leaf map φ_a carries, isometrically, C_i onto $\varphi_a(C_i)$ and so $L(p)$ onto $L(\varphi_a(p))$. This fact proves Lemma 2, since $I_0(T^{n-1})$ is the transitive group of parallel translations on T^{n-1}.

PROOF OF THEOREM. 1) The case where H-orbits are 1-dimensional. Then, M is of Type I (see [2]) and $I_0(M)$ is the transitive group of parallel translations. We can see $\bar{H} = I_0(M)$ easily.

2) The case where the H-orbits are 2-dimensional and where there exists a γ-orbit whose closure manifold coincides with M. Then, $I_0(M)$ is the same one as in 1) above and similarly we obtain $\bar{H} = I_0(M)$. We can thus see that M is of Type II.

3) The case where the H-orbits are 2-dimensional and where there is no γ-orbit whose closure manifold coincides with M. Then, by Theorem 2, the closure manifold of every γ-orbit is homeomorphic to a 2-dimensional torus and the induced metric is Euclidean. This closure manifold coincides with a $\bar{\gamma}$-orbit in M, where $\bar{\gamma}$ denotes the closure (in $\bar{H}$) of γ. The group $\bar{\gamma}$ must be a toral subgroup of $\bar{H}$. Let $\bar{\gamma}_p$ denote the isotropy subgroup of $\bar{\gamma}$ at $p \in M$. Then, for any $q \in \bar{\gamma}(p)$, we have $\bar{\gamma}_p = \bar{\gamma}_q$. So $\bar{\gamma}_p$ leaves $\bar{\gamma}(p)$ pointwise invariant. Since, moreover, every $\bar{\gamma}$-orbit has dimension 2 and M is orientable, the group $\bar{\gamma}_p$ consists of the identity only. It follows hence that the group $\bar{\gamma}$ has dimension 2 and acts, in each $\bar{\gamma}$-orbit, as the transitive group of parallel translations. Accordingly, M is regarded as a foliated manifold with the $\bar{\gamma}$-orbits as its leaves. And the metric on M is bundle-like and M has the structure of a fiber bundle over a circle, with the $\bar{\gamma}$-orbits as its fibers. Thus M satisfies the conditions in Lemma 2. Therefore, M is expressed as a Riemannian manifold constructed from the metric product $T^2 \times I$ by identifying (x, c) with $(\psi(x), 0)$ for all $x \in T^2$ and for some $\psi \in I_0(T^2)$. Since M reduces to a 3-dimensional torus with Euclidean metric, $I_0(M)$ is the transitive group of parallel translations. We have thus $\bar{H} = I_0(M)$. It is now obvious that M is of Type III.

The conclusions above complete the proof of our Theorem.

REMARK. By using Lemma 2, we may prove the following theorem: An n-dimensional compact connected Lie group G is abelian if and only if G has an $(n-1)$-dimensional abelian analytic subgroup. As the necessity is evident, we shall prove the sufficiency. Let K denote an $(n-1)$-dimensional abelian analytic subgroup of G and $\bar{K}$ the closure (in G) of K. If $\bar{K}=G$, the sufficiency follows immediately. So we consider the case $\bar{K}\neq G$. Then, $\bar{K}=K$ and K is compact. Hence K is a toral subgroup of G. We introduce on G a left invariant Riemannian metric. Since then every left translation on G reduces to an isometry on G, we may also treat G as a group of isometries on the Riemannian manifold G. Every K-orbit coincides with a right coset of K. The foliated manifold G, with the K-orbits as the leaves, satisfies the same condition as N in Lemma 2. Accordingly, the metric on G must be Euclidean and G homeomorphic to a torus. And, the group G coincides with the transitive group of parallel translations on G. This shows that G is abelian. The sufficiency has been thus proved. From this theorem and the previous one, we have: A 3-dimensional compact connected Lie group is abelian if and only if it has a non-closed analytic subgroup.

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