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ERGODIC THEOREMS FOR SEMI-GROUPS IN L_p, $1 < p < \infty$

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1. **Introduction.** In what follows we shall assume p fixed, $1 < p < \infty$. Let $(X, \mathscr{M}, m)$ be a σ-finite measure space and let $\{T_t; t \geqq 0\}$ be a semi-group of positive linear operators in $L_p(X) = L_p(X, \mathscr{M}, m)$ which is strongly integrable over every finite interval. It is then known (cf. [2], p. 686) that for each $f \in L_p(X)$ there exists a scalar function $T_t f(X)$, measurable with respect to the product of Lebesgue measure and m, such that for almost all t, $T_t f(x)$ belongs to the equivalence class of $T_t f$. Moreover there exists a set $E(f)$ with $m(E(f)) = 0$, dependent on f but independent of t, such that if $x \notin E(f)$ then $T_t f(x)$ is integrable on every finite interval $[a, b]$ and the integral $\int_a^b T_t f(x)dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f dt$. We write $S_a^b f(x)$ for $\int_a^b T_t f(x)dt$. The purpose of this note is to investigate the almost everywhere convergence of $S_0^b f(x)/S_0^b g(x)$ and $S_0^b f(x)/b$ as $b \uparrow \infty$.

2. **Preliminaries.** If $A \in \mathscr{M}$ then $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. A set $A \in \mathscr{M}$ is called *closed* under a positive linear operator T on $L_p(X)$ if $f \in L_p(A)$ implies $Tf \in L_p(A)$. The adjoint operator of T is denoted by T^*.

PROPOSITION. *If T is a positive linear operator on $L_p(X)$ such that $\sup_n \|(1/n)\sum_{k=0}^{n-1} T^k\|_p < \infty$ and $\lim_n \|(1/n)T^n f\|_p = 0$ for every $f \in L_p(X)$, then the space X uniquely decomposes into two measurable sets Y and Z such that*

(a) *Z is closed under T,*

(b) *if $f \in L_p(Z)$ then $\lim_n \|(1/n)\sum_{k=0}^{n-1} T^k f\|_p = 0$,*

(c) *there exists a nonnegative function u in $L_q(Y)$ such that $u > 0$ a.e. on Y and $T^*u = u$, where $q = p/(p-1)$.*

PROOF. We may choose a nonnegative function u in $L_q(X)$ such that $T^*u = u$ and if $0 \leqq v \in L_q(X)$ is invariant under T^* then $\operatorname{supp} v \subset \operatorname{supp} u$. Let $Y = \operatorname{supp} u$ and $Z = X - Y$. Since $T^*u = u$, (a) is obvious. To see (b), let $0 \leqq g \in L_p(Z)$. Then the mean ergodic theorem ([2], p. 661) implies that strong-$\lim_n (1/n)\sum_{k=0}^{n-1} T^k g = g_0$ for some $0 \leqq g_0 \in L_p(Z)$ with $Tg_0 = g_0$. Here if we assume that $\|g_0\|_p > 0$, then $\int g_0 v dm > 0$ for some $0 \leqq v \in L_q(X)$. Since the mapping $f \to \lim_n \int((1/n)\sum_{k=0}^{n-1} T^k f) v dm$ is a positive linear functional, there exists a nonnegative function v_0 in $L_q(X)$ such that

$$\lim_n \int\Big(\frac{1}{n}\sum_{k=0}^{n-1} T^k f\Big) v dm = \int f v_0 dm \quad \text{for any} \quad f \in L_p(X)\,.$$

It is then clear that $T^* v_0 = v_0$, and hence supp $v_0 \subset Y$. Therefore $\int g_0 v dm = \int g v_0 dm = 0$. This is a contradiction, and the proof is complete.

COROLLARY 1. *For any $f \in L_p(X)$, the limit*

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x)$$

exists and is finite a.e. on Y.

PROOF. For $uf \in L_1(Y)$, where $f \in L_p(Y)$, define $U(uf) = u(Tf)$. Since $\{uf; f \in L_p(Y)\}$ is a dense subspace of $L_1(Y)$ and $\|U(uf)\|_1 = \|u(Tf)\|_1 = \|(T^*u)f\|_1 = \|uf\|_1$, U may be considered to be a positive linear operator on $L_1(Y)$ with $\|U\|_1 = 1$. Let g be any strictly positive function in $L_p(Y)$, and let strong-$\lim_n (1/n)\sum_{k=0}^{n-1} T^k g = g_0$ for some $g_0 \in L_p(X)$. Then it follows that

$$\lim_n \Big\|\frac{1}{n}\sum_{k=0}^{n-1} U^k(ug) - ug_0\Big\|_1 = 0\,.$$

Hence the ergodic theorem of [5] implies that for any $f \in L_p(Y)$, the limit

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(x) = \frac{1}{u(x)} \lim_n \frac{1}{n}\sum_{k=0}^{n-1} U^k(uf)(x)$$

exists and is finite a.e. on Y. This completes the proof.

COROLLARY 2. *If $f \in L_p(X)$ and $0 \leqq g \in L_p(X)$, then,*
(i) *for each fixed integer j,*

$$\lim_n T^{n+j} f(x) / \sum_{k=0}^{n} T^k g(x) = 0$$

a.e. on $Y \cap \{x; \sum_{k=0}^{\infty} T^k g(x) > 0\}$,
(ii) *the limit*

$$\lim_n \sum_{k=0}^{n} T^k f(x) / \sum_{k=0}^{n} T^k g(x)$$

exists and is finite a.e. on $Y \cap \{x; \sum_{k=0}^{\infty} T^k g(x) > 0\}$.

If there exists a function $0 \leqq g \in L_p(Z)$ such that the set $\{x; \sum_{k=0}^{\infty} T^k g(x) = \infty\}$ is nonnull, then the ratio theorem fails on this set.

PROOF. The first statement of the corollary follows from [1], since

$$\frac{T^{n+j}f(x)}{\sum_{k=0}^{n} T^k g(x)} = \frac{U^{n+j}(uf)(x)}{\sum_{k=0}^{n} U^k(ug)(x)} \qquad \text{a.e.}$$

and

$$\frac{\sum_{k=0}^{n} T^k f(x)}{\sum_{k=0}^{n} T^k g(x)} = \frac{\sum_{k=0}^{n} U^k(uf)(x)}{\sum_{k=0}^{n} U^k(ug)(x)} \qquad \text{a.e.}$$

The second statement follows from the same argument as in [3], p. 77, and we omit the details.

3. Theorems. Let $\{T_t; t \geqq 0\}$ be a semi-group of positive linear operators in $L_p(X)$ strongly integrable over every finite interval, such that

$$\sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T_k \right\|_p < \infty \quad \text{and} \quad \lim_n \left\| \frac{1}{n} T_n f \right\|_p = 0$$

for each $f \in L_p(X)$. Then, by Proposition, the space X uniquely decomposes into two disjoint measurable sets Y and Z such that (a) Z is closed under T_1, (b) if $f \in L_p(Z)$ then $\lim_n \|(1/n) \sum_{k=0}^{n-1} T_k f\|_p = 0$, and (c) there exists a nonnegative function u in $L_q(Y)$ with $u > 0$ a.e. on Y and $T_1^* u = u$. The main results of this note are the following two individual ergodic theorems.

THEOREM 1. *For any $f \in L_p(X)$, the limit*

$$\lim_{b \uparrow \infty} S_0^b f(x)/b$$

exists and is finite a.e. on Y.

THEOREM 2. *If $f \in L_p(X)$ and $0 \leqq g \in L_p(X)$, then the limit*

$$\lim_{b \uparrow \infty} S_0^b f(x)/S_0^b g(x)$$

exists and is finite a.e. on $Y \cap \{x; S_0^\infty g(x) > 0\}$.

PROOF OF THEOREM 1. We may assume that f is nonnegative. Let $f' = \int_0^1 T_t f dt$. For each $b > 0$, write $b = n + r$, where $n = [b]$ and $0 \leqq r < 1$. Then, as in [2], p. 688, we have

$$S_0^b f(x)/b = \frac{n}{b}\left(\frac{1}{n}\sum_{k=0}^{n-1} T_k f'(x) + \frac{1}{n} T_n\left(\int_0^r T_t f dt\right)(x)\right) \qquad \text{a.e.} .$$

Since $0 \leqq \int_0^r T_t f dt \leqq \int_0^1 T_t f dt = f' \in L_p(X)$, it follows from Corollary 1 that

$$0 \leqq \lim_n \frac{1}{n} T_n\left(\int_0^r T_t f dt\right)(x) \leqq \lim_n \frac{1}{n} T_n f'(x) = 0 \qquad \text{a.e.}$$

on Y and uniformly on the interval $0 \leqq r \leqq 1$. Hence Corollary 1 completes the proof.

PROOF OF THEOREM 2. We may assume that f is nonnegative. Then, as in [4], p. 660, we have

$$\frac{\sum_{k=0}^{n-1} T_k f'(x)}{\sum_{k=0}^{n} T_k g'(x)} \leqq \frac{S_0^b f(x)}{S_0^b g(x)} \leqq \frac{\sum_{k=0}^{n} T_k f'(x)}{\sum_{k=0}^{n-1} T_k g'(x)} \qquad \text{a.e.} .$$

Since the first and last terms of the above formula converge to the same finite limit on the set $Y \cap \{x; S_0^\infty g(x) > 0\}$ by Corollary 2, the proof is complete.

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