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GENERATORS AND MAXIMAL IDEALS IN SOME ALGEBRAS OF HOLOMORPHIC FUNCTIONS

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1. **Introduction.** Let D be the unit disk $\{|z| < 1\}$. A holomorphic function $f(z)$ in D is said to belong to the *class N of functions of bounded characteristic* if

$$(1.1) \qquad T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = O(1) \text{ as } r \to 1\,.$$

A function $f(z) \in N$ is said to belong to the class N^+ [2, p. 25] if

$$(1.2) \qquad \lim_{r\to 1}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta\,.$$

The class N^+ can be considered as an F-space in the sense of Banach [1, p. 51], with the metric [9]

$$(1.3) \qquad \rho(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|)\, d\theta \ \text{ for } f, g \in N^+\,.$$

N^+ is easily seen to be a topological algebra with respect to this metric (1.3). N^+ is neither locally convex nor locally bounded, but has sufficiently many continuous linear functionals to form a dual system $\langle (N^+)^*, N^+\rangle$ in the sense of Dieudonné and Mackey [7, p. 88].

On the other hand, we defined a Fréchet space F^+ which contains N^+ [9]. We say that a holomorphic function $f(z)$ in D belongs to the class F^+ if

$$(1.4) \qquad M(r, f) = \max_{|z|=r} |f(z)| \leqq K_f \exp[\omega_f(r)/(1-r)]$$

with a constant $K_f > 0$ and a continuous function $\omega_f(r)$, $0 \leqq r < 1$, depending on $f \in F^+$, such that $\omega_f(r) \downarrow 0$ as $r \to 1$. A holomorphic function $f(z) = \sum a_n z^n$ belongs to F^+ if and only if

$$(1.5) \qquad \|f\|_{Fc} = \int_0^1 \exp\left[\frac{-c}{1-r}\right] M(r, f)\,dr < \infty$$

for each $c > 0$. (1.4) is equivalent to

$$(1.6) \qquad a_n = O(\exp[o(\sqrt{n})]) \qquad \text{as } n \to \infty\,.$$

F^+ is a countably normed (locally convex) Fréchet space with the system of (semi-)norms $\{\|\cdot\|_{Fc}\}_{c>0}$. F^+ is the second dual space for the space N^+ [11], and is a nuclear as well as a Montel space [11]. We can easily see that F^+ is a topological algebra.

In this note, we will characterize generators of the algebra F^+, following to the methods of Hörmander [3], Kelleher and Taylor [5], [6]. Although they treat mainly with several variables, we confine here ourselves only to one variable case. Generalizations to several variables are concerns of our further study.

In §§ 5–6, we will determine closed and other maximal ideals in F^+.

2. **Generators for F^+.** Let $f_1, \cdots, f_N \in F^+$. The ideal in F^+ generated by $\vec{f} = (f_1, \cdots, f_N)$ is denoted as $I(f_1, \cdots, f_N)$. We write

$$\|\vec{f}(z)\|^2 = |f_1(z)|^2 + \cdots + |f_N(z)|^2 , \qquad z \in D . \tag{2.1}$$

If $u \in F^+$ belongs to the ideal $I(f_1, \cdots, f_N)$, then it is easily seen that there exist a constant $K > 0$ and a continuous function $\omega(r)$, $\omega(r) \downarrow 0$ as $r \to 1$, such that

$$|u(z)| \leqq K \|\vec{f}(z)\| \exp[\omega(r)/(1-r)] , \qquad |z| = r . \tag{2.2}$$

THEOREM 1. *If $u \in F^+$ satisfies* (2.2), *then we have*

$$u^2 \in I(f_1, \cdots, f_N) .$$

As a corollary of Theorem 1, we have

THEOREM 2. *Let $f_1, \cdots, f_N \in F^+$. In order that there exist $g_1, \cdots, g_N \in F^+$ such that*

$$f_1 g_1 + \cdots + f_N g_N = 1 , \tag{2.3}$$

it is necessary and sufficient that

$$|f_1(z)| + \cdots + |f_N(z)| \geqq \delta \exp[-\omega(r)/(1-r)] \tag{2.4}$$

($r = |z|$) *for some constant $\delta > 0$ and for some continuous function $\omega(r)$, $\omega(r) \downarrow 0$ as $r \to 1$.*

In contrast to Theorem 1, we have

THEOREM 1*. (2.2) *does not imply that $u \in I(f_1, \cdots, f_N)$ for $u \in F^+$.*

For the proof, we follow to the method of Rao [8].

In connection with Theorem 2, we have

THEOREM 2*. *Let $f_1, \cdots, f_N \in N^+$. Then,* (2.4) *is not sufficient for $f_1, \cdots, f_N$ to be generators of N^+. That is,* (2.4) *does not imply* (2.3) *in N^+.*

In contrast to the case of Banach algebras, we have

THEOREM 3. *Maximal ideals in F^+ are not necessarily closed.*

Hence, in § 6, we will use somewhat strange method for compactifying in order to put it in a one-to-one correspondence with the maximal ideal space of F^+.

3. Proof of Theorem 1.

LEMMA 1. *Let $\omega_1(r)$ be a continuous function, $\omega_1(r) \downarrow 0$ as $r \to 1$. Then we can find a continuous function $\omega(r)$ such that $\omega(r) \geqq \omega_1(r)$ and $\omega(r) \geqq \sqrt{1-r}$,*

$$\omega(r) \downarrow 0 , \quad \omega(r)/(1-r) \uparrow \infty \quad \text{as } r \to 1 \tag{3.1}$$

$\omega(r)/(1-r)$ is convex.

PROOF. We can suppose that $\omega_1(r)$ is continuously differentiable and $\omega_1'(r) < 0$ for $0 \leqq r < 1$.

Let $r_0 = 0$. Let r_1 be a number, $1/2 < r_1 < 1$, and put

$$a_1 = \omega_1(r_0)/(1-r_1)^2 ,$$

$$b_1 = -a_1 r_1 + \omega_1(r_0)/(1-r_1) .$$

Let r_2 be such that $r_2 > r_1$ and

$$a_1 r_2 + b_1 = \omega_1(r_1)/(1-r_2) .$$

Then $r_2 < 1$. Suppose $\{r_k\}_{k=0}^{n}$, $r_k < r_{k+1}$, and $\{a_k\}_{k=1}^{n-1}$, $\{b_k\}_{k=1}^{n-1}$ be determined. Then, put

$$a_n = \omega_1(r_{n-1})/(1-r_n)^2 , \tag{3.2}$$

$$b_n = -a_n r_n + \omega_1(r_{n-1})/(1-r_n) , \tag{3.2'}$$

and let r_{n+1} be such that $r_{n+1} > r_n$ and

$$a_n r_{n+1} + b_n = \omega_1(r_n)/(1-r_{n+1}) , \tag{3.2''}$$

then $r_{n+1} < 1$. We will show that $r_n \uparrow 1$. For that purpose, we put

$$\rho = \lim_{n\to\infty} r_n . \tag{3.3}$$

We have

$$\begin{aligned} a_n &= (r_{n+1} - r_n)^{-1}(\omega_1(r_n)/(1-r_{n+1}) - \omega_1(r_{n-1})/(1-r_n)) \\ &= \frac{1}{1-r_{n+1}} \frac{\omega_1(r_n) - \omega_1(r_{n-1})}{r_n - r_{n-1}} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} + \frac{\omega_1(r_{n-1})}{(1-r_{n+1})(1-r_n)} . \end{aligned} \tag{3.4}$$

If ρ in (3.3) would be $\rho < 1$, we would have, letting $n \to \infty$ in (3.4),

$$\omega_1(\rho)/(1-\rho)^2 = (1-\rho)^{-1}\omega_1'(\rho) \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} + \omega_1(\rho)/(1-\rho)^2$$

hence

$$\lim_{n\to\infty} ((r_n - r_{n-1})/(r_{n+1} - r_n)) = 0$$

since $\omega_1'(\rho) < 0$. Then, for $\varepsilon < 1$, we would have

$$r_{n+1} - r_n > (1/\varepsilon)^{n-n_0}(r_{n_0} - r_{n_0-1}) , \quad n \geqq n_0 \quad \text{for an } n_0 .$$

Letting $n \longrightarrow \infty$, we obtain a contradiction. Hence we must have

$$\lim_{n\to\infty} r_n = 1 .$$

Having proved that $r_n \uparrow 1$, we define

$$\omega(r) = (1-r)(a_n r + b_n) \quad \text{for } r_n \leqq r \leqq r_{n+1} , \ n = 0, 1, \cdots . \tag{3.5}$$

Then, since

$$a_{n+1} > a_n \quad \text{and} \quad \omega(r) > \sqrt{1-r}$$

we have

$$\omega(r)/(1-r) \text{ is convex and } \omega(r)/(1-r) \uparrow \infty .$$

Further,

$$\omega(r_n) = \omega_1(r_{n-1}) , \qquad \omega(r_{n+1}) = \omega_1(r_n) \downarrow 0$$

and

$$\omega(r) = v_n(r) \quad \text{for} \quad r_n \leqq r \leqq r_{n+1} ,$$

where

$$v_n(r) = -a_n r^2 + (a_n - b_n)r + b_n .$$

Since

$$v_n'(r_n) = -2a_n r_n + (a_n - b_n) = 0 ,$$

we get that

$$\omega(r) \text{ is concave and monotone decreasing for } r_n \leqq r \leqq r_{n+1} .$$

Thus $\omega(r) \downarrow 0$, and our Lemma 1 is proved. q.e.d.

By the Lemma 1, functions $\omega(r)$ in the below may be supposed to satisfy the condition (3.1).

LEMMA 2. *There exists a constant K such that for any $z \in D$, $|z - \zeta| \leqq K \exp[-\omega(r)/(1-r)]$, $r = |z|$, implies $\zeta \in D$ and moreover*

$$\omega(\rho)/(1-\rho) \leqq 2\omega(r)/(1-r), \qquad \rho = |\zeta| .$$

PROOF. Since $\exp[-\omega(r)/(1-r)] \leqq \exp[-1/\sqrt{1-r}]$, we have

$$\exp[-\omega(r)/(1-r)] \leqq (1-r)/2, \qquad r \geqq R$$

with an $R < 1$. Put $K = (1-R)/2$. Then we have that, if

$$z \in D, \qquad |z - \zeta| \leqq K \exp[-\omega(r)/(1-r)],$$

we get

$$\rho = |\zeta| \leqq (1+r)/2 < 1 . \tag{3.6$_1$}$$

If $\rho \geqq r$, we have, as $1-\rho \geqq (1-r)/2$,

$$\omega(\rho)/(1-\rho) \leqq 2\omega(r)/(1-r) . \tag{3.6$_2$}$$

If $\rho < r$, we have by (3.1)

$$\omega(\rho)/(1-\rho) \leqq \omega(r)/(1-r) . \tag{3.6$_3$}$$

(3.6_{1-3}) give the lemma. q.e.d.

We note that $f \in F^+$ implies $f' \in F^+$ [9, Theorem 6].

LEMMA 3. *If f is holomorphic in D, then f belongs to F^+ if and only if for some $\omega(r)$ satisfying* (3.1)

$$(\|f\|_\omega)^2 = \frac{1}{\pi}\iint_D |f(re^{i\theta})|^2 \exp\left[\frac{-\omega(r)}{1-r}\right] r dr d\theta < \infty . \tag{3.7}$$

PROOF. $f \in F^+$ obviously satisfies (3.7) with some $\omega(r)$. On the other hand, it follows that the mean value of $|f|$ over the disk with center at $z \in D$:

$$\{\zeta; |\zeta - z| \leqq K \exp[-\omega(r)/(1-r)]\} \subset D$$

is bounded by

$$(1/K)\,\|f\|_\omega \exp[2\omega(r)/(1-r)] .$$

By the subharmonicity of $|f|$, this gives also a bound for $|f(z)|$, $|z| = r$, which shows that $f \in F^+$ by (1.4).

LEMMA 4. *Let g be a form of type* (0, 1) *in D with locally square summable coefficient $g(r, \theta)$, and let $\phi(r, \theta)$ be a subharmonic function in D such that*

$$\iint_D |g(r,\theta)|^2 e^{-\phi(r,\theta)} r dr d\theta < \infty .$$

It follows that there is a function f (a form of type (0, 0)) *with $\bar\partial f = g$, and*

$$\iint_D |f(r, \theta)|^2 e^{-\phi(r,\theta)}(1 + r^2)^{-2} r dr d\theta$$
$$\leqq \iint_D |g(r, \theta)|^2 e^{-\phi(r,\theta)} r dr d\theta .$$

Proof is found in [3, p. 945, Lemma 4].

For non-negative integers p and q, we shall denote by L_q^p the set of all differential forms h of type $(0, q)$ with values in $\Lambda^p C^N$, such that for some function $\omega(r)$ satisfying (3.1),

$$\iint_D |h(r, \theta)|^2 \exp\left[\frac{-\omega(r)}{1 - r}\right] r dr d\theta < \infty . \tag{3.8}$$

In other words, for each p-tuple $S = (i_1, \cdots, i_p)$, $1 \leqq i_1, \cdots, i_p \leqq N$, h has a component h_S which is a differential form of type $(0, q)$ such that h_S is skew symmetric in S and

$$\iint_D |h_S(r, \theta)|^2 \exp\left[\frac{-\omega(r)}{1 - r}\right] r dr d\theta < \infty .$$

Note that $L_q^p = 0$ if $p > N$ or $q > 1$.

Now $\bar\partial$-operator acts componentwise on the elements of L_q^p and yields a linear mapping $\bar\partial$: $L_q^p \to \{(0, q + 1)$-forms with values in $\Lambda^p C^N\}$, such that $\bar\partial^2 = 0$. Furthermore, the interior product P_f by $\vec{f} = (f_1 \cdots, f_N)$ maps L_q^{p+1} into L_q^p: If $h \in L_q^{p+1}$ then

$$(P_f h)_S = \sum_{j=1}^{N} h_{Sj} f_j \quad \text{for} \quad S = (i_1, \cdots, i_p) . \tag{3.9}$$

We define $P_f L_q^0 = 0$. Clearly $P_f^2 = 0$ and P_f commutes with $\bar\partial$ since $f_1, \cdots, f_N$ are holomorphic. So, we have a double complex.

LEMMA 5. *For every $h \in L_1^p$, the equation $\bar\partial g = h$ has a solution $g \in L_0^p$.*

Proof. This follows immediately from the Lemma 4.

LEMMA 6. *For any $v \in C^2(D)$ we have for $0 \leqq r < 1$,*

$$\int_0^r t^{-1} S(t) dt = \frac{1}{2\pi} \int_0^{2\pi} v(r, \theta) d\theta - v(0) ,$$

where

$$S(t) = \frac{1}{2\pi} \iint_{|z| \leq t} \Delta v \, dx dy , \qquad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} .$$

Proof is a simple consequence of Green's formula. See [4, p. 231, Lemma 3.3].

Lemma 7. *Let $f_1, \cdots, f_N \in F^+$. Then, if we put*

$$w_{ij}(z) = (f_i(z)f_j'(z) - f_j(z)f_i'(z))/\|\vec{f}(z)\|^2\,, \quad i, j = 1, \cdots, N\,,$$

we have

$$\iint_D |w_{ij}(z)|^2 \exp\left[\frac{-\omega(r)}{1-r}\right] r\,dr\,d\theta < \infty$$

for a function $\omega(r)$ satisfying (3.1).

Proof. At first we suppose that $f_1, \cdots, f_N$ have no common zeros. Then

$$v(z) = 2\log\|\vec{f}(z)\| \in C^2(D)\,, \tag{3.10}$$

and, if we write

$$w(z) = \frac{1}{4}\Delta v = \sum_{i,j=1}^{N} |f_i(z)f_j'(z) - f_j(z)f_i'(z)|^2/\|\vec{f}(z)\|^4\,,$$

it suffices to prove that

$$\iint_D w(z)\exp\left[\frac{-\omega(r)}{1-r}\right] r\,dr\,d\theta < \infty\,.$$

We apply Lemma 6 for the function v in (3.10). Then

$$\int_0^r t^{-1}S(t)dt \leqq \frac{\lambda(r)}{1-r}$$

for a continuous function $\lambda(r)$ satisfying (3.1). Now $S(t)$ is non-negative and increasing, since $v(z)$ in (3.10) is subharmonic, so

$$\int_0^r \frac{S(t)}{t}\,dt \geqq \int_{r^2}^r \frac{S(t)}{t}\,dt \geqq \frac{S(r^2)}{r}\,r(1-r)\,,$$

thus

$$S(r^2) \leqq \frac{\lambda(r)}{(1-r)^2} \leqq \frac{4\lambda(r)}{(1-r^2)^2} \quad \text{hence} \quad S(r) \leqq \left(\frac{\omega(r)}{1-r}\right)^2$$

with a continuous function $\omega(r)$ satisfying (3.1).

Then, writing $\omega(r)/(1-r) = p(r)$,

$$\begin{aligned}\iint_D w(z)\exp\left[\frac{-\omega(r)}{1-r}\right] r\,dr\,d\theta &\leqq \iint_{p(r)\leqq 2} + \sum_{n=1}^{\infty}\iint_{2^n \leqq p(r) \leqq 2^{n+1}} \\ &\leqq K_0 + \sum_{n=1}^{\infty} e^{-2^n}\iint_{p(r)\leqq 2^{n+1}} \Delta v\,dx\,dy \\ &\leqq K_0 + \sum_{n=1}^{\infty} 2^{2n+2}\exp[-2^n] < \infty\,.\end{aligned}$$

For the case where $f_1, \cdots, f_N$ have common zeros, the desired conclusion may be deduced via a standard argument by considering $v_\varepsilon(z) = \log(\|\vec{f}(z)\|^2 + \varepsilon^2)$ and letting $\varepsilon \to 0$. q.e.d.

PROOF OF THEOREM 1. Suppose $u \in F^+$ satisfies (2.2). Let $\alpha = (\alpha_1, \cdots, \alpha_N) \in L_0^1$ be such that

$$\alpha_i = u^2\bar{f}_i/\|\vec{f}\|^2, \qquad i = 1, \cdots, N.$$

Then

$$\bar{\partial}\alpha_i = \|\vec{f}\|^{-4} u^2 \sum_{j=1}^{N} f_j\overline{(f_j\partial f_i - f_i\partial f_j)}.$$

If we put

$$\beta_{ij} = \|\vec{f}\|^{-4} u^2\overline{(f_j\partial f_i - f_i\partial f_j)},$$

then we get $\beta = (\beta_{ij}) \in L_1^2$ by Lemma 7. Clearly, $\bar{\partial}\alpha = P_f\beta$, and there exists $\gamma \in L_0^2$ with $\bar{\partial}\gamma = \beta$ by Lemma 5. Then, if we put

$$g = \alpha - P_f\gamma \in L_0^1,$$

then $\bar{\partial}g = 0$, hence $g_j \in F^+$, $j = 1, \cdots, N$, by Lemma 3, and

$$P_f g = u^2, \quad \text{which shows that} \quad u^2 \in I(f_1, \cdots, f_N). \qquad \text{q.e.d.}$$

4. Proofs of Theorem 1*, 2* and 3.

PROOF OF THEOREM 1*. Let $f, g \in F^+$. If we take in (2.2) $N = 2$, $f_1 = f^2, f_2 = g^2$ and $u = fg$, then (2.2) holds. If it were true that (2.2) would imply $u \in I(f_1, \cdots, f_N)$, we would have $fg \in I(f^2, g^2)$ for any $f, g \in F^+$. We will show that this is not the case for some f and g.

Suppose $fg \in I(f^2, g^2)$, i.e., $fg = Af^2 + Bg^2$ with $A, B \in F^+$. Then

$$Af^2/g = f - Bg \in F^+, \qquad Bg^2/f = g - Af \in F^+. \tag{4.1}$$

We put

$$f(z) = \prod_{k=1}^{\infty} ((z - z_k)/(1 - \bar{z}_k z)),$$

where

$$z_k = 1 - b^k \quad \text{with a constant } b,\ 0 < b < 1/3,$$

and

$$g(z) = \exp\left[-c\frac{1+z}{1-z}\right] \quad \text{with a constant} \quad c > 0.$$

Then by (4.1), B/f is holomorphic.

Then, as we shall see shortly later, if $H(z)$ is holomorphic in D,

$$f \times H \in F^+ \quad \text{implies} \quad H \in F^+. \tag{4.2}$$

Thus

$$A/g = p \in F^+ \quad \text{since} \quad (A/g)f^2 \in F^+ ,$$
$$B/f = q \in F^+ \quad \text{since} \quad B \in F^+ .$$

Hence

$$1 = p \times f + q \times g . \tag{4.3}$$

This is impossible, since $p(z_k)f(z_k) = 0$ and $q(z_k)g(z_k) \to 0$ as seen from (1.4) and the definition of $g(z)$.

Now we will show (4.2). First we note that

$$|f(z)| \geqq \prod ||z| - |z_k||/(1 - |z_k||z|) .$$

Put

$$r^{(n)} = 1 - b^n(1 + b)/2 = (z_n + z_{n+1})/2 .$$

Then, if $|z| = r^{(n)}$,

$$||z| - |z_k|| = |b^k - b^n(1 + b)/2| ,$$
$$1 - |z_k||z| \leqq b^k + b^n(1 + b)/2 .$$

Thus, if $|z| = r^{(n)}$,

$$\begin{aligned} |f(z)| = \prod_{k\leqq n}\prod_{k>n} &\geqq \prod_{k\leqq n} \frac{1 - b^{n-k}(1 + b)/2}{1 + b^{n-k}(1 + b)/2} \prod_{k>n} \frac{1 - b^{k-n} \times 2/(1 + b)}{1 + b^{k-n} \times 2/(1 + b)} \\ &\geqq \prod_{m\geqq 0} \frac{1 - b^m(1 + b)/2}{1 + b^m(1 + b)/2} \prod_{m\geqq 0} \frac{1 - b^m \times 2b/(1 + b)}{1 + b^m \times 2b/(1 + b)} = K > 0 . \end{aligned} \tag{4.4}$$

Let

$$h(z) = f(z)H(z) \in F^+ .$$

Then, for any constant $a > 0$,

$$M(r, h) \exp\left[\frac{-a}{1 - r}\right] \to 0 \quad \text{as } r \to 1 .$$

By (4.4), we have

$$M(r^{(n)}, H) \leqq M(r^{(n)}, h)/K .$$

Thus, for $r^{(n-1)} \leqq r \leqq r^{(n)}$

$$M(r, H) \leqq M(r^{(n)}, h)/K .$$

Hence, for $r^{(n-1)} \leqq r \leqq r^{(n)}$,

$$M(r, H) \exp\left[\frac{-a}{1 - r}\right] \leqq K^{-1}M(r^{(n)}, h) \exp[-a/(1 - r^{(n-1)})]$$

$$\leqq K^{-1}M(r^{(n)}, h)\exp[-ab/(1-r^{(n)})] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty ,$$

hence

$$M(r, H)\exp\left[\frac{-a}{1-r}\right] \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 1$$

for any $a > 0$, which shows that $H \in F^+$. q.e.d.

PROOF OF THEOREM 2*. Let $\nu(t)$, $0 \leqq t \leqq 2\pi$, be a continuous and monotone increasing function such that $\nu(0) = 0$, $\nu(2\pi) = 1$, and $\nu'(t) = 0$ almost everywhere on $[0, 2\pi]$. We put, for $2n\pi \leqq t \leqq (2n+2)\pi$,

$$\mu(t) = n + \nu(t - 2n\pi) , \qquad n = 0, \pm 1, \pm 2, \cdots$$

and

$$f(z) = \exp\left[-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)\right] .$$

Then $f(z) \in H^\infty \subset N^+$ and

$$1/f(z) \in F^+ ,$$

as shown in [12, Proof of Theorem 1]. Therefore, $f(z)$ satisfies (2.4) but does not generate N^+, since $f(z)$ is not invertible in N^+ while $f(z)$ is invertible in F^+.

PROOF OF THEOREM 3. Put

$$E = \left\{\exp\left[-c\frac{1+z}{1-z}\right];\ c > 0\right\} .$$

Then $E \subset N^+ \subset F^+$. If we write $I = \bigcup_{f \in F^+} fE$, I is a proper ideal. It is easy to see that

$$\exp\left[-c\frac{1+z}{1-z}\right] \longrightarrow 1 \quad \text{as} \quad c \longrightarrow 0 .$$

Hence the maximal ideal containing I is not closed.

5. Closed maximal ideals in F^+. Let A be a topological algebra with identity 1, locally convex and commutative, over the complex number field $\boldsymbol{C}$. Topology of A is defined by a countable family of semi-norms $\{\|\cdot\|_\alpha\}_{\alpha \in I}$, which are supposed to satisfy that $\|1\|_\alpha = 1$ and for $a, b \in A$

$$\|ab\|_\alpha \leqq \|a\|_\alpha \|b\|_\alpha \quad \text{for every} \quad \alpha \in I . \tag{5.1}$$

For an $\alpha \in I$, let $E_\alpha = \{a \in A;\ \|a\|_\alpha = 0\}$. E_α is obviously an ideal in A. For $a \in A$, we write $a^\wedge = a + E_\alpha \in A/E_\alpha$. Then A/E_α is a normed space with $\|a^\wedge\|_\alpha = \|a\|_\alpha$. We have, by (5.1), for $a, b \in A$

(5.1′) $$\| a^\wedge b^\wedge \|_\alpha \leqq \| a^\wedge \|_\alpha \| b^\wedge \|_\alpha .$$

The completion of A/E_α with res. to the norm $\|\cdot\|_\alpha$ is denoted as A^*_α.

LEMMA 8. *Let $g \in A$. Suppose $1-\mu g$ is invertible for $|\mu| < \delta$. Then, for any $\alpha \in I$ there is a $\delta(\alpha) > 0$ such that $\|(1-\mu g)^{-1}\|_\alpha$ is bounded for $|\mu| \leqq \delta(\alpha)$.*

PROOF. Put $\delta(\alpha) = \min(\delta, 1/2\|g\|_\alpha)$ and

$$h_n = 1 + \mu g + \cdots + \mu^n g^n \in A , \qquad h_n^\wedge \in A/E_\alpha .$$

$\{h_n^\wedge\}$ is a Cauchy sequence in A^*_α if $|\mu| \leqq \delta(\alpha)$. Then

$$h^\wedge = \lim_{n\to\infty} h_n^\wedge \in A^*_\alpha .$$

For a fixed μ, we define a linear operator T on A/E_α by

$$Ta^\wedge = (1-\mu g)^\wedge a^\wedge \in A/E_\alpha \quad \text{for } a \in A .$$

T is continuous on A/E_α by (5.1′), and continuously extended on A^*_α. Then

$$Th^\wedge = \lim_{n\to\infty} Th_n^\wedge = \lim_{n\to\infty} ((1-\mu g)h_n)^\wedge = \lim_{n\to\infty} (1-\mu^{n+1}g^{n+1})^\wedge = 1^\wedge .$$

Thus $h^\wedge = ((1-\mu g)^{-1})^\wedge \in A/E_\alpha$. Then we have, for $|\mu| \leqq \delta(\alpha)$

$$\|(1-\mu g)^{-1}\|_\alpha = \lim_{n\to\infty} \|h_n^\wedge\|_\alpha = \lim_{n\to\infty} \|h_n\|_\alpha \leqq 1 + \sum_{n=1}^{\infty} \|\mu g\|_\alpha^n \leqq 2 .$$

LEMMA 9. *Let $f \in A$ and $\lambda_0 \in C$. Suppose $\lambda - f$ is invertible for $|\lambda - \lambda_0| < \delta$, $\delta > 0$. Then $(\lambda - f)^{-1}$ is continuous with respect to λ.*

PROOF. Put $(\lambda_0 - f)^{-1} = g$ and $\mu = \lambda_0 - \lambda$. Then

$$\begin{aligned} &(\lambda - f)^{-1} - (\lambda_0 - f)^{-1} \\ &= (\lambda_0 - f)^{-1}[(1-\mu g)^{-1} - 1] \\ &= \mu g(\lambda_0 - f)^{-1}(1-\mu g)^{-1} = \mu g^2(1-\mu g)^{-1} . \end{aligned}$$

Then for any $\alpha \in I$, if $|\mu| \leqq \delta(\alpha)$,

$$\begin{aligned} &\|(\lambda - f)^{-1} - (\lambda_0 - f)^{-1}\|_\alpha \\ &\leqq |\mu| \|g\|_\alpha^2 \|(1-\mu g)^{-1}\|_\alpha \\ &\leqq 2|\mu| \|g\|_\alpha^2 \to 0 \quad \text{as} \quad \lambda \to \lambda_0 , \quad \mu \to 0 , \end{aligned}$$

hence $(\lambda - f)^{-1}$ is continuous.

LEMMA 10. *For any $f \in A$, there is a number λ_f such that $\lambda_f - f$ is not invertible.*

PROOF. Suppose $\lambda - f$ were invertible for any $\lambda \in C$. Then $(\lambda - f)^{-1}$

is continuous with respect to λ. Let L be a continuous linear functional on A. Then

$$G(\lambda) = L((\lambda - f)^{-1})$$

is an entire function. For,

$$G(\lambda) - G(\lambda_0) = -(\lambda - \lambda_0)L((\lambda - f)^{-1}(\lambda_0 - f)^{-1}),$$

hence, by Lemma 9, we obtain

$$G'(\lambda_0) = -L((\lambda_0 - f)^{-1}(\lambda_0 - f)^{-1}).$$

Further, by the continuity of L,

$$|G(\lambda)| = |L((\lambda - f)^{-1})| \leqq K\|(\lambda - f)^{-1}\|_\alpha$$

with an $\alpha \in I$ and a constant K. Thus, by Lemma 8, if $|\lambda| > 2\|f\|_\alpha$,

$$|G(\lambda)| \leqq K|\lambda|^{-1}\|(1 - f/\lambda)^{-1}\|_\alpha \leqq 2K/|\lambda| \to 0$$

as $|\lambda| \to \infty$. Therefore $G(\lambda) \equiv 0$, i.e.,

$$L((\lambda - f)^{-1}) \equiv 0 \quad \text{for} \quad \lambda \in C$$

for any continuous linear functional L on A, which is absurd.

As a characterization of closed maximal ideals we have, in analogy with the well known theorem of Igusa [4], the following

THEOREM 4. *Let M be a maximal ideal in F^+. The following conditions for M are equivalent:*

(i) *M is closed in the topology of uniform convergence on every disk $|z| \leqq r$, $r < 1$.*

(ii) *$F^+/M \cong C$.*

(iii) *M corresponds to a point $z_0 \in D$, i.e., M consists of all functions of F^+ which vanish at z_0.*

PROOF. (i) $\to$ (ii): Obviously, $F^+/M \supset C$. For $f \in F^+$, we denote

$$[f] = f + M \in F^+/M.$$

We introduce the family of semi-norms in F^+/M as follows:

$$\|[f]\|_r = \inf_{h \in M}(\max_{|z|=r}|f(z) + h(z)|), \qquad 0 \leqq r < 1.$$

Then clearly

$$\|[fg]\|_r \leqq \|[f]\|_r\|[g]\|_r, \qquad 0 \leqq r < 1.$$

By Lemma 10, to each $[f] \in F^+/M$, there corresponds a number $\lambda \in C$ such that $\lambda - [f]$ is not invertible. But, since F^+/M is a field by the maximality of M, $\lambda - f$ must belong to M, i.e., $\lambda \in [f]$. Thus we obtain

$$F^+/M \cong C .$$

(ii) $\longrightarrow$ (iii): Let z_0 be the coset $[z] \in F^+/M$. Then $z - z_0 \in M$, hence $z_0 \in D$.

For each $f(z) \in F^+$, we have

$$f(z) - f(z_0) = A(z)(z - z_0) . \tag{5.2}$$

As easily seen, $A(z) \in F^+$, thus $f(z) - f(z_0) \in M$. If $f(z) \in M$, then $f(z_0) \in M$, whence $f(z_0) = 0$. Thus M corresponds to the point $z_0 \in D$.

(iii) $\longrightarrow$ (i): This is evident from the theorem of Hurwitz.

6. Maximal ideals in F^+. Now we will study some structures of maximal ideal space of the algebra F^+.

The complex w-sphere is denoted by W. Let Q be the set of all continuous functions $\omega(r)$, $0 \leqq r < 1$, satisfying (3.1).

Taking a function $f(z) \in F^+$, we define a topology $\tau_Q(f)$ in W.

For a number $\varepsilon > 0$ and a function $\omega(r) \in Q$, we define neighborhood $U(a)$ of $a \in W$ as follows:

(A) $a \neq \infty$.

A(i) Suppose there is a point $z_0 \in D$ such that $f(z_0) = a$. Then we put for a number $\eta > 0$,

$$U(a) = U(a; \varepsilon, \omega, z_0, \eta) = \Big\{w; w = f(z), \text{ where } |z - z_0| < \eta \text{ and}$$
$$\exp\left[\frac{\omega(|z|)}{1 - |z|}\right] |f(z) - a| < \varepsilon\Big\}$$

A(ii) Suppose there is a point ζ_0, $|\zeta_0| = 1$, such that

$$\varliminf_{z \to \zeta_0} \exp\left[\frac{\omega(|z|)}{1 - |z|}\right] |f(z) - a| = 0 .$$

Then we put for a number $\eta > 0$,

$$U(a) = U(a; \varepsilon, \omega, \zeta_0, \eta) = \Big\{w; w = a + \rho e^{i\theta}, 0 \leqq \theta \leqq 2\pi , \text{ and}$$
$$\rho < \varepsilon \exp\left[\frac{-\omega(|z|)}{1 - |z|}\right] \text{ for a point } z \in D \text{ such that}$$
$$|z - \zeta_0| < \eta, \ \exp\left[\frac{\omega(|z|)}{1 - |z|}\right] |f(z) - a| < \varepsilon\Big\} \cup \{a\} .$$

A(iii) Suppose there is neither z_0 in A(i) nor ζ_0 in A(ii). We put

$$U(a) = U(a; \varepsilon, \omega) = \{a\} .$$

(B) $a = \infty$.

B(i) Suppose there is a point ζ_0, $|\zeta_0| = 1$, such that

$$\varlimsup_{z\to\zeta_0} \exp\left[\frac{-\omega(|z|)}{1-|z|}\right]|f(z)| = \infty .$$

Then we put for a number $\eta > 0$,

$$U(\infty) = U(\infty;\varepsilon,\omega,\zeta_0,\eta) = \{w; w = \rho e^{i\theta}, 0 \leqq \theta \leqq 2\pi , \text{ and}$$

$$\rho > (1/\varepsilon)\exp\left[\frac{\omega(|z|)}{1-|z|}\right] \text{ for a point } z \in D \text{ such that}$$

$$|z-\zeta_0| < \eta,\ \exp\left[\frac{-\omega(|z|)}{1-|z|}\right]|f(z)| > 1/\varepsilon\Big\} \cup \{\infty\} .$$

B(ii) Suppose there is no point ζ_0 in B(ii). Then

$$U(\infty) = U(\infty;\varepsilon,\omega) = \{\infty\} .$$

By this system of neighborhoods, W becomes a Hausdorff space. We note that the topology depends on the function $f(z)$.

Let $f(D) \subset W$ be the range of $f(z)$ in D, and $(f(D))^a$ be the closure of $f(D)$ with respect to the topology determined by f. Since $f(z) \in F^+$, $(f(D))^a$ does not contain ∞. We compactify $(f(D))^a$ as follows:

Let P_f be an (abstract) element. Neighborhoods of P_f are defined to be open sets (in the sense of the usual Riemann sphere topology) containing $W-(f(D))^a$.

Then, $A_f = (f(D))^a \cup \{P_f\}$ is obviously compact. We note that $A_f - \{P_f\}$ satisfies the Hausdorff separation axiom, although A_f does not. A_f might be considered, in a sense, as an Alexandroff compactification of $(f(D))^a$.

Further, let C_0 be the set of all continuous complex valued functions with compact supports in D.

Put

(6.1) $$T = \prod_{f\in F^+} A_f \cdot \prod_{\phi\in C_0} W_\phi \quad (W_\phi = W \text{ with the usual Riemann sphere topology})$$

T is compact with the Tychonoff topology. We denote by π_f or π_ϕ the projection of T on A_f or on W_ϕ, respectively. We write, for $z \in D$,

$$\psi(z) = \{f(z), \phi(z)\}_{f\in F^+,\,\phi\in C_0}$$

ψ is a continuous and one-to-one mapping from D into T. We write the closure of $\psi(D)$ in T as D^*. Then D^* is compact and $\psi(D)$ is dense in D^*.

ψ is an open mapping. To see this, for $z_0 \in D$, let U be a relatively compact neighborhood of z_0, and ϕ be a function of C_0 with support contained in U and $\phi(z_0) \neq 0$. Put

$$V = \{p \in D^*; \pi_\phi(p) \neq 0\} .$$

V is a neighborhood of $\psi(z_0)$ on D^*. $V - \psi(\bar{U})$ is an open set, as we shall see shortly later. But we have

$$(V - \psi(\bar{U})) \cap \psi(D) = \text{void} ,$$

hence $V - \psi(\bar{U})$ is void, for $\psi(D)$ is dense in D^*. Therefore we obtain

$$V \subset \psi(\bar{U}) \subset \psi(D) , \qquad V \subset \psi(U) ,$$

and ψ is an open mapping.

Now we will show that $V - \psi(\bar{U})$ is open, i.e., $\psi(\bar{U})$ is closed. Let $q \notin \psi(\bar{U})$. If there is an $f_0 \in F^+$ such that $\pi_{f_0}(q) \neq P_{f_0}$, then there is a neighborhood $U(\pi_{f_0}(q))$ such that $U(\pi_{f_0}(q)) \cap \pi_{f_0}(\psi(\bar{U})) = \text{void}$. If $\pi_f(q) = P_f$ for any $f \in F^+$, then there is, for an $f_0 \in F^+$, a neighborhood $U(P_{f_0})$ such that $U(P_{f_0}) \cap \pi_{f_0}(\psi(\bar{U})) = \text{void}$, since $\pi_{f_0}(\psi(\bar{U})) = f_0(\bar{U})$ is compact in $f_0(D)$. Thus, if $U(q)$ is a neighborhood of q such that $\pi_{f_0}(U(q)) = U(\pi_{f_0}(q))$, then $U(q) \cap \psi(\bar{U}) = \text{void}$, and $(\psi(\bar{U}))^C$ is open, hence $\psi(\bar{U})$ is closed.

Thus ψ is homeomorphic, and D and $\psi(D)$ may be identified.

π_f is the continuous extension of f onto D^*. For $a, b \in D^*$, $a \neq b$, there is an $f \in F^+$ with

$$\pi_f(a) \neq \pi_f(b) ,$$

since for any point $p \in D^* - \psi(D)$ we have $\phi(p) = 0$ for each $\phi \in C_0$.

We put

$$P = \prod_{f \in F^+} \{P_f\} \cdot \prod_{\phi \in C_0} W_\phi$$

and

$$D^{**} = D^* - P .$$

Let $\mathfrak{M}$ be the set of all maximal ideals in F^+. Then

THEOREM 5. *Elements of $\mathfrak{M}$ and points of the space D^{**} correspond in a one-to-one way.*

PROOF. Let z_0 be a point of D. It is easy to see that the set of all functions $f(z) \in F^+$ with $f(z_0) = 0$ forms a maximal ideal in F^+.

Let J be a maximal ideal in F^+. We suppose that there are no common zeros in D for functions of J.

Let $f_{\alpha_1}, \cdots, f_{\alpha_N}$ be functions of J. Thus, by Theorem 2, we have for any $\omega(r) \in Q$,

$$\inf_{r_0 \leq r < 1} \exp [\omega(r)/(1-r)](|f_{\alpha_1}(z)| + \cdots + |f_{\alpha_N}(z)|) = 0 \tag{6.3}$$

$(r = |z|)$ for any $r_0 < 1$. Thus, there is a sequence $\{z_n\} \subset D$, $r_n = |z_n| \to 1$, such that

$$\liminf_{n\to\infty} \inf_{k \geqq n} \exp [\omega(r_k)/(1 - r_k)](|f_{\alpha_1}(z_k)| + \cdots + |f_{\alpha_N}(z_k)|) = 0 .$$

We denote by $E(\alpha_1, \cdots, \alpha_N; \omega)$ the set of points of D^* such that

$$\zeta^* \in E(\alpha_1, \cdots, \alpha_N; \omega) \text{ if for any neighborhood } U(\zeta^*) ,$$
$$\inf_{z \in U(\zeta^*) \cap D} \exp [\omega(|z|)/(1 - |z|)](|f_{\alpha_1}(z)| + \cdots + |f_{\alpha_N}(z)|) = 0 .$$

$E(\alpha_1, \cdots, \alpha_N; \omega)$ is a closed non-void subset of the compact space D^*, for every $\omega(r) \in Q$, by Theorem 2, since J is a proper ideal. If $\omega_1(r), \cdots, \omega_M(r) \in Q$, we have

$$E(\alpha_1, \cdots, \alpha_N; \omega_1) \cap \cdots \cap E(\alpha_1, \cdots, \alpha_N; \omega_M)$$
$$\supset E(\alpha_1, \cdots, \alpha_N; \omega_1 + \cdots + \omega_M) \neq \text{void} .$$

Hence

$$E(\alpha_1, \cdots, \alpha_N) = \bigcap_{\omega \in Q} E(\alpha_1, \cdots, \alpha_N; \omega)$$

is non-void. Since

$$E(\alpha_1, \cdots, \alpha_N) \cap E(\alpha'_1, \cdots, \alpha'_K) \supset E(\alpha_1, \cdots, \alpha_N, \alpha'_1, \cdots, \alpha'_K) \neq \text{void} ,$$

we have

$$E = \bigcap_{(\alpha_1, \cdots, \alpha_N)} E(\alpha_1, \cdots, \alpha_N)$$

is non-void.

Let $\zeta^* \in E$ and $M(\zeta^*)$ be the set of all functions $f \in F^+$ such that

$$\exp\left[\frac{\omega(|z|)}{1 - |z|}\right] |f(z)| \to 0 \quad \text{for each} \quad \omega(r) \in Q , \tag{6.4}$$

as $z \to \zeta^*$ in D^*, $z \in D$.

$M(\zeta^*)$ is obviously a proper ideal. Take a function $f \in J$. For any $\varepsilon > 0$ and $\omega(r) \in Q$, we choose a neighborhood $U(\zeta^*)$ as

$$\pi_f(U(\zeta^*)) = U(\pi_f(\zeta^*); \varepsilon, \omega, \zeta_0, \eta)$$

as defined in A(ii) with suitable ζ_0, $|\zeta_0| = 1$, and $\eta > 0$. Thus, if $z \in U(\zeta^*)$,

$$\exp [\omega(|z|)/(1 - |z|)] \, |f(z) - \pi_f(\zeta^*)| < \varepsilon .$$

But, by the definition of the set E, we have $\pi_f(\zeta^*) = 0$, hence f satisfies (6.4) and $J \subset M(\zeta^*)$, hence $J = M(\zeta^*)$.

We have that $E \subset D^{**}$. E contains only one point, since the extensions of functions of F^+ separate points of D^*. Thus we obtain the proof of our theorem.

REFERENCES

[1] N. DUNFORD AND J. T. SCHWARTZ, Linear operators, part I, Interscience Publishers Inc., New York, 1964.

[2] P. L. DUREN, Theory of H^p spaces, Academic Press, New York and London, 1970.

[3] L. HÖRMANDER, Generators for some rings of analytic functions, Bull. Amer. Math. Soc., 73 (1967), 943-949.

[4] J. IGUSA, On a property of the domain of regularity, Mem. Coll. Sci. University of Kyoto, Ser. A, 27 (1952), 95-97.

[5] J. J. KELLEHER AND B. A. TAYLOR, Finitely generated ideals in rings of analytic functions, Math. Ann., 193 (1971), 225-237.

[6] J. J. KELLEHER AND B. A. TAYLOR, Closed ideals in locally convex algebras of analytic functions, J. Reine Angew. Math., 255 (1972), 190-209.

[7] G. KÖTHE, Topologische Lineare Räume I, Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[8] K. V. RAJESWARA RAO, On a generalized corona problem, J. Analyse Math., 18 (1967), 277-278.

[9] N. YANAGIHARA, Multipliers and linear functionals for the class N^+, Trans. Amer. Math. Soc., 180 (1973), 449-461.

[10] N. YANAGIHARA, The containing Fréchet space for the class N^+, Duke Math. J., 40 (1973), 93-103.

[11] N. YANAGIHARA, The second dual space for the space N^+, Proc. Japan Acad., 49 (1973), 33-36.

[12] N. YANAGIHARA, The class N^+ of holomorphic functions and its containing Fréchet space F^+, Boll. Un. Mat. Ital. (Bologna), 8 (1973), 230-245.

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