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A THEOREM ON UNIFORMITY OF PRIME SURFACES OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES

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1. Introduction. Let f be a non-constant entire function of two complex variables x and y. Let z be a complex parameter. An irreducible component of an analytic surface in the space (x, y), defined by the equation $f(x, y) = z$, is called a prime surface of f with the value z and is denoted by S_z. A prime surface S_z is said to be parabolic and of type (g, n) when it is parabolic, of genus g and it has n boundary components as a Riemann surface. A prime surface S_z is said to be of finite type if its genus and the number of its boundary components are finite. Moreover, a prime surface S_z is called algebraic if it is of finite type and is parabolic.

The class of all entire functions whose prime surfaces are all parabolic is called the class (P). The class of all entire functions whose prime surfaces are all algebraic is called the class (A).

The following theorem is due to Nishino ([4]; Theorem I, p. 263, cf. [3]; Theorem p. 271).

THEOREM. *A function f of the class* (P) *belongs to the class* (A) *if it has "sufficiently many" algebraic prime surfaces, that is, if the set of values taken by f on its algebraic prime surfaces is of positive capacity.*

Recently, Yamaguchi proved the following theorem ([6]; Theorem 4, p. 433).

THEOREM. *If every prime surface of f is schlicht and if f has "sufficiently many" parabolic prime surfaces, then f belongs to the class* (P).

From above two theorems, it follows that, if every prime surface of f is schlicht and if f has "sufficiently many" algebraic prime surfaces, then f belongs to the class (A). In this article, one proves the following theorem[1] which is a generalization of the fact stated just above.

THEOREM. *If f has "sufficiently many" schlicht algebraic prime*

[1] Professor H. Yamaguchi informed me by the letter that H. Saitô also proved the same result independently.

surfaces, then f belongs to the class (A).

2. **Lemmas.** Let f be a non-constant entire function of two complex variables x and y. Let S_0 be a prime surface of f with a value a_0. Assume that S_0 is of finite type $(0, n)$ and that grad $f = (\partial f/\partial x, \partial f/\partial y)$ is not zero at every point on S_0. Take a point $P_0 \in S_0$, which is fixed as the origin of S_0.

Generally, Q^r signifies an open ball $|x|^2 + |y|^2 < r^2$. Consider an open ball Q^{r_0} and denote by S_0^0 the connected component of $S_0 \cap Q^{r_0}$ containing P_0. One can suppose that S_0^0 is of type $(0, n)$ and that n closed curves γ_i^0 $(i = 1, \cdots, n)$ limiting S_0^0 in S_0 are all simple.

Consider an analytic retraction ζ_0 about S_0 defined in a neighborhood V of S_0 in the sense of Nishino ([4]; p. 224). Then one can take a normal tube Σ_Γ about S_0 (see, Nishino [1]; p. 72) such that ζ_0 $(S_z \cap V) \supset S_0^0$ for every prime surface S_z in Σ_Γ. One can suppose that Γ is the part of the analytic surface $L = \zeta_0^{-1}(P_0)$ given by the inequality $|f - a_0| < \rho$. Let Γ^* signify the disk $|z - a_0| < \rho$ on the z-plane. Put $\Sigma_\Gamma^0 = \zeta_0^{-1}(S_0^0) \cap \Sigma_\Gamma$ and put $S_z^0 = \zeta_0^{-1}(S_0^0) \cap S_z$, $P_z = \zeta_0^{-1}(P_0) \cap S_z$ and $\gamma_i^z = \zeta_0^{-1}(\gamma_i^0) \cap S_z$ $(i = 1, \cdots, n)$ for each prime surface S_z in Σ_Γ.

Now, in the above situation, one can prove the following lemma.

LEMMA 1. *For every point z belonging to a set of positive capacity in Γ^*, assume that S_z in Σ_Γ is of type* $(0, n)$ *and is parabolic. Then every prime surface S_z in Σ_Γ is of type* $(0, n)$ *and is parabolic and* grad f *is not zero at every point on S_z.*

REMARK. It is seen that in Lemma 1, the words "type $(0, n)$" can be replaced by the words "type (g, n)". (cf. Yamaguchi [6]; pp. 428–430).

PROOF OF LEMMA 1. One proceeds along the line of Nishino ([4]; pp. 243–263). Consider an open ball Q^r $(r_0 < r)$ containing Σ_Γ^0. For each prime surface S_z of f in Σ_Γ, let S_z^r be the connected component of $S_z \cap Q^r$ containing P_z. Denote by Σ_Γ^r the union of all S_z^r for $z \in \Gamma^*$. There are at most a finite number of S_z^r in Σ_Γ^r which has at least a singular point on $\bar{S}_z^r$. Denote them by S_j^r $(j = 1, \cdots, \alpha)$ and put $a_j = f(S_j^r)$. Let D^r be the union of all S_z^r in Σ_Γ^r for $z \in \Gamma_r^* = \Gamma^* - \bigcup \{a_j\}$. Put $D_0^r = D^r \cap \Sigma_\Gamma^0$ and $\Gamma_r = \Gamma \cap D^r$.

Now, for each S_z^r in D^r, form the $(0, n)$-covering $\tilde{S}_z^r$ of S_z^r with respect to S_z^0. Denote by $\tilde{D}^r$ the union of all $\tilde{S}_z^r$ for $z \in \Gamma_r^*$. By forming an analytic retraction ζ_z about each S_z^r in D^r, one can naturally define a topology in $\tilde{D}^r$. Thus $\tilde{D}^r$ is a "domaine multivalent sans point critique intérieur étalé au-dessus de D^r" and it is a two dimensional Stein manifold. Moreover, $\tilde{S}_z^r$ is a non-singular analytic surface of type $(0, n)$ in $\tilde{D}^r$.

One can suppose that L is given by the analytic line $x = 0$ and $\partial f/\partial y$ is not zero at every point in a neighborhood of Γ. The domain R^r of holomorphy of the function obtained by the resolution of the equation $f(x, y) - z = 0$ in $(x, y) \in D^r$ and in $z \in \Gamma_r^*$ with respect to y, is a "domaine multivalent étalé" over the cylinder domain (Γ_r^*, C), where C is the complex plane $|x| < +\infty$. Then, R^r is analytically equivalent to D^r. For a $z' \in \Gamma_r^*$, denote by $R_{z'}^r$ the analytic surface in R^r which corresponds to $S_{z'}^r$ in D^r. Then $R_{z'}^r$ is on the analytic line $z = z'$. Let O_z be the point in R_z^r which corresponds to the point P_z. One can construct, by the projection, the covering $\tilde{R}^r$ of R^r and the covering $\tilde{R}_z^r$ of R_z^r as the images of $\tilde{D}^r$ and of $\tilde{S}_z^r$, respectively. Denote by $\tilde{O}_z$ the image of P_z.

By Yamaguchi's lemma ([6]; Lemma 2, p. 426), it follows that the Robin constant $\lambda_r(z)$ of $\tilde{R}_z^r$ at $\tilde{O}_z$ with respect to the local coordinate x, is a superharmonic function in Γ_r^*.

For a prime surface S_z in Σ_Γ such that grad f is not zero at every point on S_z, form the $(0, n)$-covering $\tilde{S}_z$ of S_z with respect to S_z^0. Denote by $\tilde{R}_z$ the image of $\tilde{S}_z$. Let $\lambda(z)$ be the Robin constant of $\tilde{R}_z$ at $\tilde{O}_z$ with respect to the local coordinate x. Then $\lambda(z) = \lim_{r\to\infty} \lambda_r(z)$ and $\tilde{S}_z$ is parabolic if and only if $\lambda(z)$ is infinite.

Now, for a prime surface S_c in Σ_Γ such that grad f is not zero at every point on S_c, suppose that S_c is not of type $(0, n)$ or not parabolic. Then, from Nishino's theorem ([4]; Theorem 4, p. 242), $\tilde{S}_c$ is not parabolic. Hence $\lambda(c) < +\infty$. Therefore, by the similar method to that of Nishino ([4]; pp. 259–261), one can arrive at a contradiction. Thus, every prime surface S_z in Σ_Γ such that grad f is not zero at every point on S_z, is of type $(0, n)$ and is parabolic. Hence, by the same reasoning as that in Nishino ([2]; pp. 264–269), there is no prime surface S_z in Σ_Γ such that grad f is zero at a point on S_z. Therefore, every prime surface S_z in Σ_Γ is of type $(0, n)$ and is parabolic and grad f is not zero at every point on S_z. q.e.d.

Next one can prove the following lemma.

LEMMA 2. *Let S_{z_0} be a prime surface of order* 1 *of f with a value z_0, let Σ_{Γ_0} be a normal tube about S_{z_0} and let Γ_0^* be a disk $|z - z_0| < \rho_0$ on the z-plane such that $f_{|\Gamma_0}: \Gamma_0 \to \Gamma_0^*$ is analytically isomorphic. If the set of $z \in \Gamma_0^*$ such that S_z in Σ_{Γ_0} is parabolic and is of type* $(0, n)$ *has an interior point, then all prime surfaces in Σ_{Γ_0} are parabolic and their types are at most* $(0, n)$.

PROOF. Let Δ^* be a non-empty connected component of the interior of the set of $z \in \Gamma_0^*$ such that S_z in Σ_{Γ_0} is parabolic and is of type $(0, n)$.

First, it will be proved that $\varDelta^*$ has no exterior point in $\varGamma_0^*$. For this purpose, it is sufficient to show that assumption that $\varDelta^*$ has an exterior point in $\varGamma_0^*$ induces a contradiction.

Now, assume that $\varDelta^*$ has an exterior point in $\varGamma_0^*$. Denote by $\varDelta_0^*$ the interior of the closure of $\varDelta^*$. Let m_0 be the maximum of m such that S_z in $\varSigma_{\varGamma_0}$ is of type $(0, m)$, where $z \in \partial\varDelta_0^* \cap \varGamma_0^*$. Then, from Nishino's theorem ([4]; Theorem 5, p. 252), one can see that $1 \leqq m_0 \leqq n-1$ and that there is an $a_0 \in \partial\varDelta_0^* \cap \varGamma_0^*$ such that every prime surface S_z in $\varSigma_{\varGamma_0}$ is of type $(0, m_0)$ for $z \in \partial\varDelta_0^* \cap \varGamma^*$ if one takes a sufficiently small disk $\varGamma^* = \{z; |z-a_0| < \rho\} \Subset \varGamma_0^*$. Moreover, one can suppose that grad f is not zero at every point on the prime surface S_{a_0} in $\varSigma_{\varGamma_0}$. Note that $\partial\varDelta_0^* \cap \varGamma^*$ contains a continuum. This can be easily verified from the fact that $\varDelta^*$ is connected and has an exterior point in $\varGamma_0^*$. Put $\varGamma = f^{-1}(\varGamma^*) \cap \varGamma_0$ and $P_z = S_z \cap \varGamma$ for $z \in \varGamma^*$.

Consider an open ball Q^{r_0} such that the connected component $S_{a_0}^0$ of $S_{a_0} \cap Q^{r_0}$ containing P_{a_0} is of type $(0, m_0)$ and such that m_0 closed curves γ_i^0 $(i = 1, \cdots, m_0)$ limiting $S_{a_0}^0$ in S_{a_0} are all simple. Consider an analytic retraction ζ_0 about S_{a_0} defined in a neighborhood V of S_{a_0}. Then, taking a sufficiently small $\varGamma^*$, one can suppose that $\zeta_0(S_z \cap V) \supset S_{a_0}^0$ for every prime surface S_z in $\varSigma_\varGamma$. Moreover, one can suppose that $\varGamma$ is the part of the analytic surface $L = \zeta_0^{-1}(P_{a_0})$ given by the inequality $|f - a_0| < \rho$. Put $\varSigma_\varGamma^0 = \zeta_0^{-1}(S_{a_0}^0) \cap \varSigma_\varGamma$ and put $S_z^0 = \zeta_0^{-1}(S_{a_0}^0) \cap S_z$, $\gamma_i^z = \zeta_0^{-1}(\gamma_i^0) \cap S_z$ $(i = 1, \cdots, m_0)$ for each prime surface S_z in $\varSigma_\varGamma$. Then, S_z^0 is of type $(0, m_0)$ and is limited by m_0 simple closed curves γ_i^z $(i = 1, \cdots, m_0)$ in S_z.

Consider another open ball Q^r $(r_0 < r)$ containing $\varSigma_\varGamma^0$. For each prime surface S_z in $\varSigma_\varGamma$, let S_z^r be the connected component of $S_z \cap Q^r$ containing P_z. There are at most a finite number of S_z^r in $\varSigma_\varGamma$ which has at least a singular point on $\bar{S}_z^r$. Denote them by S_j^r $(j = 1, \cdots, \alpha)$ and put $a_j = f(S_j^r)$. Denote by D^r the union of all S_z^r for $z \in \varGamma_r^* = \varGamma^* - \bigcup\{a_j\}$. Put $D_0^r = D^r \cap \varSigma_\varGamma^0$ and $\varGamma_r = \varGamma \cap D^r$. Now, for each prime surface S_z^r in D^r, put $\hat{S}_z^r = S_z^r$ if $z \in \varGamma_r^* \cap \bar{\varDelta}_0^*$ and put $\hat{S}_z^r = \tilde{S}_z^r$ (= the $(0, m_0)$-covering of S_z^r with respect to S_z^0) if $z \in \varGamma_r^* - \bar{\varDelta}_0^*$. Denote by $\hat{D}^r$ the union of all $\hat{S}_z^r$ for $z \in \varGamma_r^*$.

Define a topology in $\hat{D}^r$ as follows:

1) The case when $\hat{P} \in \hat{S}_a^r$ for $a \in \varGamma_r^* \cap \varDelta_0^*$; consider a neighborhood $V_{\hat{P}}$ of $\hat{P}$ in C^2 such that $V_{\hat{P}}$ is contained in $\bigcup_{z \in \varGamma_r^* \cap \varDelta_0^*} S_z^r$. Then such a $V_{\hat{P}}$ is regarded as a neighborhood of $\hat{P}$ in $\hat{D}^r$.

2) The case when $\hat{P} \in \hat{S}_a^r$ for $a \in \varGamma_r^* - \bar{\varDelta}_0^*$; forming an analytic retraction ζ_a about S_a^r, one can naturally define neighborhoods of $\hat{P}$ in $\hat{D}^r$ along Nishino's argument ([3]; pp. 249-251).

3) The case when $\hat{P} \in \hat{S}_a^r$ for $a \in \Gamma_r^* \cap \partial\Delta_0^*$; take a path l_a from P_a to $\hat{P}$ on S_a^r and take an open, connected and simply connected set δ on S_a^r such that $\hat{P} \in \delta$, $\delta \Subset S_a^{r'}$ and $l_a \subset S_a^{r'}$ for a certain r' $(r_0 < r' < r)$. Consider an analytic retraction ζ_a about S_a^r defined in a neighborhood V of S_a^r. Take a sufficiently small disk $\Gamma_1^*: |z - a| < \rho_1$ such that $\Gamma_1^* \Subset \Gamma_r^*$ and $\zeta_a(V \cap S_z^r) \supset S_a^{r'}$ for $z \in \Gamma_1^*$. Let $V_{\hat{P}}$ be the union of $\zeta_a^{-1}(\delta) \cap S_z^r$ for $z \in \Gamma_1^*$. Then, for each point Q in $\zeta_a^{-1}(\delta) \cap S_z^r$ where $z \in \Gamma_1^* - \bar{\Delta}_0^*$, there is a uniquely determined point $\hat{Q}$ of $\hat{S}_z^r$ represented by a pair (Q, m) such that $\zeta_a(Q)$ is in δ and $\zeta_a(m) = \sigma \cdot l_a$, where σ is a path from $\hat{P}$ to $\zeta_a(Q)$ on δ. For each point Q in $\zeta_a^{-1}(\delta) \cap S_z^r$ where $z \in \Gamma_1^* \cap \bar{\Delta}_0^*$, one obtains $\hat{Q} = Q$ by definition. Denote by $\hat{V}_{\hat{P}}$ the set of all these points $\hat{Q}$ and regard $\hat{V}_{\hat{P}}$ as a neighborhood of $\hat{P}$ in $\hat{D}^r$.

From this, one can define a Hausdorff topology in $\hat{D}^r$. Moreover, one can show that $\hat{D}^r$ is a "domaine multivalent sans point critique intérieur étalé au-dessus de D^r" and that it is a two dimensional Stein manifold. These facts are proved by the similar argument to that in the case of $\tilde{D}^r$ of Lemma 1.

Hence, if, for a $c \in \Gamma^* \cap \partial\Delta_0^*$, S_c in Σ_Γ is not parabolic and grad f is not zero at every point on S_c, one obtains a contradiction by the same way as that in the proof of Lemma 1. Therefore, every prime surface S_c in Σ_Γ such that $c \in \Gamma^* \cap \partial\Delta_0^*$ and such that grad f is not zero at every point on S_c, is of type $(0, m_0)$ and is parabolic.

One can immediately see that the set of $c \in \Gamma^* \cap \partial\Delta_0^*$ such that S_c in Σ_Γ is type $(0, m_0)$ and is parabolic, is of positive capacity. Hence every prime surface S_c in Σ_Γ is of type $(0, m_0)$ and is parabolic by Lemma 1. This is a contradiction, because a_0 is a boundary point of Δ_0^* in Γ_0^*. Therefore, $\Gamma_0^* \subset \bar{\Delta}^*$. Thus every prime surface S_z in Σ_{Γ_0} is at most of type $(0, n)$. Hence Yamaguchi's theorem ([6]; Theorem 2, p. 427) implies that every prime surface S_z in Σ_{Γ_0} is parabolic. q.e.d.

3. **Proof of Theorem.** Now, using Lemmas 1 and 2, one can prove Theorem, stated in § 1, in the following way.

By the Borel-Lebesgue theorem, one can cover the (x, y)-space by a countably many normal tubes Σ_{Γ_i} $(i = 1, 2, \cdots)$ about prime surfaces S_i of order 1 of f except for at most a countable number of prime surfaces of higher order of f.

By the assumption of Theorem, one can take a positive integer n and a normal tube Σ_{Γ_0} about a prime surface S_0 of f such that Σ_{Γ_0} satisfies the conditions of Lemma 1. Therefore, every prime surface S_z in Σ_{Γ_0} is parabolic and is of type $(0, n)$ and grad f is not zero at every point on S_z. From Lemmas 1 and 2, every prime surface S_z in Σ_{Γ_i} $(i = 1, 2, \cdots)$

is non-singular and parabolic and is of order 1 and of type $(0, n)$ except for values z belonging to a set of capacity zero. Hence types of all prime surfaces of f are at most $(0, n)$. Therefore, every prime surface of f is parabolic by Yamaguchi's theorem ([6]; Theorem 4, p. 433). Therefore, if f has "sufficiently many" schlicht algebraic prime surfaces, then f belongs to the class (A). Thus the proof of Theorem is complete.

REMARK. From the above Theorem and Suzuki's theorem ([5]; Theorem 6, p. 253), the following proposition is immediately obtained.

Under the assumption of Theorem, every prime surface S_z of f is of type $(0, n)$ except for at most $(n-1)$ values z and type of every exceptional prime surface is at most $(0, n)$.

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