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TWO REMARKS ON CONTACT METRIC STRUCTURES

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1. **Introduction.** This paper is a continuation of the author's paper [1] in which it was shown that there are no flat contact metric structures on a contact manifold of dimension $\geqq 5$. Although this result is a non-existence theorem, the argument yields some positive results on certain contact metric manifolds. Here we prove two such results.

THEOREM A. *A contact metric manifold M^{2n+1} is a K-contact manifold if and only if the Ricci curvature in the direction of the characteristic vector field ξ is equal to $2n$.*

THEOREM B. *Let M^{2n+1} be a contact metric manifold and suppose that $R(X, Y)\xi = 0$ for all vector fields X and Y. Then M^{2n+1} is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature* 4.

In Section 4 we discuss the contact structure on the tangent sphere bundle of a flat Riemannian manifold as an example of Theorem B.

2. **Proof of Theorem A.** Throughout this paper we use the same notation and terminology that was used in [1]. Recall that a contact metric structure (φ, ξ, η, g) is said to be *K-contact* if the vector field ξ is Killing. For a contact metric structure one automatically has

$$(\mathscr{L}_\xi g)(X, \xi) = \xi\eta(X) - \eta([\xi, X]) = (\mathscr{L}_\xi \eta)(X) = 0$$

by the invariance of the contact form η under the 1-parameter group of ξ. Since $d\eta$ is also invariant and $d\eta(X, Y) = g(X, \varphi Y)$, we see that ξ is Killing if and only if $\mathscr{L}_\xi\varphi = 0$.

It is well known that on a K-contact manifold the sectional curvature of a plane section containing ξ is equal to 1, [3]. Thus we shall only prove the sufficiency in Theorem A and do so by showing that the operator $h = (1/2)\mathscr{L}_\xi\varphi$ vanishes.

In [1] the following general formulas for a contact metric structure (φ, ξ, η, g) were obtained

$$\nabla_X\xi = -\varphi X - \varphi h X \tag{2.1}$$

and

$$\frac{1}{2}(R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi) = h^2 X + \varphi^2 X . \tag{2.2}$$

Now let $\{X_i, X_{n+i}, \xi\}$ $i = 1, \cdots, n$ be a local orthonormal basis such that $X_{n+i} = \varphi X_i$. Then as $h\xi = 0$ and $\varphi\xi = 0$, taking the inner product of (2.2) with X belonging to the basis and summing we see that

$$\operatorname{tr} h^2 = 2n - g(Q\xi, \xi)$$

where Q is the Ricci curvature operator.

Now if the Ricci curvature in the direction ξ is equal to $2n$, we have $\operatorname{tr} h^2 = 0$. It was also shown in [1] that h is a symmetric operator and hence its eigenvalues are real. The eigenvalues of h^2 are therefore non-negative so that $\operatorname{tr} h^2 = 0$ implies that $h = 0$ as desired.

3. Proof of Theorem B. From equation (2.2) we see that the condition $R(X, Y)\xi = 0$ for all vector fields X, Y implies that $h^2 = -\varphi^2$; in particular note that $h\xi = 0$ and rank $h = 2n$. Thus the non-zero eigenvalues of h are ± 1 and their eigenvectors are orthogonal to ξ. Now from $d\eta(X, Y) = (1/2)(g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)) = g(X, \varphi Y)$ and equation (2.1) one can easily see that $\varphi h + h\varphi = 0$. Thus if X is an eigenvector of $+1$ (respectively -1), φX is an eigenvector of -1 (respectively $+1$). Consequently the contact distribution D defined by $\eta = 0$ is decomposed into the orthogonal eigenspaces of ± 1 which we denote by $[+1]$ and $[-1]$ respectively. We denote by $[-1] \oplus [\xi]$ the distribution spanned by $[-1]$ and the vector field ξ. Note that equation (2.1) says that $\nabla_X \xi = 0$ for $X \in [-1]$ and $\nabla_X \xi = -2\varphi X$ for $X \in [+1]$.

In [1] it was shown that $[-1]$, $[-1] \oplus [\xi]$ and $[+1]$ are integrable. Thus M^{2n+1} is locally the product of an integral submanifold M^{n+1} of $[-1] \oplus [\xi]$ and an integral submanifold M^n of $[+1]$. In particular we can choose coordinates $(u^0, \cdots, u^{2n})$ such that $\partial/\partial u^0, \cdots, \partial/\partial u^n \in [-1] \oplus [\xi]$ and $X_i = \partial/\partial u^{n+i} \in [+1]$, $i = 1, \cdots, n$. In [1] the following formulas were obtained

$$\nabla_{\varphi X_i} \varphi X_j = 0 , \tag{3.1}$$

$$g(\nabla_{\varphi X_i} X_j, X_k) = 0 , \tag{3.2}$$

and

$$g(\nabla_{X_k} \nabla_{X_i} \varphi X_j, \varphi X_l) - g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) = -4g(X_i, X_j) g(X_k, X_l) . \tag{3.3}$$

Now since $\{\varphi X_i, \xi\}$ is a local basis of tangent vector fields on M^{n+1}, equation (3.1) and $R(X, Y)\xi = 0$ show that M^{n+1} is flat.

Next notice that $\nabla_{\varphi X_i} X_j = 0$. For, we have equation (3.2), $g(\nabla_{\varphi X_i} X_j, \varphi X_k) = -g(X_j, \nabla_{\varphi X_i} \varphi X_k) = 0$ by equation (3.1) and $g(\nabla_{\varphi X_i} X_j, \xi) = -g(X_j,$

$\nabla_{\varphi X_i}\xi) = 0$ by equation (2.1). Now interchanging i and k in (3.3) and subtracting we have

$$g(R(X_k, X_i)\varphi X_j, \varphi X_l) - g(R(X_k, X_i)X_j, X_l)$$
$$= -4g(X_i, X_j)g(X_k, X_l) + 4g(X_k, X_j)g(X_i, X_l) .$$

Using $\nabla_{\varphi X_i}X_j = 0$ and $[\varphi X_i, \varphi X_k] = 0$ we see that $g(R(X_k, X_i)\varphi X_j, \varphi X_l) = g(R(\varphi X_j, \varphi X_l)X_k, X_i) = 0$ and hence $g(R(X_k, X_i)X_j, X_l) = 4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l))$ completing the proof.

4. The tangent sphere bundle. In this section we shall show that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$. Y. Tashiro [6] showed that this structure was K-contact if and only if the base manifold has positive constant curvature 1 in which case the structure is Sasakian.

We first give some preliminaries on the tangent bundle. Let M be an $(n + 1)$-dimensional manifold and $\bar{\pi}: TM \rightarrow M$ its tangent bundle. If $(x^1, \cdots, x^{n+1})$ are local coordinates on M, we set $q^i = x^i \circ \bar{\pi}$; then $(q^1, \cdots, q^{n+1})$ together with the fibre coordinates $(v^1, \cdots, v^{n+1})$ form local coordinates on TM. If X is a vector field on M, its *vertical lift* X^V is defined by $X^V\omega = \omega(X)\circ\bar{\pi}$ where ω is a 1-form which on the left side of this equation is regarded as a function on TM. For an affine connection D on M one defines the horizontal lift X^H of X; see Yano and Ishihara [7] or Dombrowski [2] for details. The connection map $K: TTM \rightarrow TM$ is then given by $KX^H = 0$ and $K(X^V{}_Z) = X_{\bar{\pi}(Z)}$, $Z \in TM$. Similarly TM admits an almost complex structure defined by $JX^H = X^V$ and $JX^V = -X^H$. Dombrowski [2] shows that J is integrable if and only if D has vanishing curvature $\underline{R}$ and torsion.

If now G is the Riemannian metric on M and D its Riemannian connection, we define a Riemannian metric $\bar{g}$ on TM, called the *Sasaki metric* [4], by

$$\bar{g}(X, Y) = G(\bar{\pi}_*X, \bar{\pi}_*Y) + G(KX, KY) \tag{4.1}$$

where here X and Y are vectors on TM. The Riemannian connection $\bar{\nabla}$ of $\bar{g}$ is given at a point $Z \in TM$ by

$$\begin{aligned}
&(\bar{\nabla}_{X^H}Y^H)_Z = (D_XY)^H{}_Z - \frac{1}{2}(\underline{R}(X, Y)Z)^V \\
&(\bar{\nabla}_{X^H}Y^V)_Z = -\frac{1}{2}(\underline{R}(Y, Z)X)^H + (D_XY)^V{}_Z \\
&(\bar{\nabla}_{X^V}Y^H)_Z = -\frac{1}{2}(\underline{R}(X, Z)Y)^H \\
&\bar{\nabla}_{X^V}Y^V = 0 .
\end{aligned} \tag{4.2}$$

The curvature tensor of $\bar{\nabla}$ will be denoted by $\bar{R}$. $\bar{g}$ is a Hermitian metric for the almost complex structure J. On TM we define a 1-form β by the local expression $\beta = \sum G_{ij}v^j dq^i$. As is well known β induces a contact structure on the tangent sphere bundle T_1M. Moreover $2d\beta$ is the fundamental 2-form of the almost Hermitian structure (J, G). Summing up we see that (J, G) is an almost Kähler structure on TM which is Kählerian if and only if M is flat [2, 5].

As is customary we regard T_1M as the bundle of unit tangent vectors; however owing to the factor 1/2 in the coboundary formula for $d\eta$, a homothetic change of metric will be made. (If one adopts the convention that the 1/2 does not appear, this change is not necessary, but to be consistent the odd-dimensional sphere as a standard example of a K-contact manifold should then be taken as a sphere of radius 2.)

Now T_1M is a hypersurface of TM and the vector field $v^i(\partial/\partial v^i)$ is a unit normal as well as the position vector for a point Z in a fibre of TM. ι will denote the immersion, $\pi = \bar{\pi}\circ\iota$ the projection map and $g' = \iota^*\bar{g}$ the induced metric. Define φ', ξ', and η' on T_1M by, $\iota_*\xi' = -JN$ and $J\iota_*X = \iota_*\varphi'X + \eta'(X)N$; $(\varphi', \xi', \eta', g')$ is then an almost contact metric structure. Moreover η' is the contact form on T_1M induced from β on TM as one can easily check. However $g'(X, \varphi'Y) = 2d\eta'(X, Y)$, so that $(\varphi', \xi', \eta', g')$ is not a contact metric structure. Of course, the difficulty is easily rectified and we shall take $\eta = (1/2)\eta'$, $\xi = 2\xi'$, $\varphi = \varphi'$, $g = (1/4)g'$ as our contact metric structure on T_1M.

Let ∇ be the Riemannian connection of g. For completeness we give explicitly the covariant derivatives of ξ and φ. X and Y will denote horizontal tangent vector fields to T_1M and U and W will denote vertical tangent vector fields. Using $\iota_*\xi = 2v^i(\partial/\partial x^i)^H$ and equations (4.2) we obtain at a point $Z \in T_1M \subset TM$

$$(\iota_*\nabla_X\xi)_Z = -(\underline{R}(\pi_*X, Z)Z)^V \tag{4.3}$$

$$(\iota_*\nabla_U\xi)_Z = -2\iota_*\varphi U_Z - (\underline{R}(K\iota_*U, Z)Z)^H\,, \tag{4.4}$$

$$(\iota_*(\nabla_X\varphi)Y)_Z = -\frac{1}{2}(\underline{R}(\pi_*X, Z)\pi_*Y)^H$$

$$(\iota_*(\nabla_X\varphi)U)_Z = \frac{1}{2}\tan(\underline{R}(\pi_*X, Z)K\iota_*U)^V$$

$$(\iota_*(\nabla_U\varphi)X)_Z = -2\eta(X)\iota_*U_Z + \frac{1}{2}\tan(\underline{R}(K\iota_*U, Z)\pi_*X)^V$$

$$(\iota_*(\nabla_U\varphi)W)_Z = 2g(U, W)\iota_*\xi_Z + \frac{1}{2}(\underline{R}(K\iota_*U, Z)K\iota_*W)^H \tag{4.5}$$

where tan denotes the tangential part.

We can now prove the following theorem.

THEOREM C. *The contact metric structure* (φ, ξ, η, g) *on* T_1M *satisfies* $R(\xi, U)\xi = 0$ *for all vertical vector fields* U *if and only if* M *is flat in which case* $R(X, Y)\xi = 0$ *for all vector fields* X *and* Y *on* T_1M.

PROOF. Using equation (4.2) we can easily obtain

$$K\bar{R}(X^H, U^V)Y^H = \frac{1}{4}\underline{R}(X, \underline{R}(U, Z)Y)Z + \frac{1}{2}\underline{R}(X, Y)U$$

for any three vector fields X, U and Y on M. If now we let U be a vertical tangent vector field on T_1M, then $R(\xi, U)\xi = 0$ implies that

$$\underline{R}(Z, \underline{R}(K\iota_* U, Z)Z)Z = 0$$

and hence that

$$\underline{R}(Z, \underline{R}(X, Z)Z)Z = 0$$

for all vectors X and Z on M. Therefore

$$0 = G(\underline{R}(Z, \underline{R}(Z, X)Z)Z, X) = ||\underline{R}(Z, X)Z||^2$$

that is $\underline{R}(Z, X)Z = 0$ for all vectors X and Z on M. Linearizing this and using the Bianchi identity we have that $\underline{R}(X, Y)Z = 0$ for all X, Y and Z on M.

Conversely if M is flat, equations (4.3) and (4.4) give $\nabla_X\xi = 0$ for X horizontal and $\nabla_U\xi = -2\varphi U$ for U vertical. Thus the vertical distribution on T_1M is the $[+1]$ distribution of our earlier sections and the horizontal distribution is the $[-1] \oplus [\xi]$ distribution. Moreover by the flatness of M these distributions are integrable and hence for X and Y horizontal on T_1M and U and W vertical we may take these as coordinate vectors as in Section 3. Now $R(X, Y)\xi = 0$ is trivial,

$$R(X, U)\xi = -2\nabla_X\varphi U = 0$$

by equation (3.1), and

$$\begin{aligned} R(U, W)\xi &= -2\nabla_U\varphi W + 2\nabla_W\varphi U \\ &= -2(\nabla_U\varphi)W + 2(\nabla_W\varphi)U \\ &= -4g(U, W)\xi + 4g(W, U)\xi \\ &= 0 \end{aligned}$$

by equation (4.5).

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