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ON NON-COMMUTATIVE HARDY SPACES ASSOCIATED WITH FLOWS ON FINITE VON NEUMANN ALGEBRAS

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1. Introduction. Let M be a von Neumann algebra and let $\{\alpha_t\}_{t\in R}$ be a flow by which we mean a σ-weakly continuous one-parameter group of $*$-automorphisms on M. Let $H^\infty(\alpha)$ be the set of all elements of M with non-negative spectrum with respect to $\{\alpha_t\}_{t\in R}$. Recently the structure of $H^\infty(\alpha)$ has been investigated by Kawamura-Tomiyama [9], Loebl-Muhly [11] and the author [15]. It is important to study the structure of $H^\infty(\alpha)$ in view of the role played by the disk algebra over the unit circle. Furthermore $H^\infty(\alpha)$ happens to become a subdiagonal algebra which may be regarded as a non-commutative, weak*-Dirichlet algebra. On the other hand, as a generalization of the Hardy space H^p over the unit circle, several authors studied the Hardy spaces in the L^p-space taking values in a Hilbert space ([4], [14], etc.) or a von Neumann algebra, in particular, the ring of all $n\times n$ matrices over the complex numbers ([1], [5], [6], etc.). The latter is considered as non-commutative Hardy spaces.

Our objective in this paper is to define and investigate the non-commutative Hardy spaces $H^p(\alpha)$ associated with $\{\alpha_t\}_{t\in R}$ in case M has a faithful, normal, α_t-invariant finite trace. The method is based on the theory of spectral subspaces for a flow and the non-commutative theory of integration for a finite von Neumann algebra. Now we assume that there is a faithful, normal, α_t-invariant, finite trace τ on M. Using the non-commutative integration theory with respect to τ, we consider Banach spaces $L^p(M,\tau)$, $1\leqq p<\infty$. In §2, we define $H^p(\alpha)$ and $H_0^p(\alpha)$ and study their basic properties. In §3, we show examples of $H^p(\alpha)$. In §4, we consider the doubly invariant subspace theorem for $H^\infty(\alpha)$ in $L^p(M,\tau)$ which is a generalization of Wiener's theorem. Let $\mathscr{M}$ be a closed subspace of $L^p(M,\tau)$. If $\mathscr{M}$ is a left doubly invariant subspace of $L^p(M,\tau)$ in the sence that $H^\infty(\alpha)\mathscr{M}\subseteqq\mathscr{M}$ and $H^\infty(\alpha)^*\mathscr{M}\subseteqq\mathscr{M}$, then there exists a projection e of M such that $\mathscr{M}=L^p(M,\tau)e$. In §5, we consider the simply invariant subspace theorem for $H^\infty(\alpha)$ in $L^p(M,\tau)$

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which is an extension of Beurling's theorem. Let $\mathscr{M}$ be a left simply invariant subspace in the sense that $[H_0^\infty(\alpha)\mathscr{M}]_p \subsetneqq \mathscr{M}$, where $[H_0^\infty(\alpha)\mathscr{M}]_p$ is the closed linear span of $H_0^\infty(\alpha)\mathscr{M}$ in $L^p(M, \tau)$. If $\{\alpha_t\}_{t\in R}$ is ergodic, there exists a unitary element u of M such that $\mathscr{M} = H^p(\alpha)u$.

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2. The non-commutative Hardy spaces $H^p(\alpha)$. Let M be a finite von Neumann algebra acting on a Hilbert space H. Let $\{\alpha_t\}_{t\in R}$ be a flow on M. Throughout this paper, we assume that M has a faithful, α_t-invariant, normal trace τ on M such that $\tau(1) = 1$. Such a τ exists, for example, if $\{\alpha_t\}_{t\in R}$ is a group of automorphisms leaving the center of M elementwise fixed, in particular, if M is a factor. Let $1 \leqq p < \infty$ and we write $L^p(M, \tau)$ the space of all integrable operators for the gage space (M, H, τ) such that $\tau(|x|^p) < \infty$, $|x| = (x^*x)^{1/2}$, in the sence of Segal [17]. If $p = \infty$, we identify M with $L^\infty(M, \tau)$. It is well-known that $L^p(M, \tau)$ becomes a Banach space with the L^p-norm $\|x\|_p = \tau(|x|^p)^{1/p}$, $x \in L^p(M, \tau)$ [13, Theorem 8]. We refer the reader to ([3], [13], [17]) for the basic properties of the space $L^p(M, \tau)$. Recall that $L^1(M, \tau)$ may be identified with the predual of M with respect to the pairing $\langle x, y\rangle = \tau(xy)$, $x \in L^1(M, \tau)$, $y \in M$ [3, Théorème 5]. Furthermore, in analogy with the scalar case, the dual of $L^p(M, \tau)$, $1 < p < \infty$, may be identified with $L^q(M, \tau)$, $1/p + 1/q = 1$, via the pairing $\langle x, y\rangle = \tau(xy)$, $x \in L^p(M, \tau)$, $y \in L^q(M, \tau)$ [3, Théorème 7]. Since M is finite and $\tau(1) = 1$, we have $M \subset L^q(M, \tau) \subset L^p(M, \tau)$, $1 \leqq p \leqq q \leqq \infty$ [13, Lemma 3.3] and M is dense in $L^p(M, \tau)$ with respect to the L^p-norm [3, Proposition 5].

REMARK 2.1. In the case of abelian von Neumann algebras, the concept of measurable operator just introduced is essentially equivalent to the concept of measurable function [17, Theorem 2].

PROPOSITION 2.2. *For each p, $1 \leqq p < \infty$, $\{\alpha_t\}_{t\in R}$ extends uniquely to a strongly continuous representation of $\boldsymbol{R}$ of isometries on $L^p(M, \tau)$.*

PROOF. Since τ is α_t-invariant, we have $\|\alpha_t(x)\|_p = \|x\|_p$ for $x \in M$. Therefore $\{\alpha_t\}_{t\in R}$ extends uniquely to a representation of $\boldsymbol{R}$ of isometries on $L^p(M, \tau)$ and we also denote this extension of $\{\alpha_t\}_{t\in R}$ to each $L^p(M, \tau)$ by $\{\alpha_t\}_{t\in R}$. Let $x \in L^p(M, \tau)$. For any $\varepsilon > 0$, there exists an element $a \in M$ such that $\|x - a\|_p < \varepsilon$. For any $y \in L^q(M, \tau)$, $1/p + 1/q = 1$, we have

$$\begin{aligned}|\tau((\alpha_t(x) - x)y)| &\leqq |\tau((\alpha_t(x) - \alpha_t(a))y| + |\tau((\alpha_t(a) - a)y)| + |\tau((a - x)y)| \\ &\leqq \|\alpha_t(x - a)\|_p\|y\|_q + |\tau((\alpha_t(a) - a)y)| + \|a - x\|_p\|y\|_q\,.\end{aligned}$$

Since $\{\alpha_t\}_{t\in R}$ is σ-weakly continuous, $\tau(\alpha_t(a)y)$ is a continuous function with respect to t. Thus there exists $t_0(>0)$ such that $|\tau((\alpha_t(a)-a)y)|<\varepsilon$, $|t|<t_0$. Hence we have

$$|\tau((\alpha_t(x)-x)y)|<(2\|y\|_q+1)\varepsilon\ ,\qquad |t|<t_0\ ,$$

and so $\{\alpha_t\}_{t\in R}$ is $\sigma(L^p(M,\tau), L^q(M,\tau))$-continuous. From a well-known result $\{\alpha_t\}_{t\in R}$ is strongly continuous on $L^p(M,\tau)$. This completes the proof.

Throughout this paper we denote this extension of $\{\alpha_t\}_{t\in R}$ to $L^p(M,\tau)$ by $\{\alpha_t\}_{t\in R}$ too.

Next, we define a representation $\alpha(\cdot)$ of $L^1(R)$ into the bounded operators on $L^p(M,\tau)$ by $\alpha(f)x=\int_{-\infty}^{\infty}f(t)\alpha_t(x)dt$ where $x\in L^p(M,\tau)$ and $f\in L^1(\boldsymbol{R})$. For $f\in L^1(\boldsymbol{R})$, we put $Z(f)=\{t\in R: \hat{f}(t)=0\}$, where $\hat{f}(t)=\int_{-\infty}^{\infty}e^{-ist}f(s)ds$, $t\in R$. Let $\mathrm{Sp}_\alpha(x)$ be defined as

$$\bigcap\{Z(f): f\in L^1(\boldsymbol{R}),\ \alpha(f)x=0\}\ .$$

We refer the readers to [2] for the elementary properties of spectra and spectral subspaces.

DEFINITION 2.3. For $1\leqq p\leqq\infty$, the set of all $x\in L^p(M,\tau)$ such that $\mathrm{Sp}_\alpha(x)\subset[0,\infty)$ is denoted by $H^p(\alpha)$ and is called the non-commutative Hardy space of exponent p. Further for $1\leqq p<\infty$ (resp. $p=\infty$) the L^p-norm closure (resp. σ-weak closure) of the set of all $x\in L^p(M,\tau)$ such that $\mathrm{Sp}_\alpha(x)\subset(0,\infty)$ is denoted by $H_0^p(\alpha)$.

REMARK 2.4. Let $M=L^\infty(T)$ where T is the unit circle. Let $x\in L^\infty(T)$. Putting $\alpha_t x(e^{is})=x(e^{i(s-t)})$, $s, t\in\boldsymbol{R}$, and $\tau(x)=1/2\pi\int_0^{2\pi}x(e^{it})dt$, $\{\alpha_t\}_{t\in R}$ is a flow on M and τ is a faithful, normal, α_t-invariant trace such that $\tau(1)=1$. By Remark 2.1, we have $L^p(M,\tau)=L^p(T)$. Observe that $H^p(\alpha)$ coincides with the Hardy space H^p on the unit circle T.

For a subset S of $L^p(M,\tau)$, $1\leqq p\leqq\infty$, $[S]_p$ denotes the closed (resp. σ-weakly closed if $p=\infty$) subspace of $L^p(M,\tau)$ generated by S and we put $S^\perp=\{x\in L^q(M,\tau): \tau(xy)=0,\ y\in S\}$, $1/p+1/q=1$.

PROPOSITION 2.5. *Let* $1\leqq p\leqq\infty$, $1/p+1/q=1$ *and* $x\in L^p(M,\tau)$. *The following assertions are equivalent.*

(i) $x\in H^p(\alpha)$.

(ii) $t\longrightarrow\tau(x\alpha_t(y))$ *belongs to* $H^\infty(\boldsymbol{R})$ *for every* $y\in L^q(M,\tau)$.

(iii) $\tau(xy)=0$ *for every* $y\in H_0^q(\alpha)$.

(iv) $\tau(xy)=0$ *for every* $y\in H_0^\infty(\alpha)$.

PROOF. (i) $\Rightarrow$ (ii). Let $x \in H^p(\alpha)$. For an $\varepsilon > 0$, choose a function $f \in L^1(\boldsymbol{R})$ such that $\hat{f}$ lives in $[\varepsilon, \infty)$. Then, for every $y \in L^q(M, \tau)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \tau(x\alpha_t(y))f(t)dt &= \int_{-\infty}^{\infty} \tau(\alpha_{-t}(x)y)f(t)dt \\ &= \tau\Big(\Big(\int_{-\infty}^{\infty} \alpha_{-t}(x)f(t)dt\Big)y\Big) \\ &= \tau((\alpha(\tilde{f})x)y)\end{aligned}$$

where $\tilde{f}(t) = f(-t)$, $t \in \boldsymbol{R}$. On the other hand

$$\mathrm{Sp}_\alpha(\alpha(\tilde{f})x) \subset \mathrm{Supp}\,\hat{\tilde{f}} \cap \mathrm{Sp}_\alpha(x) \subset (-\infty, -\varepsilon] \cap [0, \infty) = \varnothing .$$

Then we have $\alpha(\tilde{f})x = 0$ and so $t \mapsto \tau(x\alpha_t(y))$ belongs to $H^\infty(\boldsymbol{R})$ for every $y \in L^q(M, \tau)$.

(ii) $\Rightarrow$ (iii). We refer to [2, Proposition 5.1].

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) Suppose that $\tau(xy) = 0$ for every $y \in H_0^\infty(\alpha)$. Then $x \in H^1(\alpha)$ by [9, Lemma 2.2]. From the definition of $H^p(\alpha)$, we have $H^1(\alpha) \cap L^p(M, \tau) = H^p(\alpha)$ and $x \in H^p(\alpha)$. This completes the proof.

Put $M(\alpha) = H^\infty(\alpha) \cap H^\infty(\alpha)^*$. Then $M(\alpha)$ is a finite von Neumann algebra which consists of all fixed points in M with respect to $\{\alpha_t\}_{t\in\boldsymbol{R}}$. Since M has a faithful, normal, α_t-invariant finite trace, there exists a unique, faithful, normal, α_t-invariant projection ε of norm one of M onto $M(\alpha)$ [10, Theorem 2]. Furthermore, for each element $x \in M$, $\varepsilon(x)$ is given as the unique element of the intersection $K(x, \alpha) \cap M(\alpha)$, where $K(x, \alpha)$ denotes the σ-weakly closed convex hull of $\{\alpha_t(x)\}_{t\in\boldsymbol{R}}$. By [9, Proof of Theorem 2.4], we have $H_0^\infty(\alpha) = \{x \in H^\infty(\alpha); \varepsilon(x) = 0\}$.

PROPOSITION 2.6. *Let* $1 \leqq p < \infty$.

(i) *ε extends uniquely to a projection ε_p of norm one of $L^p(M, \tau)$ onto $L^p(M(\alpha), \tau)$.*

(ii) *$L^p(M(\alpha), \tau)$ equals the set of all fixed points of $L^p(M, \tau)$ with respect to $\{\alpha_t\}_{t\in\boldsymbol{R}}$.*

(iii) $H_0^p(\alpha) = \{x \in H^p(\alpha); \varepsilon_p(x) = 0\}$.

PROOF. (i) Let $x \in M$. Since $\varepsilon(x)$ is given as the unique element of $K(x, \alpha) \cap M(\alpha)$, there is a net $\{\psi_i\}_{i\in I}$ of convex combinations of the α_t (i.e., $\psi_i = \sum_{k=1}^{n_i} \lambda_k^{(i)}\alpha_{t_k}^{(i)}$, $\lambda_k^{(i)} \geqq 0$, $\sum_{k=1}^{n_i} \lambda_k^{(i)} = 1$) such that $\lim_i \psi_i(x) = \varepsilon(x)$ in the σ-weak topology. Let q be the conjugate index of p: $1/p + 1/q = 1$. For any $y \in L^q(M, \tau)$,

$$\begin{aligned} |\tau(\varepsilon(x)y)| &= \lim_i |\tau(\psi_i(x)y)| \\ &\leqq \overline{\lim_i} \sum_{k=1}^{n_i} \lambda_k^{(i)} |\tau(\alpha_{t_k^{(i)}}(x)y)| \\ &\leqq \overline{\lim_i} \sum_{k=1}^{n_i} \lambda_k^{(i)} \|\alpha_{t_k^{(i)}}(x)\|_p \|y\|_q \\ &= \|x\|_p \|y\|_q . \end{aligned}$$

Since $L^q(M, \tau)$ is the dual space of $L^p(M, \tau)$, we have $\|\varepsilon(x)\|_p \leqq \|x\|_p$. As M is dense in $L^p(M, \tau)$ with respect to $\|\cdot\|_p$, ε extends uniquely to a projection ε_p of norm one on $L^p(M, \tau)$. Since $L^p(M(\alpha), \tau) = [M(\alpha)]_p$, it is clear that the range of ε_p equals $L^p(M(\alpha), \tau)$.

(ii) Let F be the set of all fixed points of $L^p(M, \tau)$ with respect to $\{\alpha_t\}_{t\in R}$. Since $L^p(M(\alpha), \tau) = [M(\alpha)]_p$, it is easy to show that $L^p(M(\alpha), \tau) \subset F$. Let $x \in F$. We may assume that x is self-adjoint. Let $x = \int_{-\infty}^{\infty} \lambda de_\lambda$ be its spectral resolution. Now we can consider $\alpha_t(x) = \int_{-\infty}^{\infty} \lambda d\alpha_t(e_\lambda)$. Since the spectral resolution is unique, $e_\lambda \in M(\alpha)$ and so $x \in L^p(M(\alpha), \tau)$.

(iii) From (iii) and (iv) of Proposition 2.5, we have $H_0^p(\alpha) = [H_0^\infty(\alpha)]_p$. Since $\varepsilon(x) = 0$ for $x \in H_0^\infty(\alpha)$, we show that $H_0^p(\alpha) \subset \{x \in H^p(\alpha); \varepsilon_p(x) = 0\}$. Now suppose that there exists an element $a \in H^p(\alpha)$ such that $\varepsilon_p(a) = 0$ and $a \notin H_0^p(\alpha)$. We can find $y \in L^q(M, \tau)$ such that $\tau(ay) = 1$ and $\tau(by) = 0$ for all $b \in H_0^p(\alpha)$. Let $F(t) = \tau(\alpha_t(a)y)$. As in the proof of [9, Theorem 2.4], F is constant in $\boldsymbol{R}$, that is, $\tau(ay) = \tau(\alpha_t(a)y) = 1$. Let δ be any number such that $0 < \delta < 1/2$. Since $L^p(M, \tau) = [M]_p$, there exists $x \in M$ such that $\|a - x\|_p < \delta/\|y\|_q$. Then

$$|\tau(\alpha_t(x)y) - 1| = |\tau(\alpha_t(x)y) - \tau(\alpha_t(a)y)| < \delta .$$

Hence we have $\operatorname{Re} \tau(\alpha_t(x)y) > 1 - \delta$. We choose a net $\{\psi_i\}_{i\in I}$ as in the proof of (i). Then

$$\begin{aligned} |\tau(\varepsilon(x)y)| &= \lim_i |\tau(\psi_i(x)y)| \\ &\geqq \lim_i \sum_{k=1}^{n_i} \lambda_k^{(i)} \operatorname{Re} \tau(\alpha_{t_k^{(i)}}(x)y) > 1 - \delta . \end{aligned}$$

On the other hand

$$|\tau(\varepsilon(x)y)| = |\tau(\varepsilon_p(a)y - \varepsilon(x)y)| \leqq \|a - x\|_p \|y\|_q < \delta .$$

This is a contradiction. This completes the proof.

PROPOSITION 2.7. *Let* $1 < p < \infty$.

(i) $H_0^p(\alpha) = [H_0^\infty(\alpha)]_p$.

(ii) $H^p(\alpha) = [H^\infty(\alpha)]_p$.

(iii) $H_0^q(\alpha) = \{x \in L^p(M, \tau); \tau(xy) = 0, y \in H^\infty(\alpha)\}$.

(iv) $H^p(\alpha) = H_0^q(\alpha)^\perp$, $1/p + 1/q = 1$.

PROOF. (i) and (iv) are clear from Proposition 2.5. (ii) is clear from Proposition 2.6. (iii) is proved from (ii).

Finally we define both simply and doubly invariant subspaces for $H^\infty(\alpha)$ in $L^p(M, \tau)$.

DEFINITION 2.8. Let $\mathscr{M}$ be a closed (resp. σ-weakly closed) subspace of $L^p(M, \tau)$ (resp. M) for $1 \leqq p < \infty$ (resp. $p = \infty$). $\mathscr{M}$ is said to be left (resp. right) doubly invariant if $H^\infty(\alpha)\mathscr{M} \subseteqq \mathscr{M}$ and $H^\infty(\alpha)^*\mathscr{M} \subseteqq \mathscr{M}$ (resp. $\mathscr{M}H^\infty(\alpha) \subseteqq \mathscr{M}$ and $\mathscr{M}H^\infty(\alpha)^* \subseteqq \mathscr{M}$). If $\mathscr{M}$ is left and right doubly invariant, $\mathscr{M}$ is said to be two-sided doubly invariant. Furthermore a closed subspace $\mathscr{M}$ of $L^p(M, \tau)$, $1 \leqq p < \infty$, is said to be left (resp. right) simply invariant if $[H_0^\infty(\alpha)\mathscr{M}]_p \subsetneqq \mathscr{M}$ (resp. $[\mathscr{M}H_0^\infty(\alpha)]_p \subsetneqq \mathscr{M}$).

3. **Examples.** Let M and τ be as in §2. Let F_n be a type I_n factor and let $\{e_{ij}\}$ be a matrix unit of F_n. We denote by B the von Neumann tensor product $M \overline{\otimes} F_n$ of M and F_n. Setting $\tilde{\alpha}_t = \alpha_t \otimes 1$, we get a flow $\{\tilde{\alpha}_t\}_{t \in \boldsymbol{R}}$ on B. Let Tr be the canonical trace on F_n and let $\tau \otimes \mathrm{Tr}$ be the tensor product of τ and Tr. We denote by $L^p(M, \tau) \otimes F_n$ the algebraic tensor product of $L^p(M, \tau)$ and F_n. Then we have the following:

PROPOSITION 3.1. *For* $1 \leqq p < \infty$, $L^p(M, \tau) \otimes F_n = L^p(B, \tau \otimes \mathrm{Tr})$.

Next, we investigate the structure of $H^p(\tilde{\alpha})$. We denote by $H^p(\alpha) \otimes F_n$ the algebraic tensor product of $H^p(\alpha)$ and F_n.

PROPOSITION 3.2. *For* $1 \leqq p \leqq \infty$, $H^p(\tilde{\alpha}) = H^p(\alpha) \otimes F_n$.

PROOF. Let $x \in L^p(M, \tau) \otimes F_n$ $(x = \sum x_{ij} \otimes e_{ij}, x_{ij} \in L^p(M, \tau))$. For $f \in L^1(\boldsymbol{R})$, we have $\tilde{\alpha}(f)x = \sum (\alpha(f)x_{ij}) \otimes e_{ij}$. Thus $\tilde{\alpha}(f)x = 0$ if and only if $\alpha(f)x_{ij} = 0$ for all i, j. By the definition of spectrum, we have $\mathrm{Sp}_{\tilde{\alpha}}(x) = \bigcup \mathrm{Sp}_\alpha(x_{ij})$. Therefore $H^p(\tilde{\alpha}) = H^p(\alpha) \otimes F_n$. This completes the proof.

REMARK 3.3. Let $L^\infty(T)$ and $\{\alpha_t\}_{t \in \boldsymbol{R}}$ be as in Remark 2.4. Let $L^\infty(T, F_n)$ be the Banach space of all F_n-valued essentially bounded weak*-measurable functions on T. Then $L^\infty(T) \otimes F_n = L^\infty(T, F_n)$ [16, Theorem 1.22.13]. Moreover $L^\infty(T, F_n)$ is a type I_n von Neumann algebra with the center $L^\infty(T)1$ [16, Proposition 3.2.3]. Put $\tilde{\alpha}_t = \alpha_t \otimes 1$. Then we have $H^p(\tilde{\alpha}) = H^p \otimes F_n$ by Remark 2.4 and Proposition 3.2. The flow $\{\tilde{\alpha}_t\}_{t \in \boldsymbol{R}}$ has the period 2π and the structure of $H^\infty(\tilde{\alpha})$ was considered in

[15]. On the other hand, this space $H^p(\tilde{\alpha})$ was studied by Helson and Lowdenslager as the notion of analytic matrix-valued functions.

4. Doubly invariant subspaces. In this section we characterize doubly invariant subspaces of $L^p(M, \tau)$, $1 \leqq p \leqq \infty$.

THEOREM 4.1. *Let $\mathscr{M}$ be a closed subspace of $L^p(M, \tau)$, $1 \leqq p \leqq \infty$. Then $\mathscr{M}$ is a left (resp. right) doubly invariant subspace of $L^p(M, \tau)$ if and only if there exists a projection e of M such that $\mathscr{M} = L^p(M, \tau)e$ (resp. $eL^p(M, \tau)$).*

PROOF. Let $\mathscr{U}$ be a salf-adjoint subalgebra generated by $H^\infty(\alpha) + H^\infty(\alpha)^*$ in M. Since $H^\infty(\alpha) + H^\infty(\alpha)^*$ is σ-weakly dense in M [11, Theorem III.15], $\mathscr{U}$ is so. Suppose $\mathscr{M}$ is left doubly invariant. Then $\mathscr{M}$ is a left $\mathscr{U}$-invariant subspace in $L^p(M, \tau)$.

Case $p = \infty$. It is trivial since $\mathscr{M}$ becomes a σ-weakly closed left ideal of M.

Case $p = 2$. Let $P_{\mathscr{M}}$ be the projection of $L^2(M, \tau)$ onto $\mathscr{M}$, $L(M) = \{L_x : x \in M\}$ where $L_x(y) = xy$, $y \in L^2(M, \tau)$ and $R(M) = \{R_x : x \in M\}$ where $R_x(y) = yx$, $y \in L^2(M, \tau)$. Since $\mathscr{M}$ is left $\mathscr{U}$-invariant, $\mathscr{M}$ is left $L(M)$-invariant. Hence $P_{\mathscr{M}} \in L(M)' = R(M)$, where $L(M)'$ is the commutant of $L(M)$, and so there exists a projection e in M such that $P_{\mathscr{M}} = P_e$. Thus $\mathscr{M} = P_{\mathscr{M}} L^2(M, \tau) = L^2(M, \tau)e$.

Case $1 \leqq p < 2$. Putting $\mathscr{N} = \mathscr{M} \cap L^2(M, \tau)$, $\mathscr{N}$ is a left $\mathscr{U}$-invariant closed subspace of $L^2(M, \tau)$. According to the case $p = 2$, there exists a projection e in M such that $\mathscr{N} = L^2(M, \tau)e$. It is sufficient to show $\mathscr{M} = L^p(M, \tau)e$. $\mathscr{M} \supset L^p(M, \tau)e$ is clear. Let $x = u|x|$ be the polar decomposition of x in $\mathscr{M}$ and put $x_1 = u|x|^{p/2}$ and $x_2 = |x|^{1-(p/2)}$. Then $x_1 \in L^2(M, \tau)$ and $x_2 \in L^r(M, \tau)$ where $1/p = 1/2 + 1/r$. Putting $\mathscr{N}' = [\mathscr{U} x_1]_2$, $\mathscr{N}'$ is a left $\mathscr{U}$-invariant subspace in $L^2(M, \tau)$ and so there exists a projection f in M such that $\mathscr{N}' = L^2(M, \tau)f$. Then

$$fx_2 \in L^2(M, \tau)fx_2 = [\mathscr{U} x_1]_2 x_2 \subset [\mathscr{U} x_1 x_2]_p = [\mathscr{U} x]_p \subset \mathscr{M} .$$

On the other hand, since $r > 2$, $fx_2 \in L^r(M, \tau) \subset L^2(M, \tau)$. Therefore

$$fx_2 \in \mathscr{M} \cap L^2(M, \tau) = \mathscr{N} = L^2(M, \tau)e .$$

Thus $fx_2 = fx_2 e$. Moreover, since $x_1 \in L^2(M, \tau)f = \mathscr{N}'$, we have $x_1 = x_1 f$. Therefore

$$x = x_1 x_2 = x_1 f x_2 = x_1 f x_2 e \in L^p(M, \tau)e .$$

Hence we have $\mathscr{M} = L^p(M, \tau)e$.

Case $2 < p < \infty$. Putting $\mathscr{M}' = \{y \in L^q(M, \tau) : \tau(y^* x) = 0 \ \ (x \in \mathscr{M})\}$ where $1/p + 1/q = 1$, $\mathscr{M}'$ is a left $\mathscr{U}$-invariant subspace of $L^q(M, \tau)$.

Since $1<q<2$, we have a projection f in M such that $\mathscr{M}' = L^q(M, \tau)f$. Put $e = 1-f$ and so we have $\mathscr{M} = L^p(M, \tau)e$.

The assertion for right doubly invariant subspaces may be proved in just the same way.

This completes the proof.

COROLLARY 4.2. *Let $\mathscr{M}$ be a closed subspace of $L^p(M, \tau)$, $1 \leqq p \leqq \infty$. Then $\mathscr{M}$ is a two-sided doubly invariant subspace of $L^p(M, \tau)$ if and only if there exists a central projection e of M such that $\mathscr{M} = L^p(M, \tau)e$.*

REMARK 4.3. We suppose that M has a faithful, normal, α_t-invariant finite trace. However, even if M does not have any α_t-invariant trace, $H^\infty(\alpha) + H^\infty(\alpha)^*$ is always σ-weakly dense in M by [11, Theorem III. 15]. Thus Theorem 4.1 holds in this case.

REMARK 4.4. Let $M = L^\infty(T)$ and let A be the disk algebra over the unit circle T. Let $\mathscr{M}$ be a closed subspace of $L^2(T)$. If $\mathscr{M}$ is a doubly invariant subspace in the sense that $A\mathscr{M} \subseteqq \mathscr{M}$ and $\bar{A}\mathscr{M} \subseteqq \mathscr{M}$, where $\bar{A}$ is the conjugate functions of A, then $\mathscr{M} = C_E L^2(T)$ for some measurable set E (where C_E denotes the characteristic function of E). This result is well-known as Wiener's theorem. Furthermore, Hasumi and Srinivasan [4, 18] extended the result to L^p-spaces taking values in a Hilbert space.

5. Simply invariant subspaces. Throughout this section, we keep the notations in §2. Then $H^\infty(\alpha)$ becomes a finite subdiagonal algebra with respect to the projection ε of norm one induced by the α_t-invariance of τ. Furthermore, if $\{\alpha_t\}_{t\in \boldsymbol{R}}$ is ergodic in the sense that for $x \in M$, $\alpha_t(x) = x$ for all $t \in \boldsymbol{R}$ implies $x = \lambda 1$ for some complex number λ, $H^\infty(\alpha)$ is an antisymmmetric finite subdiagonal algebra (see [1], [8], etc.). Then Kamei in [8] has shown simply invariant subspace theorems for antisymmetric finite subdiagonal algebras in case $p = 1, 2$. In this section we precisely characterize the simply invariant subspace theorem for $H^\infty(\alpha)$ in $L^p(M, \tau)$, $1 \leqq p \leqq \infty$, if $\{\alpha_t\}_{t\in \boldsymbol{R}}$ is ergodic.

THEOREM 5.1. *Let $1 \leqq p \leqq \infty$. If $\{\alpha_t\}_{t\in \boldsymbol{R}}$ is ergodic, every left (resp. right) simply invariant subspace $\mathscr{M}$ of $L^p(M, \tau)$ is of the form $H^p(\alpha)u$ (resp. $uH^p(\alpha)$) for some unitary operator u in M.*

To show this theorem, we have the following lemmas. Throughout the remainder of this section, we suppose that $\{\alpha_t\}_{t\in \boldsymbol{R}}$ is ergodic.

LEMMA 5.2. (Kamei) *Let $x \in L^2(M, \tau)$. If $x \notin [H_0^\infty(\alpha)x]_2$, then we have $x = au$ where $u \in [H^\infty(\alpha)x]_2$ is unitary and $[H^\infty(\alpha)a]_2 = H^2(\alpha)$.*

Let $1 \leqq p < 2$. Define the number r by $1/r + 1/2 = 1/p$. Then we have the following;

LEMMA 5.3. *Let* $x \in L^p(M, \tau)$. *If* $x \notin [H_0^\infty(\alpha)x]_p$, *then we have* $|x^*|^{p/2} \notin [H_0^\infty(\alpha)|x^*|^{p/2}]_2$.

PROOF. Let $x = |x^*|u$ be the polar decomposition of x and put $x_1 = |x^*|^{1-(p/2)}u$. Assume that $|x^*|^{p/2} \in [H_0^\infty(\alpha)|x^*|^{p/2}]_2$. Then

$$x = |x^*|^{p/2}x_1 \in [H_0^\infty(\alpha)|x^*|^{p/2}]_2 x_1 \subset [H_0^\infty(\alpha)|x^*|^{p/2}x_1]_p = [H_0^\infty(\alpha)x]_p .$$

This is a contradiction. This completes the proof.

LEMMA 5.3. *If* $x \in L^p(M, \tau)$ *and* $x \notin [H_0^\infty(\alpha)x]_p$, *then* $x = zy$ *where* $y \in [H^\infty(\alpha)x]_p \cap L^r(M, \tau)$ *and* $z \in H^2(\alpha)$.

PROOF. If $x \notin [H_0^\infty(\alpha)x]_p$, we have $|x^*|^{p/2} \notin [H_0^\infty(\alpha)|x^*|^{p/2}]_2$ by Lemma 5.3 and so $|x^*|^{p/2} = zu$ where $u \in [H^\infty(\alpha)|x^*|^{p/2}]_2$ is unitary and $[H^\infty(\alpha)z]_2 = H^2(\alpha)$ by Lemma 5.2. Let $x = |x^*|v$ be the polar decomposition of x and put $y = u|x^*|^{1-(p/2)}v$. Then $y \in L^r(M, \tau) \subset L^2(M, \tau)$. Hence

$$zy = zu|x^*|^{1-(p/2)}v = |x^*|^{p/2}|x^*|^{1-(p/2)}v = |x^*|v = x .$$

Since $[H^\infty(\alpha)z]_2 = H^2(\alpha)$, for any $\varepsilon > 0$, there exists an element $a \in H^\infty(\alpha)$ such that $||az - 1||_2 < \varepsilon/||y||_r$. Thus

$$||ax - y||_p = ||azy - y||_p < ||az - 1||_2||y||_r < \varepsilon .$$

Therefore $y \in [H^\infty(\alpha)x]_p$. This completes the proof.

PROOF OF THEOREM 5.1. Let $\mathscr{M}$ be a left simply invariant subspace of $L^p(M, \tau)$. In case $p = 2$, we have the result by [8, Theorem 1].

(1) Case $1 \leqq p < 2$. Putting $\mathscr{N} = \mathscr{M} \cap L^2(M, \tau)$, $\mathscr{N}$ is a closed subspace of $L^2(M, \tau)$. By the assumption of the left simple invariance of $\mathscr{M}$, there exists an element $x \in \mathscr{M} \backslash [H_0^\infty(\alpha)\mathscr{M}]_p$. In particular, we have $x \notin [H_0^\infty(\alpha)x]_p$ and so, by Lemma 5.4, $x = zy$ where $z \in H^2(\alpha)$ and $y \in [H^\infty(\alpha)x]_p \cap L^r(M, \tau)$. Since $H^\infty(\alpha)x \subset \mathscr{M}$, we have $y \in [H^\infty(\alpha)x]_p \subset \mathscr{M}$ and so $\mathscr{N} \neq \{0\}$. If $y \in [H_0^\infty(\alpha)\mathscr{N}]_2$ we have

$$\begin{aligned} x = zy \in H^2(\alpha)y \subset [H^\infty(\alpha)y]_p &\subset [H^\infty(\alpha)[H_0^\infty(\alpha)\mathscr{N}]_2]_p \\ &\subset [H_0^\infty(\alpha)\mathscr{N}]_p \subset [H_0^\infty(\alpha)\mathscr{M}]_p . \end{aligned}$$

This is a contradiction. Hence $\mathscr{N}$ becomes a left simply invariant subspace of $L^2(M, \tau)$. By [8, Theorem 1], there exists a unitary operator $u \in M$ such that $\mathscr{N} = H^2(\alpha)u$. Thus $H^\infty(\alpha)u \subset H^2(\alpha)u = \mathscr{N} \subset \mathscr{M}$ and so $[H^\infty(\alpha)u]_p \subset \mathscr{M}$. If $x \in \mathscr{M} \backslash [H_0^\infty(\alpha)\mathscr{M}]_p$, we have $x = zy$ where $z \in H^2(\alpha)$ and

$$y \in [H^\infty(\alpha)x]_p \cap L^r(M, \tau) \subset \mathscr{M} \cap L^r(M, \tau)$$
$$= \mathscr{N} \cap L^r(M, \tau) = H^2(\alpha)u \cap L^r(M, \tau) .$$

Hence $yu^* \in H^r(\alpha)$ and so $x = zy = zyu^*u \in H^p(\alpha)u$. Therefore $\mathscr{M}\backslash[H_0^\infty(\alpha)\mathscr{M}]_p \subset H^p(\alpha)u$. If $y \in [H_0^\infty(\alpha)\mathscr{M}]_p$, then

$$x + y \in \mathscr{M}\backslash[H_0^\infty(\alpha)\mathscr{M}]_p \subset H^p(\alpha)u .$$

Since $x \in H^p(\alpha)u$, we have $y \in H^p(\alpha)u$ and so $\mathscr{M} = H^p(\alpha)u$.

The assertion for right simply invariant subspaces in case $1 \leqq p < 2$ may be proved in just the same way.

(2) Case $2 < p \leqq \infty$. Define the number q by $1/p + 1/q = 1$. Putting

$$\mathscr{N} = \{y \in L^q(M, \tau); \tau(yx) = 0, x \in [H_0^\infty(\alpha)\mathscr{M}]_p\} ,$$

then $\mathscr{N}$ is a closed subspace of $L^q(M, \tau)$. Since $[H_0^\infty(\alpha)\mathscr{M}]_p$ is a proper subspace of $\mathscr{M}$, there exists $a \in L^q(M, \tau)$ such that $\tau(ax) = 0$, $x \in [H_0^\infty(\alpha)\mathscr{M}]_p$ and $\tau(ay) \neq 0$ for some $y \in \mathscr{M}$. Thus $a \in \mathscr{N}\backslash[\mathscr{N}H_0^\infty(\alpha)]_q$. Therefore $\mathscr{N}$ is a right simply invariant subspace of $L^q(M, \tau)$ and so there exists a unitary element $u \in M$ such that $\mathscr{N} = u^*H^q(\alpha)$. By Proposition 2.7 (iv), $[H_0^\infty(\alpha)\mathscr{M}]_p = H_0^p(\alpha)u$. If $x \in \mathscr{M}u^*$ and $y \in H_0^\infty(\alpha)$, then

$$yx \in H_0^\infty(\alpha)\mathscr{M}u^* \subset [H_0^\infty(\alpha)\mathscr{M}]_p u^* = H_0^p(\alpha)$$

and so $\tau(yx) = 0$. Thus $x \in H^p(\alpha)$ and so $\mathscr{M}u^* \subset H^p(\alpha)$. Since $H_0^p(\alpha)$ is a subspace of $H^p(\alpha)$ of codimension 1, we have $\mathscr{M} = H^p(\alpha)u$ or $\mathscr{M} = H_0^p(\alpha)u = [H_0^\infty(\alpha)\mathscr{M}]_p$. As $\mathscr{M}$ is left simply invariant, $\mathscr{M} = H^p(\alpha)u$. This completes the proof.

REMARK 5.5. The converse of this theorem is also true. If $\{\alpha_t\}_{t \in R}$ is not ergodic, there exists a α_t-invariant projection $e \in M$ such that $0 < e < 1$. Choose a unitary element $u \in M$. Putting $\mathscr{M} = H^p(\alpha)eu$, $\mathscr{M}$ is easily seen to be a left simply invariant subspace of $L^p(M, \tau)$ which is not of the form $H^p(\alpha)v$ for any unitary element $v \in M$.

REMARK 5.6. Keep the notations in Remark 2.4. Let A be the disk algebra and put $A_0 = \left\{x \in A; \int x dt = 0\right\}$. A closed subspace $\mathscr{M}$ of $L^p(T)$ is said to be simply invariant if $[A_0\mathscr{M}]_p \subsetneqq \mathscr{M}$. As $\{\alpha_t\}_{t \in R}$ in Remark 2.4 is ergodic, then every simply invariant subspace $\mathscr{M}$ of $L^p(T)$, $1 \leqq p \leqq \infty$, is of the form $H^p f$ for some unimodular function f in $L^\infty(T)$.

REMARK 5.7. Loebl-Muhly [11] showed an example such that $H^\infty(\alpha)$ becomes a reductive algebra. But our $H^\infty(\alpha)$ is not a reductive algebra on $L^2(M, \tau)$, because there is always a simply invariant subspace for $H^\infty(\alpha)$ in $L^2(M, \tau)$.

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