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A PHILLIPS-MIYADERA TYPE PERTURBATION THEOREM FOR COSINE FUNCTIONS OF OPERATORS

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1. Introduction. Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators on X to X. A one-parameter family $C = \{C(t); t \in R = (-\infty, \infty)\}$ in $B(X)$ is called a *cosine function* on X if it satisfies the following three conditions:

(i) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in R$;

(ii) $C(0) = 1$ (the identity operator);

(iii) $C(t)$ is strongly continuous in t.

The *sine function* associated with a cosine function C is a family $S = \{S(t); t \in R\}$ in $B(X)$, where $S(t)$ is defined by

$$S(t) = \int_0^t C(s)ds \, .$$

To define the (infinitesimal) *generator* A of a consine function C, set

$$D(A) = \{x \in X; C''(0)x = \lim_{h\to 0} 2h^{-2}[C(h) - 1]x \quad \text{exists}\} \, .$$

Then A is defined by $A = C''(0)$ on $D(A)$. We denote by $\rho(A)$ and $R(\lambda; A)$ the resolvent set and the resolvent of A, respectively: $R(\lambda; A) = (\lambda - A)^{-1}$, $\lambda \in \rho(A)$. In terms of the generator a cosine function is characterized by the following generation theorem established independently by Sova [10], Da Prato-Giusti [1] and Fattorini [2].

THEOREM 1.1. *Let A be a linear operator in X. Then A is the generator of a cosine function on X if and only if*

(I) *A is closed and densely defined,*

(II) *there is a constant $\omega \geqq 0$ such that for $\lambda > \omega$, $\lambda^2 \in \rho(A)$,*

(III) *there is a constant $M > 0$ such that for $\lambda > \omega$,*

$$\left\| \frac{d^m}{d\lambda^m}[\lambda R(\lambda^2; A)] \right\| \leqq \frac{Mm!}{(\lambda - \omega)^{m+1}} \, , \qquad m = 0, 1, 2, \cdots \, .$$

This theorem can be regarded as an analogue of the Hille-Yosida theorem on the generation of semigroups of class (C_0).

The purpose of this note is to prove a cosine function analogue of a theorem of Miyadera [6]-[8] (Theorem 2 in [6]) on the perturbation of semigroups of class (C_0). Here it should be noted that the proof of the Miyadera theorem is fairly simplified in the third paper [8]. In fact, the first proof in [6] depends on the Trotter-Kato convergence theorem, while the second one in [8] requires only the generation theorem. Accordingly, the proof of our result is also based on the generation theorem (Theorem 1.1 above), though we have obtained another one by using a convergence theorem of Konishi [4]. The result is related to those of Konishi-Tezuka [5] and Nagy [9].

2. Preliminaries. Let C be a cosine function on X and S be the associated sine function. Then by condition (i), we have

$$C(t) = C(-t)\,, \qquad S(t) = -S(-t)\,, \qquad t \in R\,. \tag{2.1}$$

Consequently, $C(t)C(s) = C(s)C(t)$ and hence $C(t)S(s) = S(s)C(t)$, $S(t)S(s) = S(s)S(t)$. Now let A be the generator of C. Then we have that for $x \in D(A)$ and $t \in R$,

$$AC(t)x = C(t)Ax\,, \quad C'(t)x = AS(t)x = S(t)Ax\,. \tag{2.2}$$

LEMMA 2.1. *Under the assumption of Theorem* 1.1, *we have*

$$\|C(t)\| \leqq Me^{\omega|t|}\,, \qquad t \in R\,, \tag{2.3}$$

$$\lambda R(\lambda^2; A) = \int_0^\infty e^{-\lambda t} C(t)dt\,, \qquad \lambda > \omega\,; \tag{2.4}$$

and hence

$$\|S(t)\| \leqq M|t|e^{\omega|t|}\,, \qquad t \in R\,, \tag{2.5}$$

$$R(\lambda^2; A) = \int_0^\infty e^{-\lambda t} S(t)dt\,, \qquad \lambda > \omega\,. \tag{2.6}$$

LEMMA 2.2. *The following relations hold*:

(a) $S(t+s) + S(t-s) = 2S(t)C(s)$,

(b) $S(t+s) - S(t-s) = 2C(t)S(s)$,

(c) $S(t)C(s) + C(t)S(s) = S(t+s)$,

(d) $C(t+s) - C(t-s) = 2AS(t)S(s)$,

(e) $C(t)C(s) + AS(t)S(s) = C(t+s)$.

PROOF. (b) follows from (a) and (2.1). (c) is a direct consequence of (a) and (b). (e) is clear from (d) and condition (i). So, it suffices to prove (a) and (d). By definition we have

$$\begin{aligned} 2S(t)C(s) &= \int_0^t [C(r+s) + C(r-s)]dr \\ &= \int_0^{t+s} C(r)dr - \int_0^s C(r)dr + \int_0^{t-s} C(r)dr + \int_{-s}^0 C(r)dr \\ &= S(t+s) + S(t-s)\,. \end{aligned}$$

This proves (a). Now differentiation of (b) with respect to t gives

$$C(t+s)x - C(t-s)x = 2C'(t)S(s)x\,, \qquad x \in D(A)\,.$$

Then (d) follows from (2.2) and condition (I). q.e.d.

In the rest of this section we consider a class of perturbing operators for the generator of a cosine function.

DEFINITION 2.3. Let A be the generator of a cosine function C on X, and B be a linear operator in X. Then B is said to be an operator of class (B) if

(B_1) $D(B) \supset D(A)$ and $BR(\mu^2; A) \in B(X)$ for some $\mu > \omega$,

(B_2) there exists a constant $K_0 > 0$ such that for all $x \in D(A)$,

$$\int_0^1 \|BC(t)x\|\,dt \leqq K_0\|x\|\,.$$

REMARK 2.4. Since $\rho(A)$ is nonempty, condition (B_1) is equivalent to the *relative boundedness* of B with respect to A (see e.g., Kato [3], IV-Section 1) and hence $BR(\lambda^2; A) \in B(X)$ for each $\lambda^2 \in \rho(A)$.

Now let $\lambda > 0$ and set

$$K_\lambda = \sup\left\{\int_0^1 e^{-\lambda t}\|BC(t)x\|\,dt;\ \|x\| \leqq 1,\ x \in D(A)\right\}\,. \tag{2.7}$$

Then by condition (B_2), K_λ is finite. Since K_λ is a nonnegative and monotone decreasing function of λ, $K_\infty = \lim_{\lambda\to\infty} K_\lambda$ exists and $0 \leqq K_\infty < K_0$.

LEMMA 2.5. *Let S be the sine function associated with a cosine function C on X. Suppose that B is an operator of class (B). Then for all $x \in D(A)$,*

$$\int_0^1 e^{-\lambda t}\|BS(t)x\|\,dt \leqq K_\lambda\|x\|\,, \qquad \lambda \geqq 0\,. \tag{2.8}$$

PROOF. Let $\mu^2 \in \rho(A)$. Then

$$\begin{aligned}\int_0^1 e^{-\lambda t}\|BS(t)x\|\,dt &= \int_0^1 e^{-\lambda t}\left\|BR(\mu^2; A)\int_0^t C(s)(\mu^2 - A)x\,ds\right\|dt\\ &\leqq \int_0^1\int_0^t e^{-\lambda t}\|BC(s)x\|\,ds\,dt\,.\end{aligned}$$

Since $e^{-\lambda t} \leqq e^{-\lambda s}$ on the triangle: $0 \leqq s \leqq t$, $0 \leqq t \leqq 1$, we obtain

$$\int_0^1 e^{-\lambda t}\|BS(t)x\|\,dt \leqq \int_0^1 e^{-\lambda s}\|BC(s)x\|\,ds \leqq K_\lambda\|x\|\,.$$ q.e.d.

LEMMA 2.6. *Let S and B be as in Lemma 2.5. Then for each $\lambda > \omega$,* (2.9) $$\int_0^\infty e^{-\lambda t}\|BS(t)x\|\,dt \leqq L_\lambda\|x\|\,, \qquad x\in D(A)\,,$$

where $L_\lambda = K_\lambda[1+M(2e^{\lambda-\omega}-1)(e^{\lambda-\omega}-1)^{-2}]$, *and hence*

(2.10) $$\|BR(\lambda^2; A)\| \leqq L_\lambda\,, \qquad \lambda>\omega\,.$$

PROOF. First we note that

$$\int_0^\infty e^{-\lambda t}BS(t)xdt = \sum_{k=0}^\infty \int_k^{k+1} e^{-\lambda t}BS(t)xdt\,.$$

Changing the variable of integration, we obtain

$$\begin{aligned}\int_k^{k+1} e^{-\lambda t}BS(t)xdt &= e^{-\lambda k}\int_0^1 e^{-\lambda t}BS(t+k)xdt\\ &= e^{-\lambda k}\int_0^1 e^{-\lambda t}[BS(t)C(k)x + BC(t)S(k)x]dt\,,\end{aligned}$$

where we have used Lemma 2.2(c). It then follows from (2.7) and (2.8) that

$$\int_k^{k+1} e^{-\lambda t}\|BS(t)x\|\,dt \leqq K_\lambda e^{-\lambda k}[\|C(k)x\| + \|S(k)x\|]\,.$$

In virtue of (2.3) and (2.5), we obtain (2.9):

$$\int_0^\infty e^{-\lambda t}\|BS(t)x\|\,dt \leqq K_\lambda\Big[1 + M\sum_{k=1}^\infty (1+k)e^{-k(\lambda-\omega)}\Big]\|x\|\,.$$

Finally, (2.10) follows from (2.6) and (2.9). In fact, we have that for $x\in D(A)$,

(2.11) $$BR(\lambda^2; A)x = \int_0^\infty e^{-\lambda t}BS(t)xdt\,;$$

note that $BR(\lambda^2; A)\in B(X)$ and $D(A)$ is dense in X. q.e.d.

REMARK 2.7. Some perturbation theorems for cosine functions are announced by Konishi-Tezuka [5] (see also remark after Proposition 4.2 of [4]). But they start from the inequality similar to (2.9). Comparing our assumption with that of Miyadera [6], we see that Lemmas 2.5 and 2.6 much clarify the analogy between semigroups and cosine functions. Note further that (2.7) does not in general follow from (2.8) (see Example 4.2 below).

3. Perturbation theorems. Let C be a cosine function on X, with the generator A. Then for each $x\in D(A)$, $u(t)=C(t)x$ is a unique solution to the Cauchy problem:

$$u''(t) = Au(t)\,, \qquad u(0)=x\,, \qquad u'(0)=0\,.$$

Now let B be an operator of class (B), and $v(t)$ be a solution to the perturbed Cauchy problem:

$$v''(t) = Av(t) + Bv(t)\ , \qquad v(0) = x\ , \qquad v'(0) = 0\ .$$

Denoting by S the sine function associated with C, we have

$$(d/ds)[C(t-s)v(s) + S(t-s)v'(s)] = S(t-s)Bv(s)\ .$$

Integrating this equality from $s = 0$ to $s = t$, we obtain

$$v(t) = C(t)x + \int_0^t S(t-s)Bv(s)ds\ .$$

To solve this integral equation, we apply the method of successive approximation.

We obtain a sequence $\{\bar{C}_n(t)\}$ in $B(X)$. $\bar{C}_n(t)$ is first defined on $D(A)$ as $C_n(t)$ and then extended onto X; $\bar{C}_n(t)$ denotes the extension of $C_n(t)$: $C_0(t) = C(t)$,

$$\begin{aligned} C_n(t)x &= \int_0^t \bar{C}_{n-1}(t-s)BS(s)xds \\ &= \int_0^t \bar{C}_{n-1}(s)BS(t-s)xds\ , \qquad x \in D(A)\ , \qquad n \geqq 1\ . \end{aligned}$$

We must show that $C_n(t)$ is bounded on $D(A)$.

LEMMA 3.1. *Let L_λ be as in Lemma 2.6. Then we have*

$$\|\bar{C}_n(t)\| \leqq ML_\lambda^n e^{\lambda|t|}\ , \qquad \lambda > \omega,\ n \geqq 0\ , \tag{3.1}$$

and hence

$$\int_0^\infty e^{-\lambda t}\bar{C}_n(t)dt = \lambda R(\lambda^2; A)[BR(\lambda^2; A)]^n\ . \tag{3.2}$$

PROOF. Because of $\lambda > \omega$, (2.3) implies (3.1) with $n = 0$. Now suppose that (3.1) holds. Then we have by Lemma 2.6 that for $x \in D(A)$,

$$\begin{aligned} \|C_{n+1}(t)x\| &\leqq ML_\lambda^n \left| \int_0^t e^{\lambda|t-s|}\|BS(s)x\|ds \right| \\ &\leqq ML_\lambda^n e^{\lambda|t|} \int_0^{|t|} e^{-\lambda s}\|BS(s)x\|ds \leqq ML_\lambda^{n+1}e^{\lambda|t|}\|x\|\ . \end{aligned}$$

Since $D(A)$ is dense in X, we obtain (3.1) with n replaced by $n+1$. Next we prove (3.2). We see by (2.4) that (3.2) holds for $n = 0$. So, it suffices to show that for $x \in D(A)$,

$$\int_0^\infty e^{-\lambda t}C_{n+1}(t)xdt = \int_0^\infty e^{-\lambda t}\bar{C}_n(t)BR(\lambda^2; A)xdt\ .$$

Applying the Fubini theorem, we have

$$\int_0^\infty e^{-\lambda t}C_{n+1}(t)xdt = \int_0^\infty e^{-\lambda t}\left[\int_0^t \bar{C}_n(s)BS(t-s)xds\right]dt$$
$$= \int_0^\infty e^{-\lambda s}\bar{C}_n(s)\left[\int_s^\infty e^{-\lambda(t-s)}BS(t-s)xdt\right]ds$$
$$= \int_0^\infty e^{-\lambda s}\bar{C}_n(s)\left[\int_0^\infty e^{-\lambda t}BS(t)xdt\right]ds\,.$$

So, the desired equality follows from (2.11). q.e.d.

Our main result is given by

THEOREM 3.2. *Let C be a cosine function on X, with the generator A. Suppose that B is an operator of class (B) (see Definition 2.3) and $K_\infty = \lim_{\lambda\to\infty} K_\lambda$, where K_λ is defined by (2.7). Then for each ε with $|\varepsilon| < K_\infty^{-1}$, $A+\varepsilon B$ generates a cosine function $\{C(t; A+\varepsilon B)\}$, where $C(t; A+\varepsilon B)$ is given by*

$$C(t; A+\varepsilon B) = \sum_{n=0}^\infty \varepsilon^n \bar{C}_n(t)\,. \tag{3.3}$$

Moreover, we have

$$\lim_{\varepsilon\to 0} \|C(t; A+\varepsilon B) - C(t)\| = 0\,. \tag{3.4}$$

In (3.3) and (3.4) the convergence is uniform with respect to t on each finite subinterval of $(-\infty, \infty)$.

PROOF. Let $|\varepsilon|^{-1} > K_\infty$, and L_λ be as in Lemma 2.6. Then, since $K_\infty = \lim_{\lambda\to\infty} L_\lambda$ and L_λ is monotone decreasing, we can find $\lambda_0 > \omega$ such that

$$L_\lambda \leqq L_{\lambda_0} < |\varepsilon|^{-1}\,, \qquad \lambda \geqq \lambda_0\,. \tag{3.5}$$

Then we obtain from (3.1)

$$\|\varepsilon^n \bar{C}_n(t)\| \leqq M(|\varepsilon| L_{\lambda_0})^n e^{\lambda_0|t|}\,, \qquad |\varepsilon| L_{\lambda_0} < 1\,.$$

Consequently, the series on the right of (3.3) converges uniformly in t on each compact interval:

$$\left\|\sum_{n=0}^\infty \varepsilon^n \bar{C}_n(t)\right\| \leqq Me^{\lambda_0|t|}(1 - |\varepsilon| L_{\lambda_0})^{-1}\,. \tag{3.6}$$

Since $\bar{C}_n(t)$ is strongly continuous, so is the limit, too.

Next, we show that for each $\lambda > \lambda_0$,

$$\lambda R(\lambda^2; A+\varepsilon B) = \int_0^\infty e^{-\lambda t}\sum_{n=0}^\infty \varepsilon^n \bar{C}_n(t)dt\,. \tag{3.7}$$

It follows from (2.10) and (3.5) that

$$\|\varepsilon BR(\lambda^2; A)\| \leqq |\varepsilon| L_\lambda < 1 , \qquad \lambda \geqq \lambda_0 .$$

Since $\lambda^2 - (A + \varepsilon B) = [1 - \varepsilon BR(\lambda^2; A)](\lambda^2 - A)$, we see that for $\lambda \geqq \lambda_0$, $\lambda^2 \in \rho(A + \varepsilon B)$ and

$$\lambda R(\lambda^2; A + \varepsilon B) = \lambda R(\lambda^2; A) \sum_{n=0}^{\infty} \varepsilon^n [BR(\lambda^2; A)]^n .$$

Therefore, (3.7) follows from (3.2) and the bounded convergence theorem. Now m times differentiation of (3.7) gives

$$\frac{d^m}{d\lambda^m}[\lambda R(\lambda^2; A + \varepsilon B)] = (-1)^m \int_0^\infty t^m e^{-\lambda t} \sum_{n=0}^{\infty} \varepsilon^n \bar{C}_n(t) dt .$$

It follows from (3.6) that

$$\left\| \frac{d^m}{d\lambda^m}[\lambda R(\lambda^2; A + \varepsilon B)] \right\| \leqq \frac{Mm!}{(1 - |\varepsilon| L_{\lambda_0})(\lambda - \lambda_0)^{m+1}} , \qquad m \geqq 0 .$$

We see by Theorem 1.1 and Lemma 2.1 that $A + \varepsilon B$ is the generator of a cosine function $\{C(t; A + \varepsilon B)\}$ and

$$\lambda R(\lambda^2; A + \varepsilon B) = \int_0^\infty e^{-\lambda t} C(t; A + \varepsilon B) dt .$$

Therefore, (3.3) follows from (3.7) and (3.4) is obvious. q.e.d.

In general, ε in Theorem 3.2 is supposed to be rather small. But we can take an arbitrary ε if $K_\infty = 0$, and this is the case if $B \in B(X)$.

COROLLARY 3.3 (*see* [5], *Nagy* [9]). *Let A be the generator of a cosine function on X, and let $B \in B(X)$. Then $A + B$ is also the generator of a cosine function on X.*

PROOF. Let $\lambda > \omega$. Then we have

$$\int_0^1 e^{-\lambda t} \|BC(t)x\| dt \leqq M\|B\| \int_0^1 e^{-(\lambda - \omega)t} dt \|x\|$$

and hence $K_\lambda \leqq M\|B\|(\lambda - \omega)^{-1}$. So we obtain $K_\infty = 0$. Thus, we can take $\varepsilon = 1$ in Theorem 3.2. q.e.d.

4. **Examples.** Here we consider two examples. The first one shows that in Theorem 3.2, allowable perturbing operators are not necessarily bounded (cf. [6], p. 309).

EXAMPLE 4.1. Let $X = L^1(R)$ and consider the cosine function on X defined by

$$[C(t)x](p) = \frac{1}{2}[x(p + t) + x(p - t)] . \tag{4.1}$$

Let A be the generator of $\{C(t)\}$. Then $D(A) = W_1^2(R)$ and $(Ax)(p) = x''(p)$ for $x \in D(A)$. Let $b(p)$ be a function in $L^1(R)$ but not belonging to $L^\infty(R)$. Then the multiplication operator B is defined by

$$(Bx)(p) = b(p)x(p) ,$$

whenever the product is also in $L^1(R)$. Since B is closed and $D(B) \supset D(A)$, B satisfies condition (B_1). Furthermore, we have that for all $x \in D(A)$,

$$\int_0^1 ||BC(t)x||dt \leqq ||b|| \, ||x|| .$$

Therefore, B is an operator of class (B). But, B is obviously unbounded if we further assume that $b(p)$ is not in $L^2(R)$.

The next example shows that the converse of Lemma 2.5 does not hold.

EXAMPLE 4.2. Let $C[-\infty, \infty]$ be the space of all bounded and continuous functions on R, with supremum norm. Let $X = C[-\infty, \infty]$ and $\{C(t)\}$ be the cosine function defined by (4.1). Then $D(A) = C^2[-\infty, \infty]$ and $(Ax)(p) = x''(p)$ for $x \in D(A)$. Now let $D(B) = C^1[-\infty, \infty]$ and set $(Bx)(p) = x'(p)$ for $x \in D(B)$. Then B satisfies condition (B_1). Noting that $BS(t)x = C(t)x$, we obtain

$$\int_0^1 ||BS(t)x||\,dt \leqq ||x|| , \qquad x \in X .$$

Next let $h(p)$ be a function in $C_0^\infty(R)$ such that $h(p) = 1$ for $|p| \leqq 2$ and $0 \leqq h(p) \leqq 1$ for $|p| \geqq 2$. Setting $x_n(p) = h(p) \sin n\pi p$, we have a sequence $\{x_n\}$ in $D(A)$ such that $||x_n|| = 1$. But, since $||BC(t)x_n|| \geqq |BC(t)x_n(t)| = (n\pi/2)(1 + \cos 2n\pi t)$ for $0 \leqq t \leqq 1$, we obtain

$$\int_0^1 ||BC(t)x_n||\,dt \geqq n\pi/2 .$$

Thus, condition (B_2) is not satisfied.

ADDED IN PROOF. After this paper was accepted for publication, we received a preprint of [11].

At first glance, the result in [11] seems to be somewhat different from ours. Fortunately, however, we can unify the results in both papers. In fact, the main results in both papers are corollaries of the following theorem:

THEOREM A. *Let C be a cosine function on X, with the generator A and the associated sine function S. Assume that B is a linear* *operator in X satisfying condition* (B_1) *in Definition* 2.3 *and*

(B_2') *for some $\lambda_0 > 0$ there exists a constant $L(\lambda_0) > 0$ such that for all $x \in D(A)$,*

$$\int_0^\infty e^{-\lambda_0 t}\|BS(t)x\|\,dt \leqq L(\lambda_0)\|x\| .$$

Set $L(\infty) = \lim_{\lambda\to\infty} L(\lambda)$, where

$$L(\lambda) = \sup\left\{\int_0^\infty e^{-\lambda t}\|BS(t)x\|\,dt;\ \|x\| \leqq 1,\ x \in D(A)\right\} .$$

Then for each ε with $|\varepsilon| < L(\infty)^{-1}$, $A + \varepsilon B$ is also the generator of a cosine function.

Note that the proof of Theorem A has been essentially completed in this paper.

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