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COMPACTIFICATIONS OF THE MODULI SPACES OF HYPERELLIPTIC SURFACES

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Hyperelliptic surfaces are classified into seven types [1]. In this paper we aim at constructing the moduli space of each type in the sense described in Mumford [8] and its compactification in some sense.

In Section 1, we study the structure of hyperelliptic surfaces and, by considering hyperelliptic surfaces with some base points, we get the fine moduli space M_i of each type as a quotient space of the upper half plane or the product of two copies of the upper half plane, over which Suwa [11] showed the existence of a family of hyperelliptic surfaces complete and effectively parametrized at each point.

In Section 2, we construct the compactification $\bar{M}_i$ of M_i. And as a preparation for Section 3, we describe the resolution of certain quotient singularities in terms of torus embeddings.

In Section 3, we describe "degenerate hyperelliptic surfaces" represented by the boundary points of $\bar{M}_i$.

1. We write the elements of $\boldsymbol{C}^2$ and $\boldsymbol{Z}^4$ as row vectors. Let Ω be a 2×2 matrix with coefficients in $\boldsymbol{C}$ of which the imaginary part is positive definite $\operatorname{Im}(\Omega) > 0$. Then by $A(\Omega)$ we denote the complex torus of dimension 2 with the period matrix Ω, i.e.,

$$A(\Omega) = \boldsymbol{C}^2/\boldsymbol{Z}^4\begin{pmatrix}\Omega \\ I\end{pmatrix} .$$

Let $[x, y]$ denote the point of $A(\Omega)$ which is the image of (x, y) in $\boldsymbol{C}^2$.

We denote the upper half plane by $\mathfrak{H}$, and for any element τ of $\mathfrak{H}$, we denote by $E(\tau)$ the elliptic curve with the periods 1 and τ, i.e.,

$$E(\tau) = \boldsymbol{C}/(\boldsymbol{Z}\tau + \boldsymbol{Z}) .$$

We identify the elliptic curve E with its group $\operatorname{Aut}(E)^0$ of translations, and let $[x]$ denote the point of $E(\tau)$ which is the image of x in $\boldsymbol{C}$.

DEFINITION. By a hyperelliptic surface we mean an elliptic bundle over an elliptic curve whose total space has the first Betti number $b_1 = 2$.

THEOREM 1 [11]. *Hyperelliptic surfaces are topologically classified* *into seven types, and any hyperelliptic surface can be expressed as the quotient space of an abelian surface A by the group generated by an automorphism g of A as follows, where* $\sigma_N = \exp(2\pi i/N)$ *and* $\tau, \omega \in \mathfrak{H}$:

(1) $A = A\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right)$ $\quad g\colon [x, y] \mapsto [x + 1/2, -y]$

(2) $A = A\left(\begin{pmatrix} \tau & 1/2 \\ 0 & \omega \end{pmatrix}\right)$ $\quad$ 〃

(3) $A = A\left(\begin{pmatrix} \tau & 0 \\ 0 & \sigma_3 \end{pmatrix}\right)$ $\quad g\colon [x, y] \mapsto [x + 1/3, \sigma_3 y]$

(4) $A = A\left(\begin{pmatrix} \tau & (1-\sigma_3)/3 \\ 0 & \sigma_3 \end{pmatrix}\right)$ $\quad$ 〃

(5) $A = A\left(\begin{pmatrix} \tau & 0 \\ 0 & \sigma_4 \end{pmatrix}\right)$ $\quad g\colon [x, y] \mapsto [x + 1/4, \sigma_4 y]$

(6) $A = A\left(\begin{pmatrix} \tau & (1+\sigma_4)/2 \\ 0 & \sigma_4 \end{pmatrix}\right)$ $\quad$ 〃

(7) $A = A\left(\begin{pmatrix} \tau & 0 \\ 0 & \sigma_3 \end{pmatrix}\right)$ $\quad g\colon [x, y] \mapsto [x + 1/6, -\sigma_3 y]$

We denote these surfaces by $S_1(\tau, \omega)$, $S_2(\tau, \omega)$, $S_3(\tau)$, $S_4(\tau)$, $S_5(\tau)$, $S_6(\tau)$ and $S_7(\tau)$, respectively, and denote the point of $S_i(\tau, \omega)$ or $S_i(\tau)$ which is the image of $[x, y]$ of $A(\Omega)$ by the same notation $[x, y]$.

REMARK 1. $\tilde{\mathscr{S}}_1 = \{S_1(\tau, \omega) | (\tau, \omega) \in \mathfrak{H}^2\}$ form an analytic family effectively parametrized and complete at each point of $\mathfrak{H}^2$. In fact, we can construct this as follows: Let

$$\tilde{\mathscr{A}}_1 = \boldsymbol{C}^2 \times \mathfrak{H}^2 / \{f_a | a \in \boldsymbol{Z}^4\}$$

with

$$f_a\colon (x, y, \tau, \omega) \mapsto (x, y, \tau, \omega) + a\begin{pmatrix} \tau & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{\mathscr{S}}_1 = \tilde{\mathscr{A}}_1 / g^{\boldsymbol{Z}} \qquad g\colon [x, y, \tau, \omega] \mapsto [x + 1/2, -y, \tau, \omega]\ ,$$

where $[x, y, \tau, \omega]$ denotes the point of $\tilde{\mathscr{S}}_1$ which is the image of (x, y, τ, ω) in $\boldsymbol{C}^2 \times \mathfrak{H}^2$. Then $\tilde{\mathscr{S}}_1$ is non-singular, the holomorphic map $\tilde{\pi}_1\colon \tilde{\mathscr{S}}_1 \to$

$\mathfrak{H}^2$ induced by the projection $C^2\times\mathfrak{H}^2\to\mathfrak{H}^2$ is smooth and $\tilde{\pi}_1^{-1}(\tau,\omega)\cong S_1(\tau,\omega)$.

In the other cases, we can also construct the analytic families $\tilde{\pi}_2\colon\tilde{\mathscr{S}}_2\to\mathfrak{H}^2$ and $\tilde{\pi}_i\colon\tilde{\mathscr{S}}_i\to\mathfrak{H}$ $i=3,4,5,6,7$ in the same way as above.

DEFINITION. $\Gamma=SL(2,\boldsymbol{Z})/\{\pm I\}$

$$\Gamma_o(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{Z})\,|\,c\equiv 0 \pmod N\right\}\Big/\{\pm I\}$$

$$\bar{\Gamma}_o(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{Z})\,|\,a,d\equiv 1,\ c\equiv 0 \pmod N\right\}\quad\text{for}\quad N=3,4,6$$

$$\bar{\Gamma}_o(8)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{Z})|a,d\equiv 1 \pmod 4,\quad c\equiv 0 \pmod 8\right\}$$

$$\bar{\Gamma}_o(9)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\boldsymbol{Z})|a,d\equiv 1 \pmod 3,\quad c\equiv 0 \pmod 9\right\}.$$

REMARK 2. When $N=3,4,6,8$ or 9, the projection $SL(2,\boldsymbol{Z})\to\Gamma$ obviously induces an isomorphism $\bar{\Gamma}_o(N)\to\Gamma_o(N)$.

On the other hand, the following is well known. (See, for instance, [2]).

PROPOSITION 1. *$\mathfrak{H}/\Gamma$ and $\mathfrak{H}/\Gamma_o(N)$ are punctured Riemann surfaces. Especially when $N=2,3,4,6,8$ or 9, $\mathfrak{H}/\Gamma_o(N)$ is of genus 0 with $t=2,2,3,4,4$ or 4 points removed, respectively.*

PROPOSITION 2. *Let*

$$M_1'=\mathfrak{H}/\Gamma_o(2)\times\mathfrak{H}/\Gamma\,,\qquad M_2'=(\mathfrak{H}/\Gamma_o(2))^2\,,$$
$$M_3'=M_4'=\mathfrak{H}/\Gamma_o(3)\,,\qquad M_5'=M_6'=\mathfrak{H}/\Gamma_o(4)\,,\qquad M_7'=\mathfrak{H}/\Gamma_o(6)\,.$$

Then M_i' is the space of isomorphism classes of hyperelliptic surfaces of type (i).

REMARK 3. We can show as in the proof of Theorem 2 below that M_i' is in fact the coarse moduli space for hyperelliptic surfaces of type (i) in the sense of Mumford [8].

PROOF OF PROPOSITION 2. For any hyperelliptic surface S, an abelian surface A such that $S=A/g^Z$ as in Theorem 1, is uniquely determined by S. Indeed, A is determined as the unramified covering manifold of S of degree m on which the pull back of the canonical bundle K_S of S is trivial, where m is the order of K_S in $\mathrm{Pic}(S)$. In this case g^Z is the covering transformation group and the only elements which can generate g^Z are g and g^{-1}. Hence if we set $S = A(\Omega)/g^Z$ and $S' = A(\Omega')/(g')^Z$, then S and S' are isomorphic if and only if there exists an isomorphism of complex manifolds $\varphi\colon A(\Omega') \to A(\Omega)$ such that $\varphi \circ g' = g \circ \varphi$ or $\varphi \circ g' = g^{-1} \circ \varphi$. The following facts are straightforward.

LEMMA. *$A(\Omega)$ and $A(\Omega')$ are isomorphic as complex manifolds if and only if there exists a matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4, \boldsymbol{Z})$$

such that $\Omega' = (A\Omega + B)(C\Omega + D)^{-1}$. In this case, the isomorphism $\varphi\colon A(\Omega') \to A(\Omega)$ is induced by the affine transformation

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) + (\alpha, \beta)$$

of $\boldsymbol{C}^2$ for some $(\alpha, \beta) \in \boldsymbol{C}^2$.

On the other hand, g is induced by the affine transformation

$$\tilde{g}\colon (x, y) \mapsto (x, y)\begin{pmatrix} 1 & 0 \\ 0 & \sigma'_N \end{pmatrix} + (1/N, 0)$$

of $\boldsymbol{C}^2$, where $\sigma'_N = \sigma_N$ when $N = 2, 3, 4$ and $\sigma'_6 = -\sigma_3$. Thus we have an equality

$$\{\varphi \circ g' - g^{\pm 1} \circ \varphi\}[x, y] = [\{\tilde{\varphi} \circ \tilde{g}' - \tilde{g}^{\pm 1} \circ \tilde{\varphi}\}(x, y)] .$$

Hence $\varphi \circ g' = g^{\pm 1} \circ \varphi$ if and only if

$$\{\tilde{\varphi} \circ \tilde{g}' - \tilde{g}^{\pm 1} \circ \tilde{\varphi}\}(x, y) \in \boldsymbol{Z}^4 \begin{pmatrix} \Omega \\ I \end{pmatrix}$$

for any $(x, y) \in \boldsymbol{C}^2$. When this condition is satisfied, we see by easy calculation that A, B, C and D are diagonal matrices in the case of types (1), (3), (5), (7) or triangular matrices of which the (2, 1)-entry is 0 in the case of types (2), (4), (6). Let the (1, 1)-entry of A, B, C and D be a_1, b_1, c_1 and d_1, respectively, the (2, 2)-entry of A, B, C and D be a_2, b_2, c_2 and d_2, respectively, and

$$\Omega = \begin{pmatrix} \tau & * \\ 0 & \omega \end{pmatrix} \qquad \Omega' = \begin{pmatrix} \tau' & * \\ 0 & \omega' \end{pmatrix} .$$

Then from $\Omega' = (A\Omega + B)(C\Omega + D)^{-1}$, we get

$$\tau' = \frac{a_1\tau + b_1}{c_1\tau + d_1} , \qquad \omega' = \frac{a_2\omega + b_2}{c_2\omega + d_2} ,$$

and

$$\det\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \det\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = 1 .$$

Moreover from $\{\tilde{\varphi}\circ\tilde{g}' - \tilde{g}^{\pm 1}\circ\tilde{\varphi}\}(x, y) \in \boldsymbol{Z}^4\begin{pmatrix} \Omega \\ I \end{pmatrix}$, we get $c_1 \equiv 0 \pmod{N}$ and particularly in the case of type (2) $c_2 \equiv 0 \pmod{2}$.

Conversely, assume that there exist γ and δ such that $\tau' = \gamma(\tau)$ and $\omega' = \delta(\omega)$, and let

$$\gamma = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \delta = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \qquad \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \bar{\Gamma}_0(N) .$$

Then we have

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1} , \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4, \boldsymbol{Z})$$

$$\{\tilde{\varphi}\circ\tilde{g}' - \tilde{g}\circ\tilde{\varphi}\}(x, y) \in \boldsymbol{Z}^4 \begin{pmatrix} \Omega \\ I \end{pmatrix}$$

for any $(x, y) \in \boldsymbol{C}^2$ in each case, if we choose A, B, C, D and $\tilde{\varphi}$ as follows: In the case of type (1),

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) .$$

In the case of type (2),

$$A = \begin{pmatrix} a_1 & c_2/2 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & (d_2 - a_1)/2 \\ 0 & b_2 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & -c_1/2 \\ 0 & d_2 \end{pmatrix}$$

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) + (0, c_1/8) .$$

In the case of types (3), (5) and (7),

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) .$$

In the case of type (4),

$$A = \begin{pmatrix} a_1 & -1/3 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & (a_1 - 1)/3 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & c_1/3 \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & -c_1/3 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) + (0, c_1/9) .$$

In the case of type (6),

$$A = \begin{pmatrix} a_1 & (1 - a_1)/2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} b_1 & (1 - a_1)/2 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & -c_1/2 \\ 0 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} d_1 & -c_1/2 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{\varphi}\colon (x, y) \mapsto (x, y)(C\Omega + D) + (0, c_1\sigma_4/8)\ .$$

q.e.d.

Any hyperelliptic surface is an elliptic bundle in the following unique way.

$$\varpi\colon S_i(\tau, \omega) \ni [x, y] \mapsto [2x] \in E(2\tau) \qquad i = 1, 2$$

$$\varpi\colon S_i(\tau) \ni [x, y] \mapsto [Nx] \in E(N\tau) \qquad i = 3, 4, 5, 6, 7\ ,$$

where the fibers of ϖ are $E(\omega)$, $E(\sigma_3)$, $E(\sigma_4)$ or $E(\sigma_3)$ in the case of type (1)(2), (3)(4), (5)(6) or (7), respectively. We easily see that ϖ has (1) four sections, (2) four 2-fold quasi-sections, (3) three sections, (4) three 3-fold quasi-sections, (5) two sections and a 2-fold quasi-section, (6) two 2-fold quasi-sections and a 4-fold quasi-section, (7) a section, a 2-fold quasi-section and a 3-fold quasi-section.

Now to kill automorphisms we consider hyperelliptic surfaces with specific base points. o and q are base points, while p is a collection of zero, one, two or three base points depending on the types. We require o, p and q to satisfy the following conditions which we call $(*)$ for simplicity:

In the case of type (1), o is a point on one of the four sections of the elliptic fiberation ϖ. Points p and q satisfy $f(o) = p$ and $h(o) = q$, where f and h are elements of $\mathrm{Aut}(S)^o$ and $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^o$, respectively, of order 4 with $f^2(o) = o$.

In the case of type (2), o is a point on one of the four 2-fold quasi-sections of ϖ. A pair $p = \{p_1, p_2\}$ of points satisfies $f_i(o) = p_i$ for $i = 1, 2$, where f_i is an element of $\mathrm{Aut}(S)^o$ of order 4 such that $f_i(o) = o$ and that $f_1 \neq f_2^{\pm 1}$. A point q satisfies $h(o) = q$ for an element h of $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^o$ of order 4.

In the case of type (3), o is a point on one of the three sections of ϖ. Points p and q satisfy $f(o) = p$ and $h(o) = q$, where f and h are elements of $\mathrm{Aut}(S)^o$ and $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^o$ of order 6 and 2, respectively, with $f^2(o) = o$.

In the case of type (4), o is a point on one of the three 3-fold quasi-sections of ϖ. A triple $p = \{p_1, p_2, p_3\}$ of points satisfies $f_i(o) = p_i$ for $i = 1, 2, 3$, where f_i's are mutually different elements of $\mathrm{Aut}(S)^o$ of order 9 with $f_i^3(o) = o$, and $f_1 \circ f_2 \circ f_3(o) = o$. A point q satisfies $h(o) = q$ for an element h of $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^o$ of order 2.

In the case of type (5), o is a point on one of the two sections of

ϖ. A point q satisfies $h(o) = q$, where h is an element of $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^{\circ}$ of order 4 such that $h^2(o)$ is on a section of ϖ.

In the case of type (6), o is a point on one of the two 2-fold quasi-sections of ϖ. A point q satisfies $h(o) = q$, where h is the image of $[1/4]$ under an isomorphism $E(\sigma_4) \simeq \mathrm{Aut}(\varpi^{-1}(\varpi(o)))^{\circ}$.

In the case of type (7), o is a point on the section of ϖ. A point q satisfies $h(o) = q$ for an element h of $\mathrm{Aut}(\varpi^{-1}(\varpi(o)))^{\circ}$ of order 3.

We denote the hyperelliptic surface S with these base points by $S(o, p, q)$ or $S(o, q)$.

DEFINITION. By a family $(\pi: \mathscr{S} \to T; \mathfrak{o}, \mathfrak{p}, \mathfrak{q})$ of hyperelliptic surfaces of type (1) with base points over an analytic space T, we mean a flat map $\pi: \mathscr{S} \to T$ of analytic spaces together with sections $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ such that the fibers $(\pi^{-1}(t); \mathfrak{o}(t), \mathfrak{p}(t), \mathfrak{q}(t))$ are hyperelliptic surfaces of type (1) with base points satisfying the property $(*)$ for $o = \mathfrak{o}(t)$, $p = \mathfrak{p}(t)$ and $q = \mathfrak{q}(t)$.

We can define a family of hyperelliptic surfaces of type (2), (3), (4), (5), (6) or (7) with base points in the same way as above.

On $S_i(\tau, \omega)$ and $S_i(\tau)$, we can choose base points o^*, p^* and q^* satisfying these requirements $(*)$ as follows:
In the case of type (1),

$$o^* = [0, 0] , \quad p^* = [1/4, 0] , \quad q^* = [0, 1/4] .$$

In the case of type (2),

$$o^* = [0, 0] , \quad p^* = \{[1/4, 0], [1/4 + \tau, 0]\} , \quad q^* = [0, 1/4] .$$

In the case of type (3),

$$o^* = [0, 0] , \quad p^* = [1/6, 0] , \quad q^* = [0, 1/2] .$$

In the case of type (4),

$$o^* = [0, 0] , \quad p^* = \{[1/9, 0], [1/9 + \tau, 0], [1/9 + 2\tau, 0]\} ,$$
$$q^* = [0, 1/2] .$$

In the case of type (5),

$$o^* = [0, 0] , \quad q^* = [0, (1 + \sigma_4)/4] .$$

In the case of type (6),

$$o^* = [0, 0] , \quad q^* = [0, 1/4] .$$

In the case of type (7),

$$o^* = [0, 0] , \quad q^* = [0, 1/3] .$$

Then we can verify the following:

REMARK 4. Any automorphism of $S_i(\tau, \omega)$ or $S_i(\tau)$ which fixes o^*, p^* and q^*, is the identity.

We get the following proposition by considerations similar to those in Proposition 1.

PROPOSITION 3. *The set of isomorphism classes of hyperelliptic surfaces of type* (i) *with base points is in one to one correspondence with* M_i *defined as follows*:

$$M_1 = M_2 = (\mathfrak{H}/\Gamma_o(4))^2\,, \quad M_3 = \mathfrak{H}/\Gamma_o(6)\,, \quad M_4 = \mathfrak{H}/\Gamma_o(9)\,,$$
$$M_5 = \mathfrak{H}/\Gamma_o(4)\,, \quad M_6 = \mathfrak{H}/\Gamma_o(8)\,, \quad M_7 = \mathfrak{H}/\Gamma_o(6)\,.$$

REMARK 5. There exists a family over M_i of hyperelliptic surfaces of type (i) with base points effectively parametrized and complete. Indeed, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \bar{\Gamma}_o(N)$$

which are mapped to $\gamma, \delta \in \Gamma_o(N)$, respectively, and let

$$(\gamma, \delta)\colon [x, y, \tau, \omega] \mapsto \left[\frac{x}{c\tau + d}, \frac{y}{g\omega + h}, \quad \gamma(\tau), \delta(\omega)\right] \quad \text{for} \quad i = 1, 2\,,$$

$$\gamma\colon [x, y, \tau] \mapsto \left[\frac{x}{c\tau + d}, y, \gamma(\tau)\right] \quad \text{for} \quad i = 3, 4, 5, 6, 7\,.$$

Then $(\Gamma_o(4))^2$, $(\Gamma_o(4))^2$, $\Gamma_o(6)$, $\Gamma_o(9)$, $\Gamma_o(4)$, $\Gamma_o(8)$ and $\Gamma_o(6)$ act on $\tilde{\mathcal{S}}_i$ for $i = 1, 2, 3, 4, 5, 6$ and 7, respectively without fixed point, and the actions commute with $\tilde{\pi}_i$. Let

$$\begin{array}{lll} \mathcal{S}_i = \tilde{\mathcal{S}}_i/(\Gamma_o(4))^2 & i = 1, 2 & \mathcal{S}_3 = \tilde{\mathcal{S}}_3/\Gamma_o(6) \\ \mathcal{S}_4 = \tilde{\mathcal{S}}_4/\Gamma_o(9) & \mathcal{S}_5 = \tilde{\mathcal{S}}_5/\Gamma^o(4) & \\ \mathcal{S}_6 = \tilde{\mathcal{S}}_6/\Gamma_o(8) & \mathcal{S}_7 = \tilde{\mathcal{S}}_7/\Gamma_o(6)\,, & \end{array}$$

and let

$$\pi_i\colon \mathcal{S}_i \to M_i$$

be the holomorphic map induced by $\tilde{\pi}_i$. Then π_i is smooth and $\pi_i^{-1}([\tau, \omega]) \cong S_i(\tau, \omega)$ $i = 1, 2$ or $\pi_i^{-1}([\tau]) \cong S_i(\tau)$ $i = 3, 4, 5, 6, 7$, where $[\tau, \omega]$ or $[\tau]$ denotes the points of M_i which is the image of (τ, ω) in $\mathfrak{H}^2$ or τ in $\mathfrak{H}$, respectively. Moreover, let $\mathfrak{o}^*$, $\mathfrak{p}^*$ and $\mathfrak{q}^*$ be the maps which send each point t of M_i to o^*, p^* and q^* of the fiber $\pi_i^{-1}(t)$, respectively. Then $\mathfrak{o}^*$, $\mathfrak{p}^*$ and $\mathfrak{q}^*$ are sections or sections and a quasi-section of π_i depending on i.

THEOREM 2. *$\pi_i: \mathscr{S}_i \to M_i$ is the universal family of hyperelliptic surfaces of type (i) with base points, i.e., M_i is the fine moduli space in the sense of Mumford* [8] *of hyperelliptic surfaces of type (i) with base points.*

PROOF. We prove the theorem only in the case of type (1). The proof for the other types are similar.

By Proposition 3, we have a unique map $f: T \to M_1$ for any family over an analytic space T of hyperelliptic surfaces of type (1) with base points. We will see shortly that this map f is holomorphic. Then we have a morphism $\Phi_1: \mathfrak{M}_1 \to h_{M_1}$ of contravariant functors, where $\mathfrak{M}_1$ and h_{M_1} denote the set of families of hyperelliptic surfaces over T of type (1) with base points, modulo isomorphism, and the set of holomorphic maps from T to M_1, respectively. By Remark 5, $\Phi_1(T): \mathfrak{M}_1(T) \to h_{M_1}(T)$ is surjective for any analytic space T. Thus it is enough to show that for any family $(\pi: \mathscr{S} \to T; \mathfrak{o}, \mathfrak{p}, \mathfrak{q})$ of hyperelliptic surfaces of type (1) with base points, there is an isomorphism from φ to $\mathscr{S}_1 \times_{M_1} T$ over T, which maps $\mathfrak{o}(t)$, $\mathfrak{p}(t)$ and $\mathfrak{q}(t)$ to $(\mathfrak{o}^* \circ f(t), t)$, $(\mathfrak{p}^* \circ f(t), t)$ and $(\mathfrak{q}^* \circ f(t), t)$, respectively, for each point t of T. For any point t_o of T, $\pi_1^{-1}(f(t_o)) \cong \pi^{-1}(t_o)$. Hence, by Remark 5, there is a holomorphic map f' from a connected neighborhood U of t_o to M_1 with $f'(t_o) = f(t_o)$ and an isomorphism

$$F': \mathscr{S}_{|U} \xrightarrow{\sim} \mathscr{S}_1 \times_{M_1} U$$

over U such that $F' \circ \mathfrak{o}(t_o) = (\mathfrak{o}^* \circ f'(t_o), t_o)$, $F' \circ \mathfrak{p}(t_o) = (\mathfrak{p}^* \circ f'(t_o), t_o)$ and $F' \circ \mathfrak{q}(t_o) = (\mathfrak{q}^* \circ f'(t_o), t_o)$. By Remarks 1 and 5, $\mathscr{S}_1 \times_{M_1} U$ is expressed as a quotient manifold of $\boldsymbol{C}^2 \times U$. Hence there exists an automorphism of $\mathscr{S}_1 \times_{M_1} U$ which is induced by an automorphism of $\boldsymbol{C}^2 \times U$ of the form $(x, y, t) \mapsto (x + a(t), y, t)$ and which maps $F' \circ \mathfrak{o}(t)$ to $(\mathfrak{o}^* \circ f'(t), t)$, where a is a holomorphic function on U vanishing at t_o. Composing this automorphism with F', we get an isomorphism

$$F: \mathscr{S}_{|U} \xrightarrow{\sim} \mathscr{S}_1 \times_{M_1} U$$

over U which maps $\mathfrak{o}(t)$ to $(\mathfrak{o}^* f'(t), t)$. For any $\tau, \omega \in \mathfrak{H}$, there exist only four points $p_1^* = p^*$, $p_2^* = [3/4, 0]$, $p_3^* = [(1 + 2\tau)/4, 0]$ and $p_4^* = [(3 + 2\tau/4, 0]$ on $S_1(\tau, \omega)$ such that $(o^*, p_k^*, ?)$ satisfy the property $(*)$. Therefore $F \circ \mathfrak{p}(t)$ agrees with one of these points p_k^* on $\pi^{-1}(f(t)) \cong S_1(\tau, \omega)$, where $f'(t) = [\tau, \omega]$. But these points p_k^* for $k = 1, 2, 3, 4$ on each fiber $\pi^{-1}(f'(t)) \cong S_1(\tau, \omega)$ form the sections

$$\mathfrak{p}_k^*: U \to \mathscr{S}_1 \times_{M_1} U \qquad \text{for} \quad k = 1, 2, 3, 4 \ ,$$

respectively. Hence $F \circ \mathfrak{p}$ equals one of these sections $\mathfrak{p}_k^*$, since U is connected. Thus $F \circ \mathfrak{p} = (\mathfrak{p}^* \circ f', \mathrm{id})$ on U for $F \circ \mathfrak{p}(t_o) = (\mathfrak{p}_1^* \circ f'(t_o), t_o)$ and for $\mathfrak{p}^* = \mathfrak{p}_1^*$. By the same consideration, we get $F \circ \mathfrak{q} = (\mathfrak{q}^* \circ f', \mathrm{id})$ on U. Therefore, $f = f'$ and $F \circ \mathfrak{o} = (\mathfrak{o}^* \circ f, \mathrm{id})$, $F \circ \mathfrak{p} = (\mathfrak{p}^* \circ f, \mathrm{id})$, $F \circ \mathfrak{q} = (\mathfrak{q}^* \circ f, \mathrm{id})$ on U. Since F is uniquely determined by f, we can, by Remark 5, patch up these F's defined on neighborhoods U's which cover T. Thus we get a unique isomorphism from $\mathscr{S}$ to $\mathscr{S}_1 \times_{M_1} T$ over T. q.e.d.

2. Since $\mathfrak{H}/\Gamma_o(N)$ is a complex manifold of dimension 1, it has a unique non-singular compactification $\overline{(\mathfrak{H}/\Gamma_o(N))}$, and when $N = 4, 6, 8$ or 9, it is biholomorphic to the projective line $\boldsymbol{P}^1$. We construct it explicitely, for later convenience. (See [7] I.)

There are t $\Gamma_o(N)$-equivalence classes of the cusps for $\Gamma_o(N)$ acting on $\mathfrak{H}$, where t is the same as in Proposition 1. We choose representative points $p_1, p_2, \cdots, p_t$ from each of them. Let

$$\mathfrak{H}_{p_i} = \{\omega \in \mathfrak{H} | \mathrm{Im}(\delta(\omega)) > C\} ,$$
$$\Gamma_{p_i} = \delta^{-1}\Gamma_\infty(n)\delta ,$$
$$\Gamma_\infty(n) = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \equiv 0 \pmod{n}\right\} \Big/ \{\pm I\} ,$$

where $\delta \in \Gamma$ with $\delta(p_i) = \infty$, and n is the smallest positive integer such that $\delta^{-1}\Gamma_\infty(n)\delta$ is the subgroup of $\Gamma_o(N)$. Then $\mathfrak{H}_{p_i}/\Gamma_{p_i}$ is isomorphic to a punctured disk $\Delta_{p_i}^*$, and we can regard $\Delta_{p_i}^*$ as an open set of $\mathfrak{H}/\Gamma_o(N)$ for sufficiently large C, since $\mathfrak{H}_{p_i}$ is a neighborhood of p_i, and Γ_{p_i} is the stabilizer of p_i.

Let

$$\overline{(\mathfrak{H}/\Gamma_o(N))} = \mathfrak{H}/\Gamma_o(N) \cup \Delta_{p_1} \cup \Delta_{p_2} \cup \cdots \cup \Delta_{p_t} ,$$

by the natural identifications, where Δ_{p_i} is the disk with the same radius as $\Delta_{p_i}^*$. Then $\overline{(\mathfrak{H}/\Gamma_o(N))}$ is the non-singular compactification of $\mathfrak{H}/\Gamma_o(N)$. Let

$$\alpha_{p_i}\colon \mathfrak{H}_{p_i} \to \Delta_{p_i}^*$$

be the projection. Then $\alpha_{p_i}(\omega) = \exp(2\pi i\delta(\omega)/n)$. For $N = 4, 6, 8, 9$ we have the following:

$N = 4$. The cusps of $\Gamma_o(4)$ are represented by ∞, 0 and 1/2, and $\alpha_\infty(\omega) = e(\omega)$, $\alpha_0(\omega) = e(-1/4\omega)$, $\alpha_{1/2}(\omega) = e(\alpha/(1-2\omega))$, where $e(\omega) = \exp(2\pi i\omega)$.

$N = 6$. The cusps of $\Gamma_o(6)$ are represented by ∞, 0, 1/2 and 1/3, and $\alpha_\infty(\omega) = e(\omega)$, $\alpha_0(\omega) = e(-1/6\omega)$, $\alpha_{1/2}(\omega) = e(\omega/3(1-2\omega))$, $\alpha_{1/3}(\omega) = e(\omega/2(1-3\omega))$.

$N=8$. The cusps of $\Gamma_o(8)$ are represented by ∞, 0, 1/2 and 1/4, and $\alpha_\infty(\omega)=e(\omega)$, $\alpha_0(\omega)=e(-1/8\omega)$, $\alpha_{1/2}(\omega)=e(\omega/(1-2\omega))$, $\alpha_{1/4}(\omega)=e(\omega/(1-4\omega))$.

$N=9$. The cusps of $\Gamma_o(9)$ are represented by ∞, 0, 1/3 and $-1/3$, and $\alpha_\infty(\omega)=e(\omega)$, $\alpha_0(\omega)=e(-1/9\omega)$, $\alpha_{1/3}(\omega/(1-3\omega))=e(\omega/(1-3\omega))$, $\alpha_{-1/3}(\omega)=e(\omega/(1+3\omega))$.

DEFINITION. $\bar{M}_1=\bar{M}_2=\overline{(\mathfrak{H}/\Gamma_o(4))^2}$ $\bar{M}_3=\overline{(\mathfrak{H}/\Gamma_o(6))}$ $\bar{M}_4=\overline{(\mathfrak{H}/\Gamma_o(9))}$ $\bar{M}_5=\overline{(\mathfrak{H}/\Gamma_o(4))}$ $\bar{M}_6=\overline{(\mathfrak{H}/\Gamma_o(8))}$ $\bar{M}_7=\overline{(\mathfrak{H}/\Gamma_o(6))}$.

When $i=3$ through 7, $\bar{M}_i$ is the unique compactification of M_i, but $\bar{M}_1$ or $\bar{M}_2$ is only one of many other possible compactifications of M_1 or M_2, respectively.

Our final goal is to show that a possibly degenerate hyperelliptic surface with base points "naturally" corresponds to each point of $\bar{M}_i$ in the following way: For any point t_o of $\bar{M}_i$, there exists a neighborhood U of t_o, a finite covering $\varphi\colon V\to U$ with $\varphi^{-1}(t_o)$ consisting of a point s_o, and a family $\pi\colon\mathscr{S}\to V$ with sections $\mathfrak{o}, \mathfrak{q}$ and a section or a quasi-section $\mathfrak{p}$ such that

$$\mathscr{S}_{|\varphi^{-1}(U\cap M_i)}\xrightarrow{\sim}\mathscr{S}_i\times_{M_i}\varphi^{-1}(U\cap M_i)$$

over $\varphi^{-1}(U\cap M_i)$, where $\pi_i\colon\mathscr{S}_i\to M_i$ is the family in Remark 5. In this case, we say that $\pi^{-1}(s_o)(\mathfrak{o}(s_o), \mathfrak{p}(s_o), \mathfrak{q}(s_o))$ corresponds to $t_o=\varphi(s_o)$.

It is enough for our purpose to construct families with base points over finite covering spaces of $(\mathfrak{H}/\Gamma_o(4))\times\Delta_d$, $\Delta_d\times(\mathfrak{H}/\Gamma_o(4))$ and $\Delta_d\times\Delta_{d'}$ or Δ_d's, which together with M_i cover $\bar{M}_i$, when $i=1, 2$ or $i=3$ through 7, respectively. We construct these families in the next section as follows: First, we construct degenerating families $\mathscr{A}$ of abelian surfaces and automorphisms $\mathfrak{g}$ of $\mathscr{A}$. Secondly, we construct, if possible, a non-singular model $\mathscr{S}$, flat over the base space, of $\mathscr{A}/\mathfrak{g}^{\mathbf{Z}}$. Finally, we add base sections to these families.

For these constructions, we need the following. We use the theory of torus embeddings in the same notations as in Oda [5]. Especially we denote by φ_* the morphism of torus embeddings which corresponds to a morphism φ of r.p.p. decompositions.

(I) Families of abelian surfaces

(1) Let $N=\mathbf{Z}^4$ with $\mathbf{Z}$-basis $\{n_1, n_2, n_3, n_4\}$, and let $N'=\mathbf{Z}^2$ with $\mathbf{Z}$-basis $\{n_1', n_2'\}$. Let $X=T_N\operatorname{emb}(\Sigma)$, where $\Sigma=\{$all the faces of $\sigma_{i,j}\,|\,i, j\in\mathbf{Z}\}$ with

$$\sigma_{i,j}=\mathbf{R}_0(n_3+in_1)+\mathbf{R}_0(n_3+(i+1)n_1)+\mathbf{R}_0(n_4+jn_2)+\mathbf{R}_0(n_4+(j+1)n_2)\ .$$

Then X is non-singular. On the other hand, let $Y=T_{N'}\operatorname{emb}(\Delta)$, where

$\Lambda = \{$the faces of $\boldsymbol{R}_0 n'_1 + \boldsymbol{R}_0 n'_2\}$. Then $Y \cong \boldsymbol{C}^2$. Let φ be the morphism of r.p.p. decompositions from (N, Σ) to (N', Λ) defined by

$$\varphi\colon \sum_{i=1}^{4} a_i n_i \mapsto a_3 n'_1 + a_4 n'_2 \,.$$

Thus we have an equivariant morphism $\varphi_*\colon X \to Y \cong \boldsymbol{C}^2$. Let λ, κ, ν and μ be the automorphisms of (N, Σ) defined by

$$\lambda\colon \sum_{i=1}^{4} a_i n_i \mapsto (a_1 + a_3)n_1 + \sum_{i=2}^{4} a_i n_i$$

$$\kappa\colon \sum_{i=1}^{4} a_i n_i \mapsto (a_2 + a_4)n_2 + \sum_{i=1,3,4} a_i n_i$$

$$\nu\colon \sum_{i=1}^{4} a_i n_i \mapsto -a_1 n_1 + \sum_{i=2}^{4} a_i n_i$$

$$\mu\colon \sum_{i=1}^{4} a_i n_i \mapsto -a_2 n_2 + \sum_{i=1,3,4} a_i n_i \,.$$

Then $\varphi_* \circ \lambda_* = \varphi_*$, $\varphi_* \circ \kappa_* = \varphi_*$, $\varphi_* \circ \nu_* = \varphi_*$ and $\varphi_* \circ \mu_* = \varphi_*$. Let θ and ρ be the extensions to $T_N\mathrm{emb}(\Sigma)$ of the actions $(v, w, s, t) \mapsto (-v, w, s, t)$ and $(v, w, s, t) \mapsto (v, -w, s, t)$ of T_N, respectively. Let

$$\mathscr{A}_1 = X_{|D}/\lambda_*^{\boldsymbol{Z}} \times \kappa_*^{2\boldsymbol{Z}} \qquad \mathscr{A}_2 = X_{|D}/\lambda_*^{4\boldsymbol{Z}} \times \kappa_*^{2\boldsymbol{Z}}$$
$$\mathscr{A}_3 = X_{|D}/\lambda_*^{\boldsymbol{Z}} \times \kappa_*^{4\boldsymbol{Z}} \qquad \mathscr{A}_4 = X_{|D}/\lambda_*^{4\boldsymbol{Z}} \times \kappa_*^{4\boldsymbol{Z}}$$
$$\mathscr{A}_5 = X_{|D}/(\rho \circ \lambda_*)^{\boldsymbol{Z}} \times \kappa_*^{2\boldsymbol{Z}} \qquad \mathscr{A}_6 = X_{|D}/(\theta \circ \rho)^{\boldsymbol{Z}} \times \lambda_*^{2\boldsymbol{Z}} \times \kappa_*^{2\boldsymbol{Z}}$$
$$\mathscr{A}_7 = X_{|D}/(\lambda_* \circ \kappa_*^2)^{\boldsymbol{Z}} \times \kappa_*^{4\boldsymbol{Z}} \qquad \mathscr{A}_8 = X_{|D}/\lambda_*^{2\boldsymbol{Z}} \times (\theta \circ \kappa_*^2)^{\boldsymbol{Z}} \,,$$

where D is the unit polydisk in $T_{N'}\mathrm{emb}(\Lambda)$ and $X_{|D} = \varphi_*^{-1}(D)$. When $i \neq 6$, $\mathscr{A}_i$ is non-singular, but $\mathscr{A}_6$ has four isolated singular points some neighborhoods of which are isomorphic to $\mathrm{Spec}\{\boldsymbol{C}[x^2, y^2, z^2, w^2, xy, xz, xw, yz, yw, zw]\}$. If an automorphism ε of X induces an automorphism of $\mathscr{A}_i$ by natural projection $X \to \mathscr{A}_i$, we denote the induced automorphism by $[\varepsilon]$. Let $\varpi_i\colon \mathscr{A}_i \to D$ be the holomorphic map induced by φ_*. Then for non-zero s and t, $\varpi_i^{-1}(s, t)$ is an abelian surface, while $\varpi_i^{-1}(s, 0)$ and $\varpi_i^{-1}(0, t)$ consist of components each of which is isomorphic to a product $E \times \boldsymbol{P}^1$ of an elliptic curve and a line. These components cross along elliptic curves. When $i \neq 6$, $\varpi_i^{-1}(0, 0)$ consists of components each of which is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and these components intersect along fibers and sections. $\varpi_6^{-1}(0, 0)$ consists of four components each of which is isomorphic to

$$V = \boldsymbol{P}^1 \times \boldsymbol{P}^1/h^{\boldsymbol{Z}} \quad \text{with} \quad h\colon (\eta, \xi) \mapsto (-\eta, -\xi) \,.$$

(2) Let $\phi_n\colon \mathscr{E}_n \to \Delta = \{s \in \boldsymbol{C} \mid |s| < 1\}$ be the family of elliptic curves, whose general fiber $\phi_n^{-1}(s)$ has the periods 1 and $n(\log s)/2\pi i$, and $\phi_n^{-1}(0)$

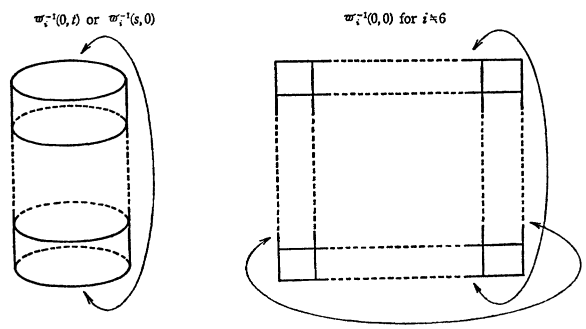


FIGURE 2-1

is a cycle of n rational curves crossing normally. Let α_k, β and γ be the automorphisms of $\mathscr{E}_n$ defined by

$$\alpha_k\colon (w, s) \mapsto (\sigma_k w, s) \quad \text{with} \quad \sigma_k = \exp(2\pi i/k) ,$$
$$\beta\colon (w, s) \mapsto (sw, s) , \qquad \gamma\colon (w, s) \mapsto (w^{-1}, s) ,$$

respectively. Let

$$\tilde{\mathscr{B}}_j = \boldsymbol{C} \times \mathfrak{H}/\{h_a^j \mid a \in \boldsymbol{Z}^2\} \text{ with}$$
$$h^1_{(a_1, a_2)}\colon (x, \tau) \mapsto (x + a_1\tau + a_2, \tau)$$
$$h^2_{(a_1, a_2)}\colon (x, \tau) \mapsto (x + 2a_1\tau + a_2, \tau) .$$

For any element $\delta \in \bar{\Gamma}_o(4)$, we define the automorphism of $\tilde{\mathscr{B}}_i$ by

$$[x, \tau] \mapsto [x/(c\tau + d), \tau] , \qquad \text{with} \quad \delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Then $\bar{\Gamma}_o(4)$ acts on $\tilde{\mathscr{B}}_i$. Let $\mathscr{B}_i = \tilde{\mathscr{B}}_i/\bar{\Gamma}_o(4)$ and let $\psi_i\colon \mathscr{B}_i \to (\mathfrak{H}/\bar{\Gamma}_o(4))$ be the holomorphic map induced by the projection $\boldsymbol{C} \times \mathfrak{H} \to \mathfrak{H}$. Clearly $\phi_k \times \psi_i\colon \mathscr{E}_k \times \mathscr{B}_i \to \Delta \times (\mathfrak{H}/\bar{\Gamma}_o(4))$ is a degenerating families of abelian surfaces.

(II) The resolution of quotient singularities.

We consider the following singularity. For a positive integer b, let

$$N_b = \boldsymbol{C}^3/h^{\boldsymbol{Z}} \quad \text{with} \quad h\colon (x, y, z) \mapsto (\sigma_b x, \sigma_b^{-1} y, \sigma_b^{-1} z) ,$$

where $\sigma_b = \exp(2\pi i/b)$. Ueno [12] constructed the canonical resolution of this singularity. Using torus embeddings, we reconstruct this resolution endowed with a fiberation different from that considered in [12].

Let $N = \boldsymbol{Z}^3$ with $\boldsymbol{Z}$-basis $\{n_1, n_2, n_3\}$, and let $n_1' = bn_1 + n_2 + n_3$, $n_2' = n_2$, $n_3' = n_3$. Let

$$\nabla = \{\text{the faces of } \sigma\}, \text{ with } \sigma = \boldsymbol{R}_0 n_1' + \boldsymbol{R}_0 n_2' + \boldsymbol{R}_0 n_3' ,$$

$$\tilde{\nabla} = \{\text{the faces of } \sigma_0, \sigma_k \text{ and } \tau_k | k = 1, \cdots, b-1\} \text{ with}$$
$$\sigma_0 = \boldsymbol{R}_0(n_1 + n_2 + n_3) + \boldsymbol{R}_0 n_2 + \boldsymbol{R}_0 n_3$$
$$\sigma_k = \boldsymbol{R}_0(kn_1 + n_2 + n_3) + \boldsymbol{R}_0((k+1)n_1 + n_2 + n_3) + \boldsymbol{R}_0 n_2$$
$$\tau_k = \boldsymbol{R}_0(kn_1 + n_2 + n_3) + \boldsymbol{R}_0((k+1)n_1 + n_2 + n_3) + \boldsymbol{R}_0 n_3 .$$

Then $\tilde{\nabla}$ is a subdivision of ∇, and $T_N\mathrm{emb}(\tilde{\nabla})$ is non-singular. $T_N\mathrm{emb}(\nabla)$ and $T_N\mathrm{emb}(\tilde{\nabla})$ are isomorphic to N_b and M in [12], respectively. ι_* agrees with T^{-1} in [12], where ι_* is the holomorphic map induced by the identity map ι of N. See Figure 2-2.

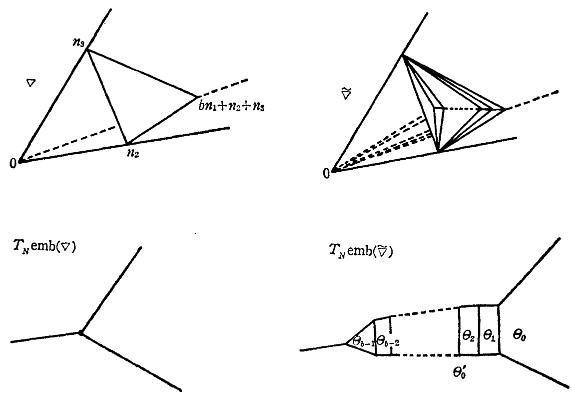


FIGURE 2-2

Next let $L = \boldsymbol{Z}$ with $\boldsymbol{Z}$-basis l, and let $\square = \{0, \boldsymbol{R}_0 l\}$. Let π be the morphism of r.p.p. decompositions defined by

$$\pi\colon (N, \tilde{\nabla}) \ni a_1 n_1 + a_2 n_2 + a_3 n_3 \mapsto a_2 l \in (L, \square) .$$

Then we obtain

$$\pi_*^{-1}(\mathrm{orb}(\boldsymbol{R}_0 l)) = \sum_{k=0}^{b-1} \Theta_k + \Theta_0' ,$$

where $\Theta_k = \overline{\mathrm{orb}(\boldsymbol{R}_0((b-k)n_1 + n_2 + n_3))}$, $\Theta_0' = \overline{\mathrm{orb}(\boldsymbol{R}_0 n_2)}$. We easily see that Θ_k and Θ_0' are isomorphic to those in [12], and intersect in the same way as in Lemma 4.6 [12]. In particular, we note that $\Theta_{b-1} \cong \boldsymbol{P}^2$ and $\Theta_k \cong \Sigma_{b-k}$ for $1 \leqq k \leqq b-2$, where Σ_d is the $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ of degree d.

3. We use the local degenerating families $\varpi_i\colon \mathscr{A}_i \to D$, $i = 1$ through 8, $\phi_k \times \psi_j\colon \mathscr{E}_k \times \mathscr{B}_j \to \Delta \times (\mathfrak{H}/\Gamma_o(4))$, $k = 1$ through 4, $j = 1, 2$, and $\phi_k\colon \mathscr{E}_k \times E \to \Delta$, $k = 1$ through 6, defined in the previous section. Most of the components of "degenerate hyperelliptic surfaces" described in this section are elliptic surfaces. In describing their singular fibers, we use the notation of Kodaira [4]. In the following, we denote by $\tilde{\Delta} \to \Delta$, sending

s to s^2, the double covering of a disk Δ.

(1-i) Over $\Delta_\infty \times (\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{A} = \mathscr{E}_1 \times \mathscr{B}_1$ and

$$\mathfrak{g}\colon \mathscr{A} \ni (p, [y, \omega]) \mapsto (\alpha_2 p, [-y, \omega]) \in \mathscr{A} .$$

Then $\mathfrak{g}$ has fixed points $\{(o, [y, \omega]) \in \mathscr{A} | y = 0, 1/2, \omega/2 \text{ or } (1+\omega)/2\}$ and by suitable coordinates in their neighborhoods, $\mathfrak{g}$ and $\phi_1 \times \psi_1$ are expressed as follows:

$$\mathfrak{g}\colon (x, y, z, w) \mapsto (-x, -y, -z, -w) ,$$
$$\phi_1 \times \psi_1\colon (x, y, z, w) \mapsto (xy, w) .$$

Thus some neighborhood of the singular points of $\mathscr{A}/\mathfrak{g}^Z$ is isomorphic to $N_2 \times (\mathfrak{H}/\Gamma_o(4))$. Let $\mathscr{S}$ be the non-singular model of $\mathscr{S}/\mathfrak{g}^Z$ as in (II) of Section 2, and let $\pi_{\mathrm{i}}\colon \mathscr{S} \to \Delta_\infty \times (\mathfrak{H}/\Gamma_o(4))$ be the holomorphic map induced by $\phi_1 \times \psi_1$. Then $\pi_{\mathrm{i}}^{-1}(0, [\omega])$ consists of five components crossing normally. A component V is isomorphic to a non-singular model of $\boldsymbol{P}^1 \times E(\omega)/\bar{g}^Z$, where $\bar{g}\colon (\eta, [y]) \mapsto (\eta, [y])$, thus is an elliptic surface over $\boldsymbol{P}^1$ with two singular fibers of type I_1^*. The other four components are isomorphic to $\boldsymbol{P}^2$. They intersect as in Figure (1-i). In particular, V intersects with itself at the points $[\infty, y]$ and $[0, y]$, where $[\eta, y]$ denotes the image in V of $(\eta, [y]) \in \boldsymbol{P}^1 \times E(\omega)$. Let

$$\mathfrak{o}\colon \Delta_\infty \times (\mathfrak{H}/\Gamma_o(4)) \ni (s, [\omega]) \mapsto [1, s, 0, \omega] \in \mathscr{S} ,$$
$$\mathfrak{p}\colon \Delta_\infty \times (\mathfrak{H}/\Gamma_o(4)) \ni (s, [\omega]) \mapsto [\sigma_4, s, 0, \omega] \in \mathscr{S} ,$$
$$\mathfrak{q}\colon \Delta_\infty \times (\mathfrak{H}/\Gamma_o(4)) \ni (s, [\omega]) \mapsto [1, s, 1/4, \omega] \in \mathscr{S} ,$$

where $[v, s, y, \omega]$ denotes the image in $\mathscr{S}$ of $([v, s], [y, \omega]) \in \mathscr{E}_1 \times \mathscr{B}_1$. Then $\mathfrak{o}(0, [\omega])$, $\mathfrak{p}(0, [\omega])$ and $\mathfrak{q}(0, [\omega])$ are the points $[1, 0]$, $[\sigma_4, 0]$ and $[1, 1/4])$ of V, respectively.

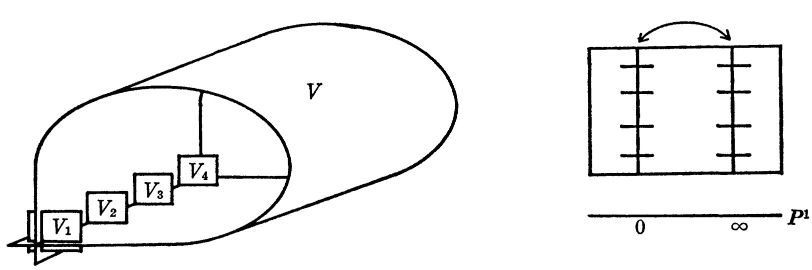


FIGURE 1-i

(1-ii) Over $\Delta_0 \times (\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{A} = \mathscr{E}_4 \times \mathscr{B}_1$ and

$$\mathfrak{g}\colon \mathscr{A} \ni (p, [y, \omega]) \mapsto (\beta^2 p, [-y, \omega]) \in \mathscr{A} .$$

Then $\mathfrak{g}$ has no fixed point, hence $\mathscr{S} = \mathscr{A}/\mathfrak{g}^{Z}$ is non-singular. Let π_{ii} be the holomorphic map from $\mathscr{S}$ to $\Delta_0 \times (\mathfrak{H}/\Gamma_o(4))$ induced by $\phi_4 \times \psi_1$. Then $\pi_{\mathrm{ii}}^{-1}(0, [\omega])$ is the analytic space consisting of two components V_1 and V_2 both isomorphic to $\boldsymbol{P}^1 \times E(\omega)$, which intersect in the following way: The points $(0, [y])$ and $(\infty, [y])$ of V_1 meet the points $(\infty, [y])$ and $(0, [-y])$, respectively. Let $\mathfrak{o}, \mathfrak{q}$ be the sections of π_{ii} defined in the same way as in (1-i), and let

$$\mathfrak{p}\colon \Delta_0 \times (\mathfrak{H}/\Gamma_o(4)) \in (s, [\omega]) \mapsto [s, s, 0, \omega] \in \mathscr{S} .$$

Then $\mathfrak{o}(0, [\omega])$, $\mathfrak{q}(0, [\omega])$ are the points $(1, [0])$, $(1, [1/4])$ of V_1, respectively, and $\mathfrak{p}(0, [\omega])$ is the point $(1, [0])$ of V_2.

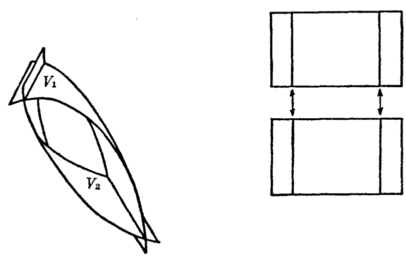


FIGURE 1-ii

(1-iii) Over $\Delta_{1/2} \times (\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{S}$ be the same manifold as in (1-i), and π_{iii} be the holomorphic map from $\mathscr{S}$ to $\Delta_0 \times (\mathfrak{H}/\Gamma_o(4))$ induced by $\phi_1 \times \psi_1$. Let $\mathfrak{o}, \mathfrak{q}$ be the sections of π_{iii} defined in the same way as in (1-i), and let

$$\mathfrak{p}\colon \Delta_{1/2} \times (\mathfrak{H}/\Gamma_o(4)) \ni (s, [\omega]) \mapsto [\sigma_4 s^{1/2}, s, 0, \omega] \in \mathscr{S} .$$

Then clearly $\pi_{\mathrm{iii}}^{-1}(0, [\omega])(\mathfrak{o}(0, [\omega]), \mathfrak{q}(0, [\omega]))$ is isomorphic to $\pi_{\mathrm{i}}^{-1}(0, [\omega])(\mathfrak{o}(0, [\omega]), \mathfrak{q}(0, [\omega]))$ in (1-i), and $\mathfrak{p}(0, [\omega])$ is a point of a component isomorphic to $\boldsymbol{P}^2$.

(1-iv) Over $(\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_\infty$.

Let $\mathscr{A} = \mathscr{B}_1 \times \mathscr{E}_2$ and let

$$\mathfrak{g}\colon \mathscr{A} \ni ([x, \tau], p) \mapsto ([x + 1/2, \tau], \gamma p) \in \mathscr{A} .$$

Then $\mathscr{S} = \mathscr{A}/\mathfrak{g}^{Z}$ is non-singular, since $\mathfrak{g}$ has no fixed point. Let $\pi_{\mathrm{iv}}\colon \mathscr{S} \to (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_\infty$ be the holomorphic map induced by $\psi_1 \times \phi_2$. Let

$$\mathfrak{o}\colon (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_\infty \ni ([\tau], t) \mapsto [0, \tau, 1, t] \in \mathscr{S} ,$$
$$\mathfrak{p}\colon (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_\infty \ni ([\tau], t) \mapsto [1/4, \tau, 1, t] \in \mathscr{S} ,$$
$$\mathfrak{q}\colon (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_\infty \ni ([\tau], t) \mapsto [0, \tau, \sigma_4, t] \in \mathscr{S} .$$

Then $\pi_{\mathrm{iv}}^{-1}([\tau], 0)$ is the analytic space consisting of two components V_1 and V_2 both isomorphic to $E(\tau) \times \boldsymbol{P}^1/\bar{g}^{Z}$, with $\bar{g}\colon ([x], \eta) \mapsto ([x + 1/2], \eta^{-1})$

meeting normally at the points $[x, 0] \in V_1$ and $[x, \infty] \in V_2$. $\mathfrak{o}([\tau], 0)$, $\mathfrak{p}([\tau], 0)$ and $\mathfrak{q}([\tau], 0)$ are the points $[0, 1]$, $[1/4, 1]$ and $[0, \sigma_4]$ of V_1, respectively.

(1-v) Over $(\mathfrak{H}/\Gamma_o(4)) \times \Delta_0$.

Let $\mathscr{A} = \mathscr{B}_1 \times \mathscr{E}_4$ and $\mathfrak{g}$ be the automorphism of $\mathscr{A}$ defined in the same way as in (1-iv). Then $\mathscr{S} = \mathscr{A}/\mathfrak{g}^Z$ is non-singular. Let $\pi_{\mathrm{v}}\colon \mathscr{S} \to (\mathfrak{H}/\Gamma_o(4)) \times \Delta_0$ be the holomorphic map induced by $\psi_1 \times \phi_4$. Then $\pi_{\mathrm{v}}^{-1}([\tau], 0)$ is the analytic space consisting of three components: V_1, V_2 both isomorphic to those of $\pi_{\mathrm{iv}}^{-1}([\tau], 0)$ in (1-iv) and V_3 isomorphic to $E(\tau) \times \boldsymbol{P}^1$ with the points $[x, \infty]$ of V_1 and $[x, 0]$ of V_2 meeting the points $([x], 0)$ and $([x], \infty)$, respectively. Let $\mathfrak{o}$, $\mathfrak{p}$ be the sections of π_{v} defined in the same way as in (1-iv), and let

$$\mathfrak{q}\colon (\mathfrak{H}/\Gamma_o(4)) \times \Delta_0 \ni ([\tau], t) \mapsto [0, \tau, t, t] \in \mathscr{S} .$$

Then $\mathfrak{q}([\tau], 0)$ is the point $([0], 1)$ of V_3.

(1-vi) Over $(\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_{1/2}$.

Let $\mathscr{S}$ be the same manifold as in (1-iv), and let $\pi_{\mathrm{vi}}\colon \mathscr{S} \to (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_{1/2}$ be the holomorphic map induced by $\psi_1 \times \phi_2$. Let $\mathfrak{o}$, $\mathfrak{p}$ be the sections of π_{vi} defined in the same way as in (1-iv), and let

$$\mathfrak{q}\colon (\mathfrak{H}/\Gamma_o(4)) \times \tilde{\Delta}_{1/2} \ni ([\tau], t) \mapsto [0, \tau, \sigma_4 t, t] \in \mathscr{S} .$$

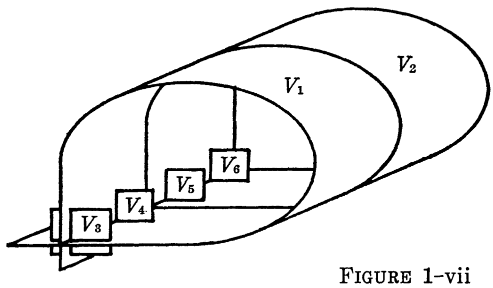
Then clearly $\pi_{\mathrm{vi}}^{-1}([\tau], 0)(\mathfrak{o}([\tau], 0), \mathfrak{p}([\tau], 0))$ is isomorphic to $\pi_{\mathrm{iv}}^{-1}([\tau], 0)(\mathfrak{o}([\tau], 0), \mathfrak{p}([\tau], 0))$ in (1-iv), and $\mathfrak{q}([\tau], 0)$ is the point $[0, \sigma_4]$ of V_2.

(1-vii) Over $\Delta_\infty \times \tilde{\Delta}_\infty$.

Let $\mathscr{A} = \mathscr{A}_{1|\Delta_\infty \times \tilde{\Delta}_\infty}$ and $\mathfrak{g} = [\theta \circ \mu_*]$. Then $\mathfrak{g}$ has fixed points $\{[0, w, s, t] \in \mathscr{A} | w = \pm 1, \pm t\}$ on $\mathscr{A}$, and some neighborhood of singular points of $\mathscr{A}/\mathfrak{g}^Z$ is isomorphic to $N_2 \times \tilde{\Delta}_\infty$. Thus we can obtain the non-singular model $\mathscr{S}$ of $\mathscr{A}/\mathfrak{g}^Z$. Let $\pi_{\mathrm{vii}}\colon \mathscr{S} \to \Delta_\infty \times \tilde{\Delta}_\infty$ be induced by ϖ_1. Then $\pi_{\mathrm{vii}}^{-1}(0, 0)$ consists of two components V_1, V_2 isomorphic to the non-singular model of $\boldsymbol{P}^1 \times \boldsymbol{P}^1/\bar{g}^Z$, where $\bar{g}\colon (\eta, \xi) \mapsto (-\eta, \xi^{-1})$, and four components V_3, V_4, V_5, V_6 isomorphic to $\boldsymbol{P}^2$. They intersect as in Figure 1-vii. In particular, the points $[\eta, 0]$ of V_1 meet the points $[\eta, \infty]$ of V_2, the points $[0, \xi]$ of V_1 (resp. V_2) meet the points $[\infty, \xi]$ of V_1 (resp. V_2). Let

$$\mathfrak{o}\colon \Delta_\infty \times \tilde{\Delta}_\infty \ni (s, t) \mapsto [1, 1, s, t] \in \mathscr{S} ,$$
$$\mathfrak{p}\colon \Delta_\infty \times \tilde{\Delta}_\infty \ni (s, t) \mapsto [\sigma_4, 1, s, t] \in \mathscr{S} ,$$
$$\mathfrak{q}\colon \Delta_\infty \times \tilde{\Delta}_\infty \ni (s, t) \mapsto [1, \sigma_4, s, t] \in \mathscr{S} .$$

Then $\mathfrak{o}(0, 0)$, $\mathfrak{p}(0, 0)$ and $\mathfrak{q}(0, 0)$ are the points $[1, 1]$, $[\sigma_4, 1]$ and $[1, \sigma_4]$ of V_1, respectively.



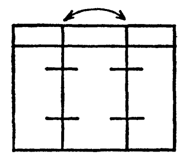


FIGURE 1-vii

(1-viii) Over $\Delta_0 \times \tilde{\Delta}_\infty$.

Let $\mathscr{S} = \mathscr{A}/\mathfrak{g}^Z$, where $\mathscr{A} = \mathscr{A}_{2|\Delta_0\times\tilde{\Delta}_\infty}$, $\mathfrak{g} = [\lambda_*^2 \circ \mu_*]$. Then $\mathscr{S}$ is non-singular. Let $\pi_{\text{viii}}: \mathscr{S} \to \Delta_0 \times \tilde{\Delta}_\infty$ be induced by ϖ_2. Then $\pi_{\text{viii}}^{-1}(0, 0)$ consists of four components V_1, V_2, V_3 and V_4 isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and intersecting in the following way: The points $(0, \xi)$, (∞, ξ) of V_1 (resp. V_3) meet the points (∞, ξ^{-1}), $(0, \xi)$ of V_2 (resp. V_4), and the points (η, ∞), $(\eta, 0)$ of V_1 (resp. V_2) meet the points (η, ∞), $(\eta, 0)$ of V_3 (resp. V_4). Let $\mathfrak{o}$, $\mathfrak{q}$ be the sections of π_{viii} defined in the same way as in (1-xii), and let

$$\mathfrak{p}: \Delta_0 \times \tilde{\Delta}_\infty \ni (s, t) \mapsto [s, 1, s, t] \in \mathscr{S} .$$

Then $\mathfrak{o}(0, 0)$, $\mathfrak{q}(0, 0)$ are the points $(1, 1)$, $(1, \sigma_4)$ of V_1 and $\mathfrak{p}(0, 0)$ is the point $(1, 1)$ of V_2.

(1-ix) Over $\Delta_{1/2} \times \tilde{\Delta}_\infty$.

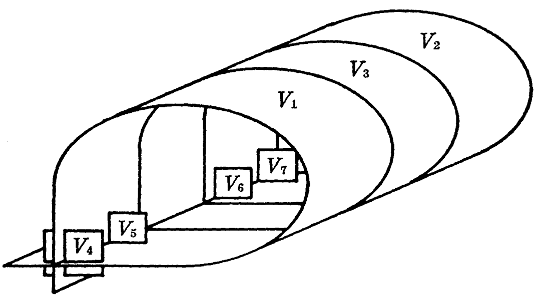
Let $\mathscr{S}$ be the same manifold as in (1-vii), and let $\pi_{\text{ix}}: \mathscr{S} \to \Delta_{1/2} \times \tilde{\Delta}_\infty$ be induced by ϖ_1. Let $\mathfrak{o}$, $\mathfrak{q}$ be the sections of π_{ix} defined in the same way as in (1-vii), and let

$$\mathfrak{p}: \Delta_{1/2} \times \Delta_\infty \ni (s, t) \mapsto [\sigma_4 s^{1/2}, 1, s, t] \in \mathscr{S} .$$

Then clearly $\pi_{\text{ix}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{q}(0, 0))$ is isomorphic to $\pi_{\text{vii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{q}(0, 0))$ in (1-vii), and $\mathfrak{p}(0, 0)$ is a point of V_3.

(1-x) Over $\Delta_\infty \times \Delta_0$.

Let $\mathscr{A} = \mathscr{A}_{3|\Delta_\infty\times\Delta_0}$ and $\mathfrak{g} = [\theta \circ \mu_*]$. Then we can obtain the non-singular model of $\mathscr{A}/\mathfrak{g}^Z$ in the same way as in (1-vii). Let $\pi_{\text{x}}: \mathscr{S} \to \Delta_\infty \times \Delta_0$ be the holomorphic map induced by ϖ_3. Then $\pi_{\text{x}}^{-1}(0, 0)$ consists of two components V_1, V_2 isomorphic to those in (1-vii) a component V_3 isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, four components V_4, V_5, V_6 and V_7 isomorphic to P^2 intersecting as in Figure 1-x. In particular, the points $[0, \xi]$ of V_1 (resp. V_2) meet the points $[\infty, \xi]$ of V_1 (resp. V_2), the points $[\eta, \infty]$ of V_1 and $[\eta, 0]$ of V_2 meet the points $(\eta, 0)$ and (η, ∞) of V_3, respectively. Let $\mathfrak{o}$, $\mathfrak{p}$ be the sections of π_{x} defined in the same way as in (1-vii), and let



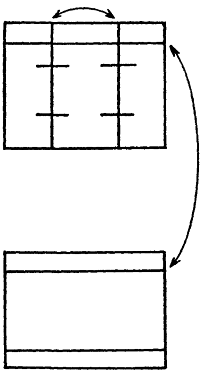


FIGURE 1-x

$$\mathfrak{q}\colon \Delta_\infty \times \Delta_0 \ni (s, t) \mapsto [1, t, s, t] \in \mathcal{S} .$$

Then $\mathfrak{o}(0, 0)$, $\mathfrak{p}(0, 0)$ are the points $[1, 1]$, $[\sigma_4, 1]$ of V_1, respectively, and $\mathfrak{q}(0, 0)$ is the point $(1, 1)$ of V_3.

(1-xi) Over $\Delta_0 \times \Delta_0$.

Let $\mathcal{S} = \mathcal{A}/\mathfrak{g}^Z$, where $\mathcal{A} = \mathcal{A}_{4|\Delta_0\times\Delta_0}$, $\mathfrak{g} = [\lambda_*^2 \circ \mu_*]$. Then $\mathcal{S}$ is non-singular. Let $\pi_{\mathrm{xi}}\colon \mathcal{S} \to \Delta_0 \times \Delta_0$ be the holomorphic map induced by ϖ_4. Then $\pi_{\mathrm{xi}}^{-1}(0, 0)$ consists of eight components isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ intersecting in the following way: The points (∞, ξ) of $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8$ meet the point $(0, \xi)$ of $V_5, V_6, V_7, V_8, (0, \xi^{-1})$ of V_4, V_1, V_3, V_2, respectively. The points (η, ∞) of $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8$ meet the points $(\eta, 0)$ of $V_2, V_3, V_4, V_1, V_6, V_7, V_8, V_5$, respectively. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{xi} defined in the same way as in (1-vii), (1-viii) and (1-x), respectively. Then $\mathfrak{o}(0, 0)$, $\mathfrak{p}(0, 0)$ and $\mathfrak{q}(0, 0)$ are the points $(1, 1)$ of V_1, V_5 and V_2, respectively.

(1-xii) Over $\Delta_{1/2} \times \Delta_0$.

Let $\mathcal{S}$ be the same manifold as in (1-x), and let $\pi_{\mathrm{xii}}\colon \mathcal{S} \to \Delta_{1/2} \times \Delta_0$ be induced by ϖ_3. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{xii} defined in the same way as in (1-vii), (1-ix) and (1-x), respectively. Then clearly $\pi_{\mathrm{xii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{q}(0, 0))$ is isomorphic to $\pi_{\mathrm{x}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{q}(0, 0))$ in (1-x), and $\mathfrak{p}(0, 0)$ is a point of V_4.

(1-xiii) Over $\Delta_\infty \times \tilde{\Delta}_{1/2}$.

Let $\mathcal{S}$ be the same manifold as in (1-vii), and let $\pi_{\mathrm{xiii}}\colon \mathcal{S} \to \Delta_\infty \times \tilde{\Delta}_{1/2}$ be the holomorphic map induced by ϖ_1. Let $\mathfrak{o}$ and $\mathfrak{p}$ be the sections of π_{xiii} defined in the same way as in (1-vii), and let

$$\mathfrak{q}\colon \Delta_\infty \times \tilde{\Delta}_{1/2} \ni (s, t) \mapsto [1, \sigma_4 t, s, t] \in \mathcal{S} .$$

Then clearly $\pi_{\mathrm{xiii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$ is isomorphic to $\pi_{\mathrm{vii}}^{-1}(0, 0)(\mathfrak{o}(0, 0),$

$\mathfrak{p}(0, 0))$ in (1-vii), and $\mathfrak{q}(0, 0)$ is the point $[1, \sigma_4]$ of V_2.

(1-xiv) Over $\Delta_0 \times \tilde{\Delta}_{1/2}$.

Let $\mathscr{S}$ be the same manifold as in (1-viii), and let $\pi_{\mathrm{xiv}}: \mathscr{S} \to \Delta_0 \times \tilde{\Delta}_{1/2}$ be the holomorphic map induced by ϖ_2. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{xiv} defined in the same way as in (1-vii), (1-viii) and (1-xiii), respectively. Then clearly $\pi_{\mathrm{xiv}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$ is isomorphic to $\pi_{\mathrm{viii}}^{-1}(0, 0)$ $(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$ in (1-viii), and $\mathfrak{q}(0, 0)$ is the point $(1, \sigma_4)$ of V_3.

(1-xv) Over $\Delta_{1/2} \times \tilde{\Delta}_{1/2}$.

Let $\mathscr{S}$ be the same manifold as in (1-ix), and let $\pi_{\mathrm{xv}}: \mathscr{S} \to \Delta_{1/2} \times \tilde{\Delta}_{1/2}$ be the holomorphic map induced by ϖ_1. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{xv} defined in the same way as in (1-vii), (1-ix) and (1-xiii), respectively. Then clearly $\pi_{\mathrm{xv}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{q}(0, 0))$ is isomorphic to $\pi_{\mathrm{xiii}}^{-1}(0, 0)(\mathfrak{o}(0$ $0), \mathfrak{q}(0, 0))$ in (1-xiii), and $\mathfrak{p}(0, 0)$ is a point of V_3.

(2-i) Over $\Delta_\infty \times (\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{A} = \mathscr{E}_2 \times \mathscr{B}_1/h^Z$, where $h: (p, [y, \omega]) \mapsto (\beta p, [y + 1/2, \omega])$. Then we can obtain the family $\pi_{\mathrm{i}}: \mathscr{S} \to \Delta_\infty \times (\mathfrak{H}/\Gamma_o(4))$ together with sections $\mathfrak{o}$, $\mathfrak{q}$ in the same way as in (1-i). Let $\mathfrak{p}$ be the quasi-section of π_{i} defined by

$$\mathfrak{p}: (s, [\omega]) \mapsto \{[\sigma_4, s, y, \omega] \mid y = 0, 1/2\} .$$

Then $\pi_{\mathrm{i}}^{-1}(0, [\omega])$ consists of the components isomorphic to those of $\pi_{\mathrm{i}}^{-1}(0, [\omega])$ in (1-i). But the points $[\infty, y]$ of V meet the points $[0, y + 1/2]$ of itself. $\mathfrak{p}(0, [\omega])$ is the pair of the points $[\sigma_4, 0]$, $[\sigma_4, 1/2]$ of V.

(2-ii) Over $\Delta_0 \times (\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{A} = \mathscr{E}_2 \times \mathscr{B}_1/h^Z$, where $h: (p, [y, \omega]) \mapsto (\alpha_2 p, [y + 1/2, \omega])$. Then we obtain the family $\pi_{\mathrm{ii}}: \mathscr{S} \to \Delta_0 \times (\mathfrak{H}/\Gamma_o(4))$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (1-ii). Let $\mathfrak{p}$ be the quasi-section of π_{ii} defined by

$$\mathfrak{p}: (s, [\omega]) \mapsto \{[\pm s^{1/2}, s, 0, \omega]\} .$$

Then $\pi_{\mathrm{ii}}^{-1}(0, [\omega])$ consists of the component V whose normalization is isomorphic to $\boldsymbol{P}^1 \times E(\omega)/\bar{g}^Z$, where $\bar{g}: (\eta, [y]) \mapsto (-\eta, [y + 1/2])$, thus an elliptic surface with two double fibers. $\mathfrak{p}(0, [\omega])$ is a point on the double

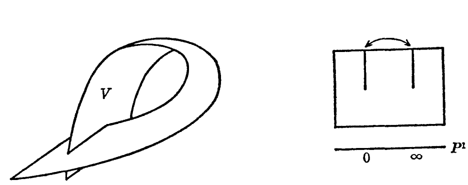


FIGURE 2–ii

curve of V.

(2-iii) Over $\Delta_{1/2}\times(\mathfrak{H}/\Gamma_o(4))$.

Let $\mathscr{A}$ be the same manifold as in (2-i), and let $\mathfrak{g}: [p, y, \omega]\mapsto [\alpha_2\beta p, -y, \omega]$. Then we obtain the family $\pi_{\mathrm{iii}}: \mathscr{S}\to\Delta_{1/2}\times(\mathfrak{H}/\Gamma_o(4))$ together with sections $\mathfrak{o}$, $\mathfrak{q}$. Let $\mathfrak{p}$ be the quasi-section of π_{iii} defined by

$$\mathfrak{p}: (s, [\omega])\mapsto\{[\sigma_4 s^{1/2}, s, y, \omega]\,|\,y=0, 1/2\}\ .$$

Then $\pi_{\mathrm{iii}}^{-1}(0, [\omega])$ is isomorphic to $\pi_{\mathrm{i}}^{-1}(0, [\omega])$ in (2-i), $\mathfrak{o}(0, [\omega])$, $\mathfrak{q}([0, [\omega])$ are the points $[1, 1/4]$, $[1, 1/2]$ of V_1, respectively, and $\mathfrak{p}(0, [\omega])$ is a pair of points one on V_2 and the other on V_4.

(2-iv) Over $(\mathfrak{H}/\Gamma_o(4))\times\tilde{\Delta}_\infty$.

Let $\mathscr{A}=\mathscr{B}_2\times\mathscr{E}_2/h^{\mathbf{Z}}$, where $h: ([x, \tau], p)\mapsto([x+\tau, \tau], \alpha_2 p)$. Then we obtain the family $\pi_{\mathrm{iv}}: \mathscr{S}\to(\mathfrak{H}/\Gamma_o(4))\times\tilde{\Delta}_\infty$ together with the sections $\mathfrak{o}$, $\mathfrak{q}$ in the same way as in (1-iv). Let $\mathfrak{p}$ be the quasi-section of π_{iv} defined by

$$\mathfrak{p}: ([\tau], t)\mapsto\{[x, \tau, 1, t]\,|\,x=1/4, 1/4+\tau\}\ .$$

Then $\pi_{\mathrm{iv}}^{-1}([\tau], 0)$ consists of two components V_1 and V_2 isomorphic to $E(2\tau)\times\boldsymbol{P}^1/\bar{h}^{\mathbf{Z}}\times\bar{g}^{\mathbf{Z}}$, where $\bar{h}: ([x], \eta)\mapsto([x+\tau], -\eta)$, $\bar{g}: ([x], \eta)\mapsto([x+1/2], \eta^{-1})$, with the points $[x, \infty]$ and $[x, 0]$ of V_1 meeting the points $[x, 0]$ and $[x, \infty]$ of V_2, respectively. $\mathfrak{p}([\tau], 0)$ is the pair of the points $[1/4, 1]$ and $[1/4+\tau, 1]$ of V_1.

(2-v) Over $(\mathfrak{H}/\Gamma_o(4))\times\Delta_0$.

Let $\mathscr{A}=\mathscr{B}_2\times\mathscr{E}_4/h^{\mathbf{Z}}$, where $h: ([x, \tau], p)\mapsto([x+\tau, \tau], \beta^2 p)$. Then we obtain the family $\pi_{\mathrm{v}}: \mathscr{S}\to(\mathfrak{H}/\Gamma_o(4))\times\Delta_0$ together with the sections $\mathfrak{o}$ and $\mathfrak{q}$, in the same way as in (1-v). Let $\mathfrak{p}$ be the quasi-section of π_{v} defined in the same way as in (2-iv). Then $\pi_{\mathrm{v}}^{-1}([\tau], 0)$ consists of components: V_1 isomorphic to that of $\pi_{\mathrm{iv}}^{-1}([\tau], 0)$ in (1-iv), and V_2 isomorphic to $E(2\tau)\times\boldsymbol{P}^1/\bar{g}^{\mathbf{Z}}$, where $\bar{g}: ([x], \xi)\mapsto([x+\tau+1/2], \xi^{-1})$. The points $[x, \infty]$ of V_1 meet the points $[x, 0]$ of V_2. And $\mathfrak{o}([\tau], 0)$, $\mathfrak{q}([\tau], 0)$ are the points $[0, 1]$ of V_1, V_2, respectively, $\mathfrak{p}([\tau], 0)$ is the pair of the points $[1/4, 1]$ and $[1/4+\tau, 1]$ of V_1.

(2-vi) Over $(\mathfrak{H}/\Gamma_o(4))\times\tilde{\Delta}_{1/2}$.

Let $\mathscr{S}$ be the same manifold as in (2-iv), and let $\pi_{\mathrm{vi}}: \mathscr{S}\to(\mathfrak{H}/\Gamma_o(4))\times\Delta_{1/2}$ be the holomorphic map induced by $\psi_2\times\phi_2$. Let $\mathfrak{o}$, $\mathfrak{q}$ and $\mathfrak{p}$ be the sections and the quasi-section of π_{vi} defined in the same way as in (1-vi) and (2-iv), respectively. Then clearly $\pi_{\mathrm{vi}}^{-1}([\tau], 0)(\mathfrak{o}([\tau], 0), \mathfrak{p}([\tau], 0))$ is isomorphic to $\pi_{\mathrm{iv}}^{-1}([\tau], 0)(\mathfrak{o}([\tau], 0), \mathfrak{p}([\tau], 0))$ in (2-iv), and $\mathfrak{q}([\tau], 0)$ is the point $[0, \sigma_4]$ of V_2.

(2-vii) Over $\Delta_\infty\times\tilde{\Delta}_\infty$.

Let $\mathscr{A}=\mathscr{A}_{5|\Delta_\infty\times\tilde{\Delta}_\infty}$, $\mathfrak{g}=[\theta\circ\mu_*]$. Then we obtain the family $\pi_{\mathrm{vii}}\colon\mathscr{S}\to\Delta_\infty\times\tilde{\Delta}_\infty$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (1-vii). Let $\mathfrak{p}$ be the quasi-section defined by

$$\mathfrak{p}\colon(s,t)\mapsto\{[\sigma_4,\pm1,s,t]\}\ .$$

Then $\pi_{\mathrm{vii}}^{-1}(0,0)$ consists of six components isomorphic to those of $\pi_{\mathrm{vii}}^{-1}(0,0)$ in (1-vii). The points $[0,\xi]$ of V_1 (resp. V_2) meet the points $[\infty,\xi]$ of V_1 (resp. V_2) and the points $[\eta,0]$ of V_1 meet the points $[\eta,\infty]$ of V_2. $\mathfrak{p}(0,0)$ is the pair of the points $[\sigma_4,1]$ and $[\sigma_4,-1]$ of V_1.

(2-viii) Over $\Delta_0\times\tilde{\Delta}_\infty$.

Let $\mathscr{S}=\mathscr{A}/\mathfrak{g}^{\boldsymbol{Z}}$, where $\mathscr{A}=\mathscr{A}_{6|\Delta_0\times\tilde{\Delta}_\infty}$, $\mathfrak{g}=[\lambda_*\circ\mu_*]$, and let $\pi_{\mathrm{viii}}\colon\mathscr{S}\to\Delta_0\times\tilde{\Delta}_\infty$ be the holomorphic map induced by ϖ_6. Let $\mathfrak{p}$ be the quasi-section of π_{viii} defined by

$$\mathfrak{p}\colon(s,t)\mapsto\{[\pm s^{1/2},1,s,t]\}\ .$$

Then $\mathscr{S}$ has two isolated singular points isomorphic to those of $\mathscr{A}_6$, and $\pi_{\mathrm{viii}}^{-1}(0,0)$ consists of two components whose normalization is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1/\bar{h}^{\boldsymbol{Z}}$, with $\bar{h}\colon(\eta,\xi)\mapsto(-\eta,-\xi)$. The points $[\eta,\infty]$ of V_1 (resp. V_2) meet the points $[\eta,0]$ of V_2 (resp. V_1), and the points $[0,\xi]$ of V_1 (resp. V_2) meet the points $[\infty,\xi]$ of V_1 (resp. V_2). $\mathfrak{p}(0,0)$ is a point on the double curve of V_1.

(2-ix) Over $\Delta_{1/2}\times\tilde{\Delta}_\infty$.

Let $\mathscr{A}$ be the same manifold as in (2-vii), and let $\mathfrak{g}=[\theta\circ\lambda\circ\mu_*]$. Then we obtain the family $\pi_{\mathrm{ix}}\colon\mathscr{S}\to\Delta_{1/2}\times\Delta_\infty$ together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (2-vii). Let $\mathfrak{p}$ be the quasi-section of π_{ix} defined by

$$\mathfrak{p}\colon(s,t)\mapsto\{[\pm\sigma_4 s^{1/2},\sigma_4,s,t]\}\ .$$

Then $\pi_{\mathrm{ix}}^{-1}(0,0)$ is isomorphic to $\pi_{\mathrm{vii}}^{-1}(0,0)$ in (2-vii), $\mathfrak{o}(0,0)$, $\mathfrak{q}(0,0)$ are the points $[1,\sigma_4]$, $[1,-1]$ of V_1, respectively, and $\mathfrak{p}(0,0)$ is a pair of points of V_3 and V_4.

(2-x) Over $\Delta_\infty\times\Delta_0$.

Let $\mathscr{A}=\mathscr{A}_{7|\Delta_\infty\times\Delta_0}$, $\mathfrak{g}=[\theta\circ\mu_*]$. Then we obtain the family $\pi_{\mathrm{x}}\colon\mathscr{S}\to\Delta_\infty\times\Delta_0$ together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (1-x). Let $\mathfrak{p}$ be the quasi-section of π defined by

$$\mathfrak{p}\colon(s,t)\mapsto\{[\sigma_4,1,s,t],[\sigma_4,t^2,s,t]\}\ .$$

Then $\pi_{\mathrm{x}}^{-1}(0,0)$ consists of seven components isomorphic to those of $\pi_{\mathrm{x}}^{-1}(0,0)$ in (1-x), and intersecting as in Figure 2-x. In particular, the points $[0,\xi]$ of V_1 (resp. $[0,\xi]$ of V_2, $[\eta,0]$ of V_1, $[\eta,0]$ of V_2, $(0,\xi)$ of V_3)

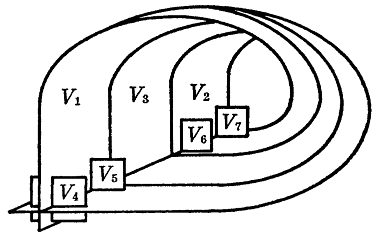


FIGURE 2-x

meet the points $[\infty, \xi]$ of V_2 (resp. $[\infty, \xi]$ of V_1, $(\eta, 0)$ of V_3, (η, ∞) of V_3, (∞, ξ) of V_3). $\mathfrak{p}(0, 0)$ is the pair of the points $[\sigma_4, 1]$ of V_1 and $(\sigma_4, 1)$ of V_3.

(2-xi) Over $\varDelta_0 \times \varDelta_0$.

Let $\mathscr{A} = \mathscr{A}_{8|\varDelta_0\times\varDelta_0}$, $\mathfrak{g} = [\lambda_* \circ \mu_*]$. Then we obtain the family $\pi_{\mathrm{xi}}: \mathscr{S} \to \varDelta_0 \times \varDelta_0$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (1-xi). Let $\mathfrak{p}$ be the quasi-section of π_{ix} defined in the same way as in (2-viii). Then $\pi_{\mathrm{xi}}^{-1}(0, 0)$ consists of four components V_1, V_2, V_3 and V_4 isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and intersect in the following way: The points (∞, ξ) of V_1 (resp. V_2) meet the points $(0, \xi)$ of V_3 (resp. V_4), the points $(0, \xi)$ of V_1 (resp. V_2) meet the points (∞, ξ^{-1}) of V_3 (resp. V_4), the points (η, ∞) of V_1 (resp. V_3) meet the points $(\eta, 0)$ of V_2 (resp. V_4), and the points $(\eta, 0)$ of V_1 (resp. V_3) meet the points $(-\eta, \infty)$ of V_2 (resp. V_4). And $\mathfrak{p}(0, 0)$ is the point obtained by identifing the point $(0, 1)$ of V_1 with the point $(\infty, 1)$ of V_3.

(2-xii) Over $\varDelta_{1/2} \times \varDelta_0$.

Let $\mathscr{A} = \mathscr{A}_{7|\varDelta_{1/2}\times\varDelta_0}$, $\mathfrak{g} = [\theta \circ \lambda_* \circ \mu_*]$. Then we obtain the family $\pi_{\mathrm{xii}}: \mathscr{S} \to \varDelta_{1/2} \times \varDelta_0$, together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (2-x). Let $\mathfrak{p}$ be the quasi-section of π_{xii} defined in the same way as in (2-ix). Then $\pi_{\mathrm{xii}}^{-1}(0, 0)$ is isomorphic to $\pi_{\mathrm{x}}^{-1}(0, 0)$ in (2-x), and $\mathfrak{p}(0, 0)$ is the pair of the points obtained by identifing the points $(0, 1)$ and $(0, -1)$ of V_3 with the points $(\infty, 1)$ and $(\infty, -1)$ of itself, respectively.

(2-xiii) Over $\varDelta_\infty \times \tilde{\varDelta}_{1/2}$.

We can obtain the family $\pi_{\mathrm{xiii}}: \mathscr{S} \to \varDelta_\infty \times \tilde{\varDelta}_{1/2}$, together with the section $\mathfrak{o}$ and the quasi-section $\mathfrak{p}$ of π_{xiii} in the same way as in (2-vii). Let $\mathfrak{q}$ be the section of π_{xiii} defined in the same way as in (1-xiii). Then clearly $\pi_{\mathrm{xiii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$ is isomorphic to $\pi_{\mathrm{vii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$, and $\mathfrak{q}(0, 0)$ is the point $[1, \sigma_4]$ of V_2.

(2-xiv) Over $\varDelta_0 \times \tilde{\varDelta}_{1/2}$.

We can obtain the family $\pi_{\mathrm{xiv}}: \mathscr{S} \to \varDelta_0 \times \tilde{\varDelta}_{1/2}$, together with the section $\mathfrak{o}$ and the quasi-section $\mathfrak{p}$ of π_{xiv} in the same way as in (2-viii). Let $\mathfrak{q}$ be the section of π_{xiv} defined in the same way as in (1-xiv). Then clearly $\pi_{\text{xiv}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$ is isomorphic to $\pi_{\text{viii}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$, and $\mathfrak{q}(0, 0)$ is the point $[1, \sigma_4]$ of V_2.

(2-xv) Over $\Delta_{1/2}\times\tilde{\Delta}_{1/2}$.

We can obtain the family $\pi_{\text{xv}}\colon \mathscr{S}\to\Delta_{1/2}\times\tilde{\Delta}_{1/2}$, together with the section $\mathfrak{o}$ and the quasi-section $\mathfrak{p}$ of π_{xv} in the same way as in (2-ix). Let $\mathfrak{q}$ be the section of π_{xv} defined in the same way as in (1-xv). Then clearly $\pi_{\text{xv}}^{-1}(0, 0)(\mathfrak{o}(0, 0), (0, 0))$ is isomorphic to $\pi_{\text{ix}}^{-1}(0, 0)(\mathfrak{o}(0, 0), \mathfrak{p}(0, 0))$, and $\mathfrak{q}(0, 0)$ is the point $[1, -\sigma_4]$ of V_2.

(3-i) Over Δ_∞.

Let $\mathscr{A} = \mathscr{E}_1\times E(\sigma_3)$, and let $\mathfrak{g}\colon (p, [y])\mapsto(\alpha_3 p, [\sigma_3 y])$ be the automorphism of $\mathscr{A}$. Then $\mathfrak{g}$ has three fixed points, and the singular points of $\mathscr{A}/\mathfrak{g}^Z$ are isomorphic to N_3 of (II) in Section 2. Thus we can obtain the family $\pi_{\text{i}}\colon\mathscr{S}\to\Delta_\infty$. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{i} which map $s\in\Delta_\infty$ to $[1, s, 0]$, $[\sigma_6, s, 0]$ and $[1, s, 1/2]\in\mathscr{S}$, respectively. Then $\pi_{\text{i}}^{-1}(0)$ consists of three components $V_{1,1}$, $V_{2,1}$, $V_{3,1}$ isomorphic to Σ_2, three components $V_{1,2}$, $V_{2,2}$, $V_{3,2}$ isomorphic to $\boldsymbol{P}^2$, and a component V isomorphic to a non-singular model of $\boldsymbol{P}^1\times E(\sigma_3)/\bar{\mathfrak{g}}^Z$, with $\bar{\mathfrak{g}}\colon(\eta, [y])\mapsto(\sigma_3\eta, [\sigma_3 y])$. Thus V is an elliptic surface over $\boldsymbol{P}^1$ with two singular fibers of types IV* and IV. These components intersect as in Figure 3-i. In particular, the points $[0, y]$ of V meet the point $[\infty, y]$ of itself. $\mathfrak{o}(0)$, $\mathfrak{p}(0)$ and $\mathfrak{q}(0)$ are the points $[1, 0]$, $[\sigma_6, 0]$ and $[1, 1/2]$ of V, respectively.

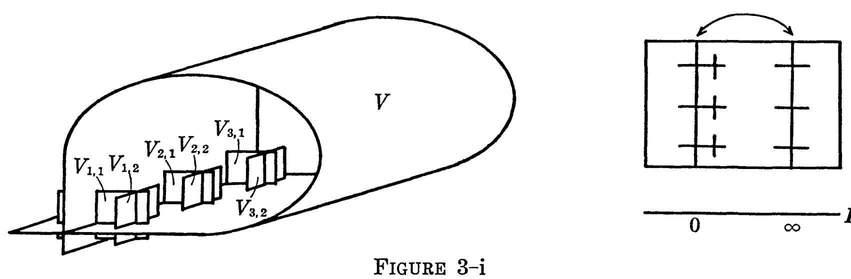


FIGURE 3-i

(3-ii) Over Δ_0.

Let $\mathscr{S} = \mathscr{A}/\mathfrak{g}^Z$, where $\mathscr{A} = \mathscr{E}_6\times E(\sigma_3)$, $\mathfrak{g}\colon(p, [y])\mapsto(\beta^2 p, [\sigma_3 y])$. Then $\mathscr{S}$ is non-singular. Let $\pi_{\text{ii}}\colon\mathscr{S}\to\Delta_0$ be the holomorphic map induced by ϕ_6. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{ii} which map $s\in\Delta_0$ to $[1, s, 0]$, $[s, s, 0]$ and $[1, s, 1/2]\in\mathscr{S}$, respectively. Then $\pi_{\text{ii}}^{-1}(0)$ consists of two components V_1 and V_2 both isomorphic to $\boldsymbol{P}^1\times E(\sigma_3)$. The points $(0, [y])$ and $(\infty, [y])$ of V_1 meet the points $(\infty, [\sigma_3 y])$ and $(0, [y])$ of V_2, respectively. (See Figure 1-ii.) $\mathfrak{o}(0)$, $\mathfrak{q}(0)$ are the points $(1, [0])$, $(1, [1/2])$ of V_1

respectively, and $\mathfrak{p}(0)$ is the point $(1, [0])$ of V_2.

(3-iii) Over $\Delta_{1/2}$.

Let $\mathscr{S} = \mathscr{A}/\mathfrak{g}^Z$, where $\mathscr{A} = \mathscr{E}_3 \times E(\sigma_3)$, $\mathfrak{g}: (p, [y]) \mapsto (\alpha_3\beta^2 p, [\sigma_3 y])$, and let $\pi_{\mathrm{iii}}: \mathscr{S} \to \Delta_{1/2}$ be the holomorphic map induced by ϕ_3. Let $\mathfrak{o}$, $\mathfrak{p}$ and $\mathfrak{q}$ be the sections of π_{iii} which map $s \in \Delta_{1/2}$ to $[1, s, 0]$, $[\sigma_6 s, s, 0]$ and $[1, s, 1/2] \in \mathscr{S}$, respectively. Then $\pi_{\mathrm{iii}}^{-1}(0)$ consists of a component V whose normalization is isomorphic to $\boldsymbol{P}^1 \times E(\sigma_3)$. The points $(0, [y])$ meet the points $(\infty, [\sigma_3 y])$. And $\mathfrak{o}(0)$, $\mathfrak{p}(0)$ and $\mathfrak{q}(0)$ are the points $(1, [0])$, $(-1, [0])$ and $(1, [1/2])$ of V, respectively.

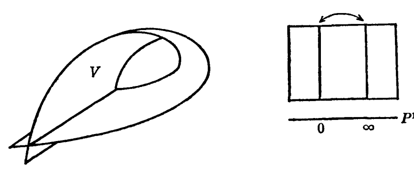


FIGURE 3-iii

(3-iv) Over $\Delta_{1/3}$.

Let $\mathscr{A} = \mathscr{E}_2 \times E(\sigma_3)$, and let $\mathfrak{g}$ be the automorphism of $\mathscr{A}$ defined in the same way as in (3-i). Then we obtain the family $\pi_{\mathrm{iv}}: \mathscr{S} \to \Delta_{1/3}$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (3-i). Let $\mathfrak{p}$ be the section of π_{iv} which maps $s \in \Delta_{1/3}$ to $[\sigma_6 s^{3/2}, s, 0] \in \mathscr{S}$. Then $\pi_{\mathrm{iv}}^{-1}(0)$ is the unramified double covering space of $\pi_{\mathrm{i}}^{-1}(0)$ in (3-i), $\mathfrak{o}(0)$, $\mathfrak{q}(0)$ are the points $[1, 0]$, $[1, 1/2]$ of V', respectively, and $\mathfrak{p}(0)$ is a point of $V'_{1,2}$, where V' and $V'_{1,2}$ are copies of V and $V_{1,2}$ of $\pi_{\mathrm{i}}^{-1}(0)$, respectively.

(4-i) Over Δ_∞.

Let $\mathscr{A} = \mathscr{E}_3 \times E(\sigma_3)/h^Z$, where $h: (p, [y]) \mapsto (\beta p, [y + (1 - \sigma_3)/3])$. We obtain the family $\pi_{\mathrm{i}}: \mathscr{S} \to \Delta_\infty$ together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (3-i). Let $\mathfrak{p}$ be the quasi-section of π_{i} which maps $s \in \Delta_\infty$ to $\{[\sigma_6, s, k(1 - \sigma_3)/3] \mid k = 0, 1, 2\} \subset \mathscr{S}$. Then $\pi_{\mathrm{i}}^{-1}(0)$ consists of the components isomorphic to those in (3-i). But the points $[0, y]$ of V meet the points $[\infty, y + (1 - \sigma_3)/3]$. $\mathfrak{p}(0)$ is the triple of the points $[\sigma_6, k(1 - \sigma_3)/3]$, $k = 0, 1, 2$, of V.

(4-ii) Over Δ_0.

Let $\mathscr{A} = \mathscr{E}_3 \times E(\sigma_3)/h^Z$, where $h: (p, [y]) \mapsto (\alpha_3 p, [y + (1 - \sigma_3)/3])$, and let $\mathfrak{g}: [p, y] \mapsto [\beta p, \sigma_3 y]$ be the automorphism of $\mathscr{A}$. Then we obtain the family $\pi_{\mathrm{ii}}: \mathscr{S} \to \Delta_0$, together with sections $\mathfrak{o}$ and $\mathfrak{q}$, in the same way as in (3-ii). Let $\mathfrak{p}$ be the quasi-section of π_{ii} which maps $s \in \Delta_0$ to $\{[\sigma_3^k s^{1/3}, s, 0] \mid k = 0, 1, 2\} \subset \mathscr{S}$. Then $\pi_{\mathrm{ii}}^{-1}(0)$ consists of the component V whose normalization is isomorphic to $\boldsymbol{P}^1 \times E(\sigma_3)/\bar{h}^Z$, with $\bar{h}: (\eta, [y]) \mapsto (\sigma_3 \eta, [y + (1 - \sigma_3)/3])$. Thus V is an elliptic surface over $\boldsymbol{P}^1$ with two

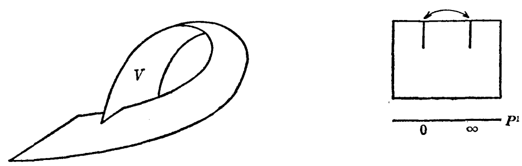


FIGURE 4-ii

triple fibers. The points $[0, y]$ meet the points $[\infty, \sigma_3 y]$. And $\mathfrak{o}(0)$, $\mathfrak{q}(0)$ and $\mathfrak{p}(0)$ are the points $[1, 0]$, $[1, 1/2]$ and a point on the double curve of V, respectively.

(4-iii) Over $\Delta_{1/3}$.

Let $\mathscr{S}$ be the same manifold as in (4-i), and let $\pi_{\text{iii}}: \mathscr{S} \to \Delta_{1/3}$ be the holomorphic map induced by ϕ_3. Let $\mathfrak{o}$, $\mathfrak{q}$ and $\mathfrak{p}$ be the sections and quasi-section of π_{iii} which maps $s \in \Delta_{1/3}$ to $[1, s, \delta]$, $[1, s, 1/2 + \delta] \in \mathscr{S}$ and $\{[\sigma_9 s^\delta, s, k(1 - \sigma_3)/3 + \delta] | k = 0, 1, 2\} \subset \mathscr{S}$, respectively, where $\delta = 1/3$. Then $\mathfrak{o}(0)$, $\mathfrak{q}(0)$ and $\mathfrak{p}(0)$ are the points $[1/3]$, $[1, 5/6]$ of V, and a point on the double curve of V, respectively.

(4-iv) Over $\Delta_{-1/3}$.

If we set $\delta = -1/3$, we obtain the family $\pi_{\text{iv}}: \mathscr{S} \to \Delta_{-1/3}$ together with the sections $\mathfrak{o}$, $\mathfrak{q}$ and the quasi-section $\mathfrak{p}$, in the same way as in (4-iii).

(5-i) Over Δ_∞.

Let $\mathscr{A} = \mathscr{E}_1 \times E(\sigma_4)$, and let $g: (p, [y]) \mapsto (\alpha_4 p, [\sigma_4 y])$ be the automorphism of $\mathscr{A}$. Then $\mathscr{A}/g^Z$ has two singular points isomorphic to N_2, and a singular point isomorphic to N_4. Thus we obtain the family $\pi_{\text{i}}: \mathscr{S} \to \Delta_\infty$. Let $\mathfrak{o}$ and $\mathfrak{q}$ be the sections of π_{i}, which map $s \in \Delta_\infty$ to $[1, s, 0]$ and $[1, s, 1/4] \in \mathscr{S}$, respectively. Then $\pi_{\text{i}}^{-1}(0)$ consists of two components $V_{1,1}$, $V_{2,1}$ isomorphic to Σ_3, two components $V_{1,2}$, $V_{2,2}$ isomorphic to Σ_2, three components $V_{1,3}$, $V_{2,3}$, V_3 isomorphic to $\boldsymbol{P}^2$, and a component V isomorphic to a non-singular model of $\boldsymbol{P}^1 \times E(\sigma_4)/\bar{g}^Z$, where $\bar{g}: (\eta, [y]) \mapsto (\sigma_4 \eta, [\sigma_4 y])$. Thus V is an elliptic surface over $\boldsymbol{P}^1$ with two singular fibers of types III* and III. These components intersect as in

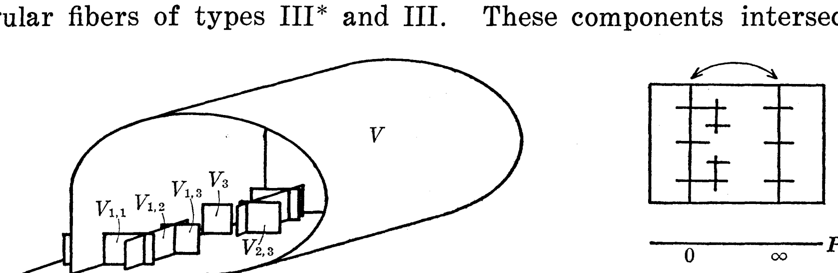


FIGURE 5-i

Figure 5-i. In particular, the points $[0, y]$ of V meet the points $[\infty, \sigma_4 y]$ of itself. And $\mathfrak{o}(0)$ and $\mathfrak{q}(0)$ are the points $[1, 0]$ and $[1, 1/4]$ of V, respectively.

(5-ii) Over Δ_0.

Let $\mathscr{S} = \mathscr{A}/\mathfrak{g}^Z$, where $\mathscr{A} = \mathscr{E}_4 \times E(\sigma_4)$, $\mathfrak{g}$: $(p, [y]) \mapsto (\beta p, [\sigma_4 y])$. Let π_{ii}: $\mathscr{S} \to \Delta_0$ be the holomorphic map induced by ϕ_4, and let $\mathfrak{o}$ and $\mathfrak{q}$ be the sections of π_{ii} defined in the same way as in (5-i). Then $\pi_{\mathrm{ii}}^{-1}(0)$ consists of a component V whose normalization is isomorphic to $\boldsymbol{P}^1 \times E(\sigma_4)$. The points $(0, [y])$ meet the points $(\infty, [\sigma_4 y])$. And $\mathfrak{o}(0)$, $\mathfrak{q}(0)$ are the points $(1, [0])$, $(1, [1/4])$ of V, respectively. (See Figure 3-iii.)

(5-iii) Over $\Delta_{1/2}$.

Let $\mathscr{A} = \mathscr{E}_2 \times E(\sigma_4)$, and let $\mathfrak{g}$: $(p, [y]) \mapsto (\alpha_4 \beta p, [\sigma_4 y])$ be the automorphism of $\mathscr{A}$. Then $\mathscr{A}/\mathfrak{g}^Z$ has four singular points isomorphic to N_2. Thus we obtain the family π_{iii}: $\mathscr{S} \to \Delta_{1/2}$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ defined in the same way as in (5-i). Then $\pi_{\mathrm{iii}}^{-1}(0)$ consists of five components isomorphic to those of $\pi_{\mathrm{i}}^{-1}(0, [\sigma_4])$ in (1-i). But the points $[\infty, y]$ of V meet the points $[0, \sigma_4 y]$ of itself.

(6-i) Over Δ_∞.

Let $\mathscr{A} = \mathscr{E}_2 \times E(\sigma_4)/h^Z$, where h: $(p, [y]) \mapsto (\beta p, [y + (1 + \sigma_4)/2])$. Then we obtain the family π_{i}: $\mathscr{S} \to \Delta_\infty$, together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (5-i). Then $\pi_{\mathrm{i}}^{-1}(0)$ consists of eight components isomorphic to those of $\pi_{\mathrm{i}}^{-1}(0)$ in (5-i). (See Figure 5-i). In particular, the points $[\infty, y]$ of V meet the points $[0, y + (1 + \sigma_4)/2]$ of itself.

(6-ii) Over Δ_0.

Let $\mathscr{A} = \mathscr{E}_4 \times E(\sigma_4)/h^Z$, where h: $(p, [y]) \mapsto (\alpha_2 p, [y + (1 + \sigma_4)/2])$, and let $\mathfrak{g}$ be the automorphism of $\mathscr{A}$ defined in the same way as in (5-ii). Then we obtain the family π_{ii}: $\mathscr{S} \to \Delta_0$, together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (5-ii). Then $\pi_{\mathrm{ii}}^{-1}(0)$ consists of the component V isomorphic to $\boldsymbol{P}^1 \times E(\sigma_4)/\bar{h}^Z$, where $\bar{h}$: $(\eta, [y]) \mapsto (-\eta, [y + (1 + \sigma_4)/2])$, with the points $[0, y]$ meeting the points $[\infty, \sigma_4 y]$. (See Figure 2-ii.)

(6-iii) Over $\Delta_{1/2}$.

Let $\mathscr{A} = \mathscr{E}_4 \times E(\sigma_4)/h^Z$, where h: $(p, [y]) \mapsto (\beta^2 p, [y + (1 + \sigma_4)2])$. Then we obtain the family π_{iii}: $\mathscr{S} \to \Delta_{1/2}$, together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (5-iii). Then $\pi_{\mathrm{iii}}^{-1}(0)$ consists of the five components isomorphic to those of $\pi_{\mathrm{iii}}^{-1}(0)$ in (5-iii). But the points $([0, y + (1 + \sigma_4)/2])$ of V meet the points $[\infty, y]$ of itself $\mathfrak{o}(0)$ and $\mathfrak{q}(0)$ are the points $[1, 1/2]$ and $[1, 3/4]$, respectively. (See Figure 1-i.)

(6-iv) Over $\Delta_{1/4}$.

Let $\mathfrak{g}\colon [p, y] \mapsto [\alpha_4\beta p, \sigma_4 y]$ be the automorphism of $\mathscr{A}$, where $\mathscr{A}$ is the same manifold as in (6-i). Then we obtain the family $\pi_{\mathrm{iv}}\colon \mathscr{S} \to \Delta_{1/4}$ together with sections $\mathfrak{o}$ and $\mathfrak{q}$ in the same way as in (6-i). Then $\pi_{\mathrm{iv}}^{-1}(0)$ is isomorphic to $\pi_{\mathrm{i}}^{-1}(0)$ in (6-i). $\mathfrak{o}(0)$ and $\mathfrak{q}(0)$ are the points $[1, 1/2]$ and $[1, \sigma_4/2]$ of V, respectively.

(7-i) Over Δ_∞.

Let $\mathscr{A} = \mathscr{E}_1 \times E(\sigma_3)$, and let $\mathfrak{g}\colon (p, [y]) \mapsto (\alpha_6 p, [-\sigma_3 y])$ be the automorphism of $\mathscr{A}$. Then $\mathscr{A}/\mathfrak{g}^{\boldsymbol{Z}}$ has three singular points isomorphic to N_6, N_3 and N_2, respectively. Thus we obtain the family $\pi_{\mathrm{i}}\colon \mathscr{S} \to \Delta_\infty$. Let $\mathfrak{o}$ and $\mathfrak{q}$ be the sections of π_{i}, which map $s \in \Delta_\infty$ to $[1, s, 0]$ and $[1, s, 1/3]$, respectively. Then $\pi_{\mathrm{i}}^{-1}(0)$ consists of components $V_{1,1}$, $V_{1,2}$, $V_{1,3}$, $V_{1,4}$, $V_{1,5}$, $V_{2,1}$, $V_{2,2}$, V_3 isomorphic to Σ_5, Σ_4, Σ_3, Σ_2, $\boldsymbol{P}^2$, Σ_2, $\boldsymbol{P}^2$, $\boldsymbol{P}^2$, respectively, and a component V isomorphic to a non-singular model of $\boldsymbol{P}^1 \times E(\sigma_3)/\bar{g}^{\boldsymbol{Z}}$, where $\bar{g}\colon (\eta, [y]) \mapsto (\sigma_6\eta, [-\sigma_3 y])$. Thus V is an elliptic surface over $\boldsymbol{P}^1$ with two singular fibers of types II* and II. These components intersect as in Figure (7-i). In particular, the points $[0, y]$ of V meet the points $[\infty, y]$ of itself. $\mathfrak{o}(0)$ and $\mathfrak{q}(0)$ are the points $[1, 0]$ and $[1, 1/3]$ of V, respectively.

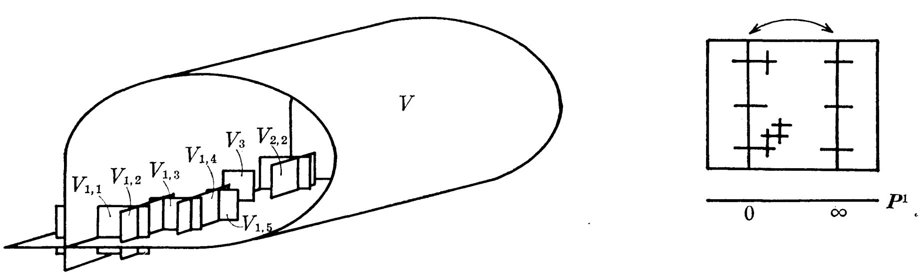


FIGURE 7-i

(7-ii) Over Δ_0.

Let $\mathscr{S} = \mathscr{A}/\mathfrak{g}^{\boldsymbol{Z}}$, where $\mathscr{A} = \mathscr{E}_6 \times E(\sigma_3)$, $\mathfrak{g}\colon (p, [y]) \mapsto (\beta^{-1}p, [-\sigma_3 y])$. Let $\pi_{\mathrm{ii}}\colon \mathscr{S} \to \Delta_0$ be the holomorphic map induced by ϕ_6, and let $\mathfrak{o}$ and $\mathfrak{q}$ be the sections of π_{ii} defined in the same way as in (7-i). Then $\pi_{\mathrm{ii}}^{-1}(0)$ consists of the component V isomorphic to $\boldsymbol{P}^1 \times E(\sigma_3)$. The points $(0, [y])$ meet the points $(\infty, [-\sigma_3 y])$.

(7-iii) Over $\Delta_{1/2}$.

Let $\mathscr{A} = \mathscr{E}_3 \times E(\sigma_3)$, and let $\mathfrak{g}\colon (p, [y]) \mapsto (\alpha_6\beta p, [-\sigma_3 y])$ be the automorphism of $\mathscr{A}$. Then $\mathscr{A}/\mathfrak{g}^{\boldsymbol{Z}}$ has four singular points isomorphic to N_2. Thus we obtain the family $\pi_{\mathrm{iii}}\colon \mathscr{S} \to \Delta_{1/2}$, together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ defined in the same way as in (7-i). Then $\pi_{\mathrm{iii}}^{-1}(0)$ consists of five components isomorphic to those of $\pi_{\mathrm{I}}^{-1}(0, [\sigma_3])$ in (1-i). But the points $[\infty, y]$ of V meet the points $[0, -\sigma_3 y]$ of itself.

(7-iv) Over $\Delta_{1/3}$.

Let $\mathscr{A} = \mathscr{E}_2 \times E(\sigma_3)$, and let $\mathfrak{g}$ be the automorphism of $\mathscr{A}$ defined in the same way as in (7-iii). Then $\mathscr{A}/\mathfrak{g}^Z$ has three singular points isomorphic to N_3. Thus we obtain the family $\pi_{\mathrm{iv}}\colon \mathscr{S} \to \Delta_{1/3}$, together with the sections $\mathfrak{o}$ and $\mathfrak{q}$ defined in the same way as in (7-i). Then $\pi_{\mathrm{iv}}^{-1}(0)$ consists of seven components isomorphic to those of $\pi_{\mathrm{I}}^{-1}(0)$ in (3-i). But the points $[\infty, y]$ of V meet the points $[0, -\sigma_3 y]$ of itself.

References

[1] F. Enriques et F. Severi, Mémoire sur les surfaces hyperelliptique, Acta Math., 32 (1909), 283–392, 33 (1910), 321–403.

[2] R. C. Gunning, Lectures on Riemann Surfaces, Princeton Univ. Press, 1965.

[3] G. Kempf et al., Toridal Embeddings, I, Lecture Notes in Math. 339, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[4] K. Kodaira, On compact analytic surfaces II, Ann. Math., 77 (1963), 563–626.

[5] T. Oda, Lectures on torus embeddings and applications (Based on joint work with K. Miyake), Tata Inst. of Fund. Res., 1978.

[6] D. Mumford, Abelian Varieties, Oxford Univ. Press, 1970.

[7] D. Mumford, A new approach to compactifing locally symmetric varieties, in Discrete subgroups of Lie groups, Oxford Univ. Press, 1975.

[8] D. Mumford, Geometric Invariant Theory, Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[9] D. Mumford and K. Suominen, Introduction to the theory of moduli, in Algebraic Geometry, Oslo, 1970.

[10] U. Persson, On degenerations of algebraic surfaces, Memoire of Amer. Math. Soc., 189 (1977).

[11] T. Suwa, On hyperelliptic surfaces, J. Fac. Sci. Univ. Tokyo Sec. I, 14 (1970), 469–476.

[12] K. Ueno, On fiber spaces of normally polarized abelian varieties of dimensions 2, I, J. Fac. Sci. Univ. Tokyo, Sec. I. A, 17 (1971), 37–95.

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