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ON THE STRUCTURE OF THE IDELE GROUPS OF ALGEBRAIC NUMBER FIELDS, II

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In this paper, we make further and more precise investigation into the idele groups of algebraic number fields than in our previous paper [4].

Here we state, as an example, a theorem obtained in §7 in a somehow weakened and simplified form, which, even so, includes the main result [4, Theorem 2] as a special case:

THEOREM. *Let L be a finite Galois extension of an algebraic number field F, and V an open subgroup of the idele group $L_A^\times$ of L which contains $L^\times \cdot L_{\infty+}^\times$ and satisfies* $(*)$ $V^\sigma = V$ *for any* $\sigma \in \mathrm{Gal}(L/F)$ *and* $(**)$ $L_A^\times = F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)$. *Then* $F_A^\times \cap V = F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)$.

Our basic tool is Terada's theorem on transfers of a finite group, which is generalized in §4.

In the final section, we point out a few results on capitulation of ideals easily derived from what we obtain the previous sections.

1. Preliminaries. For an algebraic number field F, we denote the ring of adeles of F by F_A, and the idele group by $F_A^\times$. Let $F_A^\times = F_f^\times \cdot F_\infty^\times$ be the decomposition of $F_A^\times$ into the product of its non-Archimedian part $F_f^\times$ and its Archimedian part $F_\infty^\times$. The connected component of the unity of $F_\infty^\times$ is denoted by $F_{\infty+}^\times$, and the topological closure of $F^\times \cdot F_{\infty+}^\times$ in $F_A^\times$ by $F^\#$. Let F_{ab} be the maximal abelian extension of F in the algebraic closure of F. The Artin map $[\cdot, F]: F_A^\times \to \mathrm{Gal}(F_{\mathrm{ab}}/F)$ of class field theory is an open, continuous and surjective homomorphism, whose kernel is $F^\#$.

Let K be a finite abelian extension of F, and put $\mathfrak{g} = \mathrm{Gal}(K/F)$. Then $\mathfrak{g}$ acts on $K_A^\times$ naturally. Let $G_{K,F}$ be the Weil group of the extension K over F. This is the extension of the idele class group $K_A^\times/K^\times$ by $\mathfrak{g}$, which corresponds to the canonical class $\xi_{K/F}$ in the cohomology group $H^2(\mathfrak{g}, K_A^\times/K^\times)$. (See Weil [7] and Hochschild and Nakayama [2], or Iyanaga [3, Ch. 5, §6].) There exists a surjective homomorphism $\phi_{K,F}: G_{K,F} \to \mathrm{Gal}(K_{\mathrm{ab}}/F)$ whose kernel is $K^\#/K^\times$ and whose restriction to the subgroup $K_A^\times/K^\times$ coincides with the homomorphism induced by the

Artin map $[\cdot, K]$: $K_A^\times \rightarrow \mathrm{Gal}(K_{\mathrm{ab}}/K)$. Let $V_{K,F}$ be the transfer of $G_{K,F}$ to the abelian subgroup $K_A^\times/K^\times$. Then it may naturally be regarded as a homomorphism of $G_{K,F}/G'_{K,F}$ to $K_A^\times/K^\times$ where $G'_{K,F}$ is the commutator subgroup of $G_{K,F}$. The homomorphism $\phi_{K,F}$ induces a homomorphism

$$\bar{\phi}_{K,F}\colon G_{K,F} \rightarrow \mathrm{Gal}(F_{\mathrm{ab}}/F) = \mathrm{Gal}(K_{\mathrm{ab}}/F)/\mathrm{Gal}(K_{\mathrm{ab}}/F)' .$$

By the properties (A) and (D) of Weil groups, we have a commutative diagram

$$\begin{tikzcd}
K_A^\times \arrow[r] & K_A^\times/K^\times & G_{K,F}/G'_{K,F} \arrow[l, "V_{K,F}"'] \arrow[d, "\bar{\phi}_{K,F}"] \\
F_A^\times \arrow[r] \arrow[u, hook] & F_A^\times/F^\times \arrow[r, "{[\cdot\,,\,F]}"] \arrow[u, hook] \arrow[ur, bend left] & \mathrm{Gal}(F_{\mathrm{ab}}/F).
\end{tikzcd}$$

2. **Our problem.** If an open subgroup U of $K_A^\times$ contains $K^\times \cdot K_{\infty+}^\times$, then it contains the kernel $K^\#$ of the Artin map of K, and determines a finite abelian extension K_U of K. By [4, Theorem 1], we immediately see:

PROPOSITION 1. *The abelian extension K_U of K is a Galois extension of F if and only if*

$$(*) \qquad U^\sigma = U \quad \textit{for any} \quad \sigma \in \mathfrak{g} = \mathrm{Gal}(K/F) .$$

We only consider an open subgroup U of $K_A^\times$ which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies $(*)$ in this paper.

Put $G_U = G_{K,F}/(U/K^\times)$ and $\mathfrak{G}_U = \mathrm{Gal}(K_U/F)$. Then $\phi_{K,F}$ induces an isomorphism $\phi_U\colon G_U \rightarrow \mathfrak{G}_U$. The natural map of $K_A^\times/K^\times$ onto $K_A^\times/U$ induces a homomorphism of $\mathrm{H}^2(\mathfrak{g}, K_A^\times/K^\times)$ to $\mathrm{H}^2(\mathfrak{g}, K_A^\times/U)$. Let ξ_U be the image of the canonical class $\xi_{K/F}$ by the homomorphism. Then G_U is the extension of $K_A^\times/U$ by $\mathfrak{g}$ corresponding to ξ_U. Put $K'_U = K_U \cap F_{\mathrm{ab}}$. Then by class field theory, we see that $F_A^\times/F^\times \cdot N(U)$ is isomorphic to $\mathrm{Gal}(K'_U/F)$ by the Artin map of F. Here $N = N_{K/F}\colon K_A^\times \rightarrow F_A^\times$ is the norm map of K over F.

PROPOSITION 2. *If K is an abelian extension of F, then the commutator subgroup G'_U of G_U is equal to $U \cdot N^{-1}(F^\times)/U$.*

PROOF. Since K is abelian over F, the maximal abelian extension K'_U of F in K_U contains K. Therefore, the abelian extension $K'_U = K_U \cap F_{\mathrm{ab}}$ of K corresponds to the open subgroup $U \cdot N^{-1}(F^\times)$ of $K_A^\times$ by class field theory. Note that $U \cdot N^{-1}(F^\times) = U \cdot N^{-1}(F^\#)$. The isomorphism ϕ_U now establishes the proposition.

The isomorphism $\phi_U\colon G_U \to \mathrm{Gal}(K_U/F)$ induces an isomorphism $\bar{\phi}_U\colon G_U/G'_U \to \mathrm{Gal}(K'_U/F)$. Put $\psi_U = \bar{\phi}_U^{-1} \circ [\cdot, K'_U/F]$. Let V_U be the transfer of G_U to $K_A^\times/U$. This may be considered a homomorphism of G_U/G'_U to $K_A^\times/U$. One can easily see the following by the diagram at the end of the previous section:

PROPOSITION 3. *The following diagram is commutative*:

$$\begin{tikzcd}
K_A^\times \arrow[r] & K_A^\times/U & & G_U/G'_U \arrow[ll, "V_U"'] \\
F_A^\times \arrow[r] \arrow[u, hook, "\iota"] & F_A^\times/F^\times\cdot N(U) \arrow[u, "\iota_U"] \arrow[urr, "\psi_U"] \arrow[rr, "{[\cdot\,,\, K'_U/F]}"'] & & \mathrm{Gal}(K'_U/F). \arrow[u, "\bar{\phi}_U^{-1}"']
\end{tikzcd}$$

PROBLEM. *How large is the kernel of the homomorphism ι_U of the diagram? Does the degree $[K\colon F]$ divide $[F_A^\times \cap U\colon F^\times \cdot N(U)]$?*

Let O_F be the maximal order of F. For a prime divisor $\mathfrak{p}$ of F, let $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of F, and $O_\mathfrak{p}$ the closure of O_F in $F_\mathfrak{p}$. Put $U_F = F^\times \cdot F_\infty^\times \cdot \prod_\mathfrak{p} O_\mathfrak{p}^\times$ where $\prod_\mathfrak{p}$ is the direct product over all the non-Archimedean prime divisors of F. Then $F_A^\times/U_F$ is canonically isomorphic to the absolute ideal class group of F. Define U_K for K in the same way. Suppose that K is an unramified abelian extension of F. Then $F^\times \cdot N(U_K) = U_F$. Furthermore, the subgroup $(F_A^\times \cap U_K)/U_F$ of $F_A^\times/U_F$ is canonically isomorphic to the subgroup of the absolute ideal class group of F consisting of the classes whose ideals become principal in K. Therefore if, moreover, K is a cyclic extension of F, then $[K\colon F]$ certainly divides $[F_A^\times \cap U_K : F^\times \cdot N(U_K)]$, which is just Hilbert's Theorem 94. Adachi questioned in [1] if this would be true for any unramified abelian extension K of F.

3. The subgroup $X_{K/F}(U)$ of $\mathrm{Hom}(F_A^\times/F^\times \cdot N(U),\ K_A^\times/U \cdot N^{-1}(F^\times))$. Put $H_U = K_A^\times/U$. This is a normal abelian subgroup of G_U, and is naturally regarded as a G_U-module. By Proposition 2, we have $K_A^\times/U \cdot N^{-1}(F^\times) = H_U/G'_U$. Note that the norm map N gives an isomorphism of this group onto the subgroup $F^\times \cdot N(K_A^\times)/F^\times \cdot N(U)$ of $F_A^\times/F^\times \cdot N(U)$.

PROPOSITION 4. *There is a canonical isomorphism of the abelian group* $\mathrm{Hom}(F_A^\times/F^\times \cdot N(U),\ K_A^\times/U \cdot N^{-1}(F^\times))$ *onto the cohomology group* $\mathrm{H}^1(G_U, H_U/G'_U)$.

PROOF. Obviously G_U acts on H_U/G'_U trivially. Therefore we have

$\mathrm{H}^1(G_U, H_U/G'_U) = \mathrm{Hom}\,(G_U, H_U/G'_U)$. Since H_U/G'_U is abelian, we may identify these groups with $\mathrm{Hom}\,(G_U/G'_U, H_U/G'_U)$. The isomorphism ψ_U of Proposition 3 now establishes the proposition.

Composing the isomorphism of Proposition 4 with the natural homomorphism of $\mathrm{H}^1(G_U, H_U)$ to $\mathrm{H}^1(G_U, H_U/G'_U)$, we have

$$\pi_U\colon \mathrm{H}^1(G_U, H_U) \to \mathrm{Hom}\,(F_A^\times/F^\times \cdot N(U), K_A^\times/U \cdot N^{-1}(F^\times)) .$$

We put

$$X_{K/F}(U) = \pi_U(\mathrm{H}^1(G_U, H_U)) .$$

For $f \in X_{K/F}(U)$, put

$$d(f) = |\mathrm{Coker}\,(f)| = [K_A^\times/U \cdot N^{-1}(F^\times) \colon \mathrm{Im}\,(f)] .$$

THEOREM 1. *Let K be a finite abelian extension of an algebraic number field F. Let U be an open subgroup of $K_A^\times$ which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies* $(*)$, *and $X_{K/F}(U)$ as above. Then for $f \in X_{K/F}(U)$,*

$$\{a^{d(f)} \mid a \in F_A^\times,\ a \bmod (F^\times \cdot N(U)) \in \mathrm{Ker}\,(f)\} \subset F_A^\times \cap U .$$

In this section, we reduce the theorem to Proposition 6 in §5.

Put $G = G_U$ and $H = H_U = K_A^\times/U$. Let $V_{G\to H}\colon G \to H$ be the transfer of G to H. Then it induces a homomorphism $\bar{V}\colon G/G' \to H$ where G' is the commutator subgroup of G. By Proposition 3, we have a commutative diagram

$$\begin{tikzcd}
K_A^\times \arrow[r] & H = K_A^\times/U & & G/G' \arrow[ll, "\bar{V}"'] \\
F_A^\times \arrow[u, hook, "\iota"] \arrow[r] & F_A^\times/F^\times \cdot N(U) . \arrow[u, "\iota_U"] \arrow[urr, bend right, "\psi = \psi_U"'] & &
\end{tikzcd}$$

Therefore, an element x of $F_A^\times$ belongs to U if and only if $\psi(x \bmod (F^\times \cdot N(U))) \in \mathrm{Ker}\,(\bar{V})$.

Let π be the homomorphism of $\mathrm{H}^1(G, H)$ to $\mathrm{Hom}\,(G/G', H/G') \cong \mathrm{H}^1(G, H/G')$. Then

$$X_{K/F}(U) = \{\tilde{f} \circ \psi \mid \tilde{f} \in \mathrm{Im}\,(\pi)\} .$$

Take $f = \tilde{f} \circ \psi \in X_{K/F}(U)$. For $a \in F_A^\times$, put

$$z = \psi(a \bmod (F^\times \cdot N(U))) .$$

Then it is sufficient to show that $\tilde{f}(z) = 1 \Rightarrow z^{d(f)} \in \mathrm{Ker}\,(\bar{V})$. We have $d(f) = |\mathrm{Coker}\,(f)| = |\mathrm{Coker}\,(\tilde{f})| = [H/G' \colon \mathrm{Im}\,(\tilde{f})]$ since ψ is an isomorphism. Theorem 1 now follows from Proposition 6 in §5 immediately.

By the corollary to Proposition 6, we see at once:

COROLLARY. *Let the notation and the assumptions be as in the theorem. If there exists an* f *in* $X_{K/F}(U)$ *with* $d(f) = 1$, *then the degree* $[K: F]$ *divides the index* $[F_A^\times \cap U : F^\times \cdot N(U)]$.

4. **A generalization of Terada's theorem.** Let G be a finite group, and G' the commutator subgroup. Let $\mathrm{End}\,(G)$ be the set of all the endomorphisms of G. For $\phi \in \mathrm{End}\,(G)$, put

$$H(\phi) = \langle g^{-1} \cdot \phi(g) \,|\, g \in G \rangle \cdot G' .$$

This is the subgroup of G generated by the elements of the form $g^{-1} \cdot \phi(g)$ with $g \in G$, and by the commutators of G. Denote the transfer of G to $H(\phi)$ by V_ϕ. (If ϕ is the trivial endomorphism, then $H(\phi) = G$. In this case, $V_\phi: G \to G/G'$ is the natural projection.) We generalize Terada's theorem [5] as follows:

PROPOSITION 5. *Let* ϕ *be an element of* $\mathrm{End}\,(G)$. *Then*

$$\{g \in G \,|\, g^{-1} \cdot \phi(g) \in G'\} \subset \mathrm{Ker}\,(V_\phi) .$$

PROOF. We may assume that $H(\phi)$ is abelian. In fact: The commutator subgroup $H(\phi)'$ of $H(\phi)$ is normal in G. Put $\bar{G} = G/H(\phi)'$. It is obvious that ϕ induces an endomorphism $\bar{\phi}$ of $\bar{G}$. Then $H(\bar{\phi}) = \langle \bar{g}^{-1} \cdot \bar{\phi}(\bar{g}) \,|\, \bar{g} \in \bar{G} \rangle \cdot \bar{G}'$ is equal to $H(\phi)/H(\phi)'$. As for V_ϕ, it is a homomorphism of G to the abelian group $H(\phi)/H(\phi)'$. Since $H(\phi)' \subset G' \subset \mathrm{Ker}\,(V_\phi)$, V_ϕ induces a homomorphism of $\bar{G}$ to $H(\bar{\phi})$, which coincides with the transfer $V_{\bar{\phi}}$ of $\bar{G}$ to $H(\bar{\phi})$ as is easily seen. Obviously,

$$\{g \in G \,|\, g^{-1} \cdot \phi(g) \in G'\}/H(\phi)' = \{\bar{g} \in \bar{G} \,|\, \bar{g}^{-1} \cdot \bar{\phi}(\bar{g}) \in \bar{G}'\} .$$

Thus we may replace G and $\phi \in \mathrm{End}\,(G)$ by $\bar{G}$ and $\bar{\phi} \in \mathrm{End}\,(\bar{G})$ to show the theorem. We now assume that $H(\phi)$ is abelian. (Then G has to be metabelian.)

Terada [5] showed the theorem in the case that ϕ is an *automorphism* of G. A clear and fairly simple proof is obtained by Terada [6] (with the assistance of Adachi). It should be noted that Terada's setting in [6] might seem rather special. But the proof of Reduction 1 of [6] is applicable to show our Proposition 5 for an automorphism ϕ of G, putting $\mathfrak{G} = G \cdot \langle \phi \rangle$, the semi-direct product of G and the cyclic subgroup $\langle \phi \rangle$ of $\mathrm{Aut}\,(G)$.

Now we reduce the case of $\phi \in \mathrm{End}\,(G)$ to the case of an automorphism of a certain subgroup of G. Put $H = H(\phi)$. Since $\mathrm{Ker}\,(\phi)$ is contained in H, we have $[G : H] = [\phi(G): \phi(H)]$. Put $G_1 = \phi(G)$, $H_1 = \phi(H)$

and $\phi_1 = \phi|_{G_1}$. Then ϕ_1 is an endomorphism of G_1. Since $G_1' = \phi(G')$, we have

$$H_1 = H(\phi_1) = \langle g^{-1}\cdot\phi_1(g) \mid g \in G_1\rangle \cdot G_1' .$$

Put $d = [G : H]$, and let $R = \{x_1, \cdots, x_d\}$ be a set of representatives of the cosets of G/H. Since $\phi(g) = g\cdot(g^{-1}\cdot\phi(g)) \equiv g \bmod (H)$, $\phi(R) = \{\phi(x_1), \cdots, \phi(x_d)\}$ is a set of representatives not only of G_1/H_1 but also of G/H. Let V and V_1 be the transfers of G to H and of G_1 to H_1, respectively. Then we have

$$V(\phi(g)) = V_1(\phi(g)) \qquad \text{for} \quad g \in G$$

at once if we express them according to the definition using the set of representatives $\phi(R)$.

Let g be an element of G such that $g^{-1}\cdot\phi(g)$ belongs to G'. Then $V(g) = V(\phi(g))$ because G' is contained in $\mathrm{Ker}(V)$. Therefore, we have $V(g) = V_1(\phi(g))$. Put $g_1 = \phi(g)$. Then we have $V(g) = V_1(g_1)$ and $g_1^{-1}\cdot\phi_1(g_1) \in G_1'$. Define G_n, ϕ_n, H_n and g_n for $n \geqq 2$ inductively by

$$G_n = \phi_{n-1}(G_{n-1}) , \quad \phi_n = \phi_{n-1}|_{G_n} , \quad H_n = \phi_{n-1}(H_{n-1}) \quad \text{and} \quad g_n = \phi_{n-1}(g_{n-1}) .$$

Then $\phi_n \in \mathrm{End}(G_n)$, $G_n' = \phi_{n-1}(G_{n-1}')$, $H_n = \langle g^{-1}\cdot\phi_n(g) \mid g \in G_n\rangle\cdot G_n'$ and $g_n^{-1}\cdot\phi_n(g_n) \in G_n'$. Let V_n be the transfer of G_n to H_n. Then we also have $V_{n-1}(g_{n-1}) = V_n(g_n)$. Since G is finite, the series $G \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ become stable. That is, there exists an integer m such that $G_m = G_{m+1} = \phi_m(G_m)$. Then ϕ_m is an automorphism of G_m. Therefore, by Terada's theorem, we have $V_m(g_m) = 1$. Since $V(g) = V_1(g_1) = \cdots = V_m(g_m)$, the proof of Proposition 5 is now completed.

COROLLARY. *For $\phi \in \mathrm{End}(G)$, the index $[G : H(\phi)]$ divides the index* $[\mathrm{Ker}(V_{G\to H(\phi)}) : G']$.

PROOF. Let $f: G \to G/G'$ be the map defined by $f(g) = g^{-1}\cdot\phi(g) \bmod (G')$. Then this is a homomorphism. Since $\mathrm{Im}(f) = H(\phi)/G'$, we have $[H(\phi): G'] = [G : \mathrm{Ker}(f)]$. Therefore, $[G: H(\phi)] = [G: G']\cdot[H(\phi): G']^{-1} = [G: G']\cdot[G : \mathrm{Ker}(f)]^{-1} = [\mathrm{Ker}(f): G']$. This divides $[\mathrm{Ker}(V_{G\to H(\phi)}): G']$ because $\mathrm{Ker}(f)$ is a subgroup of $\mathrm{Ker}(V_{G\to H(\phi)})$. q.e.d.

5. Cohomological interpretation. Let G be a finite metabelian group, and H an abelian subgroup containing the commutator subgroup G' of G. Since H is a normal subgroup, G acts on H through the inner automorphisms of G. The action of G induced on H/G' is trivial. Therefore

$$\mathrm{H}^1(G, H/G') = \mathrm{Hom}(G, H/G') = \mathrm{Hom}(G/G', H/G') .$$

Let π be the homomorphism

$$\pi\colon \mathrm{H}^1(G, H) \to \mathrm{H}^1(G, H/G') = \mathrm{Hom}\,(G/G', H/G')$$

induced by the natural map of H onto H/G'. For $f \in \mathrm{Im}\,(\pi)$, put

$$d(f) = |\mathrm{Coker}\,(f)| = [H/G'\colon \mathrm{Im}\,(f)]\ .$$

PROPOSITION 6. *Let G be a finite metabelian group, and H an abelian subgroup containing G'. Let $V_{G\to H}$ be the transfer of G to H. Then for $f \in \mathrm{Im}\,(\pi)$,*

$$\{x^{d(f)} \mid x \in G,\ f(x \bmod (G')) = 1\} \subset \mathrm{Ker}\,(V_{G\to H})\ .$$

PROOF. Let ζ be a cocycle in $\mathrm{Z}^1(G, H)$. Composing ζ with the natural projection of H to H/G', we have an element f of $\mathrm{Hom}\,(G, H/G')$, which is regarded as an element of $\mathrm{Hom}\,(G/G', H/G')$. Put $\phi(x) = x \cdot \zeta(x)$ for $x \in G$. Then this is an endomorphism of G. It is obvious that $H \supset H(\phi)$, and that $\mathrm{Im}\,(f) = H(\phi)/G'$. Therefore, we have $d(f) = [H\colon H(\phi)]$. Let x be an element of G such that

$$f(x \bmod (G')) = \zeta(x) \bmod (G') = 1\ .$$

Then $x^{-1} \cdot \phi(x)$ belongs to G'. Therefore by Proposition 5, we have $V_{G\to H(\phi)}(x) = 1$. As is well known,

$$V_{G\to H(\phi)}(x) = V_{H\to H(\phi)}(V_{G\to H}(x))\ .$$

Because H is abelian, we have

$$V_{H\to H(\phi)}(V_{G\to H}(x)) = V_{G\to H}(x)^{[H:H(\phi)]}\ .$$

Therefore

$$V_{G\to H}(x^{d(f)}) = V_{G\to H}(x^{[H:H(\phi)]}) = V_{G\to H(\phi)}(x) = 1\ .$$

The proposition is proved.

COROLLARY. *Let the notation and the assumptions be as in Proposition 6. If there exists an f in $\mathrm{Im}\,(\pi)$ such that $d(f) = 1$, then $[G\colon H]$ divides $[\mathrm{Ker}\,(V_{G\to H}) : G']$.*

PROOF. Suppose that $d(f) = 1$ for $f \in \mathrm{Im}\,(\pi)$. Corresponding to f, take a cocycle $\zeta \in \mathrm{Z}^1(G, H)$, and define $\phi \in \mathrm{End}\,(G)$ as in the above proof. Then $H = H(\phi)$ because $d(f) = 1$. By the corollary to Proposition 5, we have the desired result.

6. The subgroup $X^0_{K/F}(U)$ of $X_{K/F}(U)$. As was in §3, let K be a finite abelian extension of F, $\mathfrak{g} = \mathrm{Gal}\,(K/F)$ and U an open subgroup of $K_A^\times$ which contains $K^\times \cdot K^\times_{\infty+}$ and satisfies the condition $(*)$ of Proposition 1.

Let $H_U^{\mathfrak{g}} = (K_A^\times/U)^{\mathfrak{g}}$ be the subgroup of H_U consisting of the elements fixed by $\mathfrak{g}$, i.e.,

$$H_U^{\mathfrak{g}} = \{x \in H_U \mid x^\sigma = x \text{ for } {}^\forall\sigma \in \mathfrak{g}\} .$$

Composing with the natural map of $K_A^\times/U$ onto $K_A^\times/U\cdot N^{-1}(F^\times)$, we have a homomorphism,

$$\mathrm{Hom}\,(F_A^\times/F^\times\cdot N(U),\, (K_A^\times/U)^{\mathfrak{g}}) \to \mathrm{Hom}\,(F_A^\times/F^\times\cdot N(U),\, K_A^\times/U\cdot N^{-1}(F^\times)) .$$

Denote the image of this homomorphism by $X_{K/F}^0(U)$. This is a subgroup of $X_{K/F}(U)$ defined in §3. In fact: $\mathrm{H}^1(G_U, H_U^{\mathfrak{g}})$ is equal to $\mathrm{Hom}\,(G_U, H_U^{\mathfrak{g}}) = \mathrm{Hom}\,(G_U/G_U', H_U^{\mathfrak{g}})$, since G_U acts on $H_U^{\mathfrak{g}}$ trivially. Therefore, the isomorphism $\psi_U\colon F_A^\times/F^\times\cdot N(U) \to G_U/G_U'$ induces an isomorphism of $\mathrm{H}^1(G_U, H_U^{\mathfrak{g}})$ onto $\mathrm{Hom}\,(F_A^\times/F^\times\cdot N(U), H_U^{\mathfrak{g}})$. If we naturally map $\mathrm{H}^1(G_U, H_U^{\mathfrak{g}})$ onto a subgroup of $\mathrm{H}^1(G_U, H_U)$, then we see at once that it is mapped by π_U onto $X_{K/F}^0(U)$, which is, therefore, certainly a subgroup of $X_{K/F}(U)$.

7. The case of ι_U. In this section, we investigate the case of the homomorphism

$$\iota_U : F_A^\times/F^\times\cdot N(U) \to K_A^\times/U ,$$

which is induced by the inclusion map $\iota\colon F_A^\times \hookrightarrow K_A^\times$. Obviously, ι_U belongs to $\mathrm{Hom}\,(F_A^\times/F^\times\cdot N(U), (K_A^\times/U)^{\mathfrak{g}})$. Applying Theorem 1 to the image of ι_U in $X_{K/F}^0(U)$, we have

THEOREM 2. *Let K be a finite abelian extension of F. Let U be an open subgroup of $K_A^\times$ which contains $K^\times\cdot K_{\infty+}^\times$ and satisfies* (*). *Put $d(U) = [K_A^\times : F_A^\times\cdot U\cdot N^{-1}(F^\times)]$. Then*

$$\{a^{d(U)} \mid a \in F_A^\times \cap (U\cdot N^{-1}(F^\times))\} \subset F_A^\times \cap U .$$

We extend this theorem to the case of a Galois extension of F. Let L be a finite Galois extension of F, and K the maximal abelian extension of F in L. We specify the norm maps of the extensions L/F, L/K and K/F as $N_{L/F}$, $N_{L/K}$ and $N_{K/F}$, respectively. Because K is the maximal abelian extension of F in L, we have

$$F^\times\cdot N_{L/F}(L_A^\times) = F^\times\cdot N_{K/F}(K_A^\times) .$$

Therefore

$$K_A^\times = N_{L/K}(L_A^\times)\cdot N_{K/F}^{-1}(F^\times) . \tag{1}$$

Let V be an open subgroup of $L_A^\times$ which contains $L^\times\cdot L_{\infty+}^\times$ and satisfies

$$V^\sigma = V \quad \text{for any} \quad \sigma \in \mathrm{Gal}\,(L/F) . \tag{*}$$

Put $U = K^\times \cdot N_{L/K}(V)$. Then this is an open subgroup of $K_A^\times$ which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies $(*)$ for U. Put

$$d = [L : K] \,,$$
$$d(V) = [L_A^\times : F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)] \,,$$
$$d(U) = [K_A^\times : F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times)] \,.$$

Note that $U \subset V$ and $N_{K/F}^{-1}(F^\times) \subset N_{L/F}^{-1}(F^\times)$. The norm map $N_{L/K}$ induces an injective homomorphism of $L_A^\times / N_{L/F}^{-1}(F^\times)$ to $K_A^\times / N_{K/F}^{-1}(F^\times)$. Therefore, we have

$$d(V) = [N_{L/K}(L_A^\times) \cdot N_{K/F}^{-1}(F^\times) : F_A^{\times d} \cdot U \cdot N_{K/F}^{-1}(F^\times)] \,.$$

Here $F_A^{\times d} = \{a^d \mid a \in F_A^\times\} = N_{L/K}(F_A^\times)$. Then by (1), we have

$$d(V) = [K_A^\times \colon F_A^{\times d} \cdot U \cdot N_{K/F}^{-1}(F^\times)] \,.$$

Put $W = U \cdot N_{K/F}^{-1}(F^\times)$ for the simplicity, and let $e(V)$ be the exponent of the finite abelian group $F_A^\times \cdot W / F_A^{\times d} \cdot W$. Then $e(V)$ divides d. Put $d = m \cdot e(V)$. Since $e(V)$ divides $[F_A^\times \cdot W : F_A^{\times d} \cdot W]$, we see that $d(U) \cdot e(V)$ divides $d(V)$. By the choice of $e(V)$, we have

$$F_A^{\times e(V)} \subset F_A^{\times d} \cdot W = (F_A^{\times e(V)})^m \cdot W \,.$$

Therefore, as is easily seen, $[F_A^{\times e(V)} \colon F_A^{\times e(V)} \cap W]$ is relatively prime to m.

If a is an element of $F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)$, then $(a^{e(V)})^m = a^d = N_{L/K}(a)$ is an element of W. Therefore $a^{e(V)}$ belongs to W since m is relatively prime to $[F_A^{\times e(V)} \colon F_A^{\times e(V)} \cap W]$. By Theorem 2, then, we see that $(a^{e(V)})^{d(U)}$ belongs to U. We have shown the following generalization of Theorem 2.

THEOREM 3. *Let L be a finite Galois extension of F. Let V be an open subgroup of $L_A^\times$ which contains $L^\times \cdot L_{\infty+}^\times$ and satisfies $(*)$, and put $d(V) = [L_A^\times \colon F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)]$. Then we have*

$$\{a^{d(V)} \mid a \in F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)\} \subset F_A^\times \cap V \,.$$

More precisely, let K be the maximal abelian extension of F in L, and put $U = K^\times \cdot N_{L/K}(V)$ and $d(U) = [K_A^\times \colon F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times)]$. Let $e(V)$ be the exponent of

$$F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times) / F_A^{\times [L:K]} \cdot U \cdot N_{K/F}^{-1}(F^\times) \,.$$

Then $d(U) \cdot e(V)$ divides $d(V)$, and

$$\{a^{d(U) e(V)} \mid a \in F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)\} \subset F_A^\times \cap U \,.$$

Moreover, $[L : K] \cdot e(V)^{-1}$ is relatively prime to the index

$$[F_A^{\times e(V)} \colon F_A^{\times e(V)} \cap U \cdot N_{K/F}^{-1}(F^\times)] \,.$$

The case of $d(V)=1$ of Theorem 3 is worth being pointed out, by which [4, Theorem 2] is obtained at once as a special case.

THEOREM 4. *Let L be a finite Galois extension of F, and K the maximal abelian extension of F in L. Let V be an open subgroup of $L_A^\times$ which contains $L^\times \cdot L_{\infty+}^\times$, and suppose that V satisfies $(*)$ above and the following $(**)$:*

$$(**) \qquad L_A^\times = F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)\ .$$

Then the open subgroup $U = K^\times \cdot N_{L/K}(V)$ of $K_A^\times$ satisfies

$$K_A^\times = F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times)\ .$$

Furthermore, we have

$$F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times) = F_A^\times \cap U \cdot N_{K/F}^{-1}(F^\times) = F_A^\times \cap V = F_A^\times \cap U\ .$$

The degree $[L:K]$ is relatively prime to the index $[F_A^\times : F_A^\times \cap U]$. The degree $[K:F]$ divides the indices

$$[F_A^\times \cap V : F^\times \cdot N_{L/F}(V)] = [F_A^\times \cap U : F^\times \cdot N_{K/F}(U)]\ .$$

One can easily see the theorem by Theorem 3 except the last assertion, which is also easily seen by Corollary to Theorem 1.

8. **On capitulation of the ideals.** In this final section, we point out some consequences of our results obtained above, on capitulation of the ideals of F.

Let K be an unramified abelian extension of F. Let C_F be the absolute ideal class group of F, and $H_F(K)$ the subgroup of C_F corresponding to K, which consists of the classes containing norms of the ideals of K. Let $P_F(K)$ be the subgroup of C_F consisting of the classes whose ideals become principal in K. If we take the open subgroups U_F of $F_A^\times$ and U_K of $K_A^\times$ defined in the last paragraph of §2, then we may canonically identify C_F with $F_A^\times/U_F$, $H_F(K)$ with $F^\times \cdot N(K_A^\times)/U_F$ and $P_F(K)$ with $(F_A^\times \cap U_K)/U_F$.

Let $\tilde{K}$ be the absolute class field of K. This is the abelian extension K_{U_K} of K corresponding to U_K. Therefore, G_{U_K} is isomorphic to $\mathfrak{G} = \mathrm{Gal}(\tilde{K}/F)$. The maximal abelian extension K'_{U_K} of F in K_{U_K} is the absolute class field $\tilde{F}$ of F. Since K is unramified over F, we have

$$F^\times \cdot N(U_K) = F^\times \cdot N(U_K \cdot N^{-1}(F^\times)) = U_F\ .$$

Identifying $F_A^\times/F^\times \cdot N(U_K)$ with C_F, we see that the norm map N maps $K_A^\times/U_K \cdot N^{-1}(F^\times)$ isomorphically onto $H_K(K)$. The homomorphism π_{U_K} in §3 induces the homomorphism

$$\pi_{K/F} : \mathrm{H}^1(\mathrm{Gal}(\tilde{K}/F),\ C_K) \to \mathrm{Hom}(C_F,\ H_F(K))$$

where C_K is the absolute ideal class group of K. We use

$$\widetilde{X}_{K/F} = \pi_{K/F}(\mathrm{H}^1(\mathrm{Gal}\,(\widetilde{K}/F),\, C_K))$$

in place of $X_{K/F}(U_K)$, which are naturally isomorphic. For $f \in \widetilde{X}_{K/F}$, put

$$d(f) = |\mathrm{Coker}\,(f)| = [H_F(K)\colon \mathrm{Im}\,(f)]\ .$$

By Theorem 1 and its corollary, we have:

THEOREM 5. *Let K be an unramified abelian extension of F, and the notation as above. Then for any $f \in \widetilde{X}_{K/F}$,*

$$\{x^{d(f)} \mid x \in \mathrm{Ker}\,(f)\} \subset P_F(K)\ .$$

If there exists an f in $\widetilde{X}_{K/F}$ such that $d(f) = 1$, then the degree $[K:F]$ divides $|P_F(K)|$.

Put $n = [K\colon F]$, and $C_F(n) = \{x \in C_F \mid x^n = 1\}$. Then n divides $|C_F(n)|$. Put $m = |C_F(n)|/n$, and

$$C_F(n)^m = \{x^m \mid x \in C_F(n)\}\ .$$

THEOREM 6. *The notation and the assumption being as above,*

$$C_F(n)^m \subset P_F(K) \subset C_F(n)\ .$$

PROOF. We use Theorem 2 for U_K. Put $C_F^n = \{x^n \mid x \in C_F\}$. By the norm map N, $K_A^\times/U_K \cdot N^{-1}(F^\times)$ is isomorphic to $H_F(K)$, and the subgroup $F_A^\times \cdot U_K \cdot N^{-1}(F^\times)/U_K \cdot N^{-1}(F^\times)$ is isomorphically mapped onto C_F^n. Therefore $d(U_K) = [H_F(K)\colon C_F^n]$. The endomorphism $x \mapsto x^n$ of C_F gives an exact sequence

$$1 \to C_F(n) \to C_F \to C_F^n \to 1\ .$$

Therefore $|C_F(n)| = [C_F\colon C_F^n]$. Since $n = [K:F] = [C_F : H_F(K)]$, we have $m = d(U_K)$. It is obvious that, for $a \in F_A^\times$, a belongs to $F_A^\times \cap U_K \cdot N^{-1}(F^\times)$ if and only if a^n belongs to U_F. Therefore, we may canonically identify $F_A^\times \cap U_K \cdot N^{-1}(F^\times)/U_F$ with $C_F(n)$, and $\{a^{d(U_K)} \mid a \in F_A^\times \cap (U_K \cdot N^{-1}(F^\times))\}/U_F$ with $C_F(n)^m$. The theorem now follows from Theorem 2 at once.

REMARK. If K is the absolute class field $\widetilde{F}$ of F, then $n = [K:F] = C_F$, $C_F(n) = C_F$ and $m = 1$. Therefore, Theorem 6 becomes the principal ideal theorem in this case.

Instead of the cohomological formulation as in Theorem 5, we can apply Proposition 5 directly as follows:

Let $\widetilde{\widetilde{F}}$ be the second class field of F, that is, the absolute class field of $\widetilde{F}$. Put $\mathfrak{G} = \mathrm{Gal}\,(\widetilde{\widetilde{F}}/F)$. For $\phi \in \mathrm{End}\,(\mathfrak{G})$, put

$$\mathfrak{H}(\phi) = \langle g^{-1} \cdot \phi(g) \mid g \in \mathfrak{G} \rangle \cdot \mathfrak{G}' \text{ ,}$$

and let $K(\phi)$ be the subfield of $\tilde{\tilde{F}}$ corresponding to $\mathfrak{H}(\phi)$. Then $K(\phi)$ is contained in $\tilde{F}$. Denote the endomorphism of $\mathrm{Gal}(\tilde{F}/F) = \mathfrak{G}/\mathfrak{G}'$ induced by ϕ by $\bar{\phi}$. Then for $g \in \mathfrak{G}$, we have $g^{-1} \cdot \phi(g) \in \mathfrak{G}'$ if and only if $\bar{\phi}(g \bmod (\mathfrak{G}')) = g \bmod (\mathfrak{G}')$. Let $[\cdot, \tilde{F}/F]$ be the Artin map.

THEOREM 7. *Let* $\mathfrak{a}$ *be an ideal of* F. *If* $\bar{\phi}([\mathfrak{a}, \tilde{F}/F]) = [\mathfrak{a}, \tilde{F}/F]$ *for* $\phi \in \mathrm{End}(\mathfrak{G})$, *then* $\mathfrak{a}$ *becomes principal in* $K(\phi)$.

REMARK. Let α be an automorphism of the field F, and k the subfield of F fixed by α. Take an automorphism τ of $\tilde{\tilde{F}}$ such that $\tau|_F = \alpha$. Then τ induces an automorphism ϕ of $\mathfrak{G} = \mathrm{Gal}(\tilde{\tilde{F}}/F)$ through the inner automorphism of $\mathrm{Gal}(\tilde{\tilde{F}}/k)$. Since the inner automorphisms of $\mathfrak{G}$ act on $\mathrm{Gal}(\tilde{F}/F) = \mathfrak{G}/\mathfrak{G}'$ trivially, the effect of $\bar{\phi} \in \mathrm{End}(\mathrm{Gal}(\tilde{F}/F))$ is independent of the choice of τ, and determined by α. It is easy to see that $\mathfrak{H}(\phi)$ is the commutator subgroup of $\mathrm{Gal}(\tilde{\tilde{F}}/k)$. Therefore, $K(\phi)$ is the maximal abelian extension of k unramified over F. In other words, $K(\phi)$ is the genus field of the cyclic extension F over k. By [4, Theorem 1], one can now easily see that Theorem 7 gives Terada's principal ideal theorem in the genus field of [6] in this case.

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