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A GEOMETRIC CHARACTERIZATION OF A SIMPLE $K3$-SINGULARITY

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Abstract. A simple $K3$-singularity is a three-dimensional normal isolated singularity with a certain condition on the mixed Hodge structure on a good resolution. We prove here that a three-dimensional normal isolated singularity is a simple $K3$-singularity if and only if the exceptional divisor of a $\boldsymbol{Q}$-factorial terminal modification is an irreducible normal $K3$-surface.

A simple $K3$-singularity is defined in terms of the Hodge structure as a three-dimensional analogue of a simple elliptic singularity. It is well known that a simple elliptic singularity is characterized by the geometric structure of the minimal resolution (cf. [S], [I1] and [W1]). The aim of this paper is to prove that a simple $K3$-singularity is also characterized by the geometric structure of a $\boldsymbol{Q}$-factorial terminal modification which is a three-dimensional analogue of the minimal resolution (cf. [M]). This characterization should help investigations of a simple $K3$-singularity which are being carried out from various viewpoints (cf. [T], [W2], [W3] and [Y]).

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Let $f\colon \tilde{X}\to X$ be a good resolution of a normal isolated singularity (X, x), where a resolution is called a good resolution if $E=f^{-1}(x)_{\mathrm{red}}$ is a divisor with normal crossings. We decompose E into irreducible components E_i $(i=1, 2, \dots, s)$. If (X, x) is a Gorenstein singularity, then we have a presentation of canonical divisors

$$K_{\tilde{X}}=f^*K_X+\sum_{i\in I} m_iE_i-\sum_{j\in J} m_jE_j\,,$$

where $m_i\geq 0$ for any $i\in I$ and $m_j>0$ for any $j\in J$.

DEFINITION 1 (cf. [I1]). In the previous situation, the divisor $\sum_{j\in J} m_jE_j$ is called the essential divisor and denoted by E_J.

PROPOSITION 2 (cf. [I1]). *A Gorenstein isolated singularity (X, x) is purely elliptic if and only if the essential divisor E_J is a non-zero reduced divisor (i.e. $J\neq\emptyset$ and $m_j=1$ for every $j\in J$) for any good resolution f.*

A purely elliptic singularity is defined in terms of the plurigenera $\{\delta_m\}$ in [W1].

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However we do not need the concept of the plurigenera in this paper. So the reader may consider the above proposition as the definition of a Gorenstein purely elliptic singularity.

For a Gorenstein purely elliptic singularity (X, x), we define the type of the singularity according to the Hodge structure of E_J. Since E_J is a complete variety with normal crossings,

$$H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq \mathrm{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=1}^{n-1} H_{n-1}^{0,i}(E_J),$$

where $n = \dim(X, x)$ and $H_m^{p,q}(*)$ means the (p, q)-component of $\mathrm{Gr}_{p+q}^W H^m(*)$. Since (X, x) is purely elliptic singularity, we have $H^{n-1}(E_J, \mathcal{O}_{E_J}) = H^{n-1}(E, \mathcal{O}_E) = \boldsymbol{C}$ (cf. [I1, 3.7]). Therefore $H^{n-1}(E_J, \mathcal{O}_{E_J})$ must coincide with one of $H_{n-1}^{0,i}(E_J)$.

DEFINITION 3 (cf. [I1]). For an integer i $(0 \leq i \leq n-1)$, a purely elliptic singularity (X, x) is of type $(0, i)$ if $H^{n-1}(E_J, \mathcal{O}_{E_J})$ consists of the $(0, i)$-Hodge component.

EXAMPLE 1. A 2-dimensional Gorenstein isolated singularity (X, x) is purely elliptic if and only if (X, x) is either a simple elliptic singularity or a cusp singularity. They are characterized by the exceptional curves on the minimal resolutions. The exceptional curve of a simple elliptic singularity is a non-singular elliptic curve and that of a cusp singularity is a cycle of rational curves. A simple elliptic singularity is of type $(0, 1)$ and a cusp singularity is of type $(0, 0)$.

Now, we define a three-dimensional analogue of a simple elliptic singularity.

DEFINITION 4. A normal isolated singularity (X, x) of dimension three is called a simple $K3$-singularity, if (X, x) is a Gorenstein purely elliptic singularity of type $(0, 2)$.

DEFINITION 5. A projective birational morphism $g: Y \to X$ is called a partial resolution of the singularity on X, if g is an isomorphism on the outside of the singular locus of X and Y is normal. A partial resolution $g: Y \to X$ is called a $\boldsymbol{Q}$-factorial terminal modification, if Y has only $\boldsymbol{Q}$-factorial terminal singularities and the canonical divisor K_Y is g-semi-ample. Here g-semi-ample means that the natural map $g^* g_* \mathcal{O}_Y(mK_Y) \to \mathcal{O}_Y(mK_Y)$ is surjective for some m divisible by the index of Y.

REMARK. For a two-dimensional singularity, a $\boldsymbol{Q}$-factorial terminal modification is equivalent to the minimal resolution. By [M, 0.3.12], every three-dimensional normal singularity admits a $\boldsymbol{Q}$-factorial terminal modification.

LEMMA 6. *Let (X, x) be an n-dimensional Gorenstein purely elliptic singularity of type $(0, n-1)$. Then, for any good resolution $f: \tilde{X} \to X$ of the singularity, the essential divisor E_J is irreducible.*

PROOF. Assume that E_J has a decomposition $E_J = E_1 + E_2$. From the Mayer-Vietoris exact sequence

$$H^{n-2}(E_1 \cap E_2, \boldsymbol{C}) \to H^{n-1}(E_J, \boldsymbol{C}) \to H^{n-1}(E_1, \boldsymbol{C}) \oplus H^{n-1}(E_2, \boldsymbol{C}),$$

we have the exact sequence

$$\begin{array}{ccccc} \mathrm{Gr}_F^0 H^{n-2}(E_1 \cap E_2) & \to & \mathrm{Gr}_F^0 H^{n-1}(E_J) & \to & \mathrm{Gr}_F^0 H^{n-1}(E_1) \oplus \mathrm{Gr}_F^0 H^{n-1}(E_2) \\ & & \| & & \| \\ & & H^{n-1}(E_J, \mathcal{O}) & & H^{n-1}(E_1, \mathcal{O}) \oplus H^{n-1}(E_2, \mathcal{O}), \end{array}$$

where F is the Hodge filtration. The first term does not contribute to the $(0, n-1)$-component of the middle term, since $E_1 \cap E_2$ is a compact $(n-2)$-dimensional variety. Therefore, the middle term is mapped to the last one injectively. This contradicts the fact that $H^{n-1}(E_i, \mathcal{O}) = 0$ for $i = 1, 2$ (cf. [I1, Corollary 3.9]).

LEMMA 7. *Let (X, x) be an n-dimensional Gorenstein purely elliptic singularity of type $(0, n-1)$. Then, for a $\boldsymbol{Q}$-factorial terminal modification $g: Y \to X$, the exceptional set $D = g^{-1}(x)_{\mathrm{red}}$ is an irreducible divisor and $K_Y = g^*K_X - D$. Furthermore, if Y is non-singular in codimension two, then D is non-singular in codimension one.*

PROOF. First of all, we see that D is a divisor and $K_Y = g^*K_X - D'$, where D' is an effective divisor with the support exactly on D. This is proved by a slight modification of [I2, Lemma 2] as follows:
As is well-known, a projective birational morphism $g: Y \to X$ is obtained by the blowing up of some ideal sheaf on X. Therefore there are positive numbers m_i $(i = 1, 2, \dots, r, r+1, \dots, t)$ such that $L = -\sum_{i=1}^t m_i E_i$ is a relatively very ample Cartier divisor with respect to g, where all E_i's $(i = 1, 2, \dots, r)$ are the irreducible Weil divisors contained in $g^{-1}(x)$ and E_i's $(i = r+1, \dots, t)$ are the ones not contained in $g^{-1}(x)$. Since K_Y is relatively nef, $K_Y + aL$ $(a \geq 0, a \in \boldsymbol{Q})$ is relatively nef with respect to g. Denote the canonical divisor K_Y by $g^*K_X - \sum_{i=1}^r a_i E_i$ with $a_i \in \boldsymbol{Q}$. If there exists a non-positive a_i, we let a be the non-negative number $-\min_{1 \leq i \leq r}\{a_i/m_i\}$. Then we have:

$$K_Y + aL = g^*K_X - \sum_{i=r+1}^{t} am_i E_i - \sum_{i=1}^{r} \beta_i E_i,$$

where $\beta_i = 0$ for the i's such that a_i/m_i attain the minimal value, and $\beta_i > 0$ for the other i's. Here, there exists i $(1 \leq i \leq r)$ for which a_i/m_i does not attain the minimal value $-a$, otherwise the singularity (X, x) would become a canonical singularity. Let E_i be an irreducible component with $\beta_i = 0$ and $E_i \cap E_j \neq \varnothing$ for some j with $\beta_j > 0$. Let C be an irreducible curve on E_i such that $C \cap E_j \neq \varnothing$ and $C \not\subset E_j$. Then $(K_Y + aL) \cdot C < 0$, which is a contradiction. Therefore a_i's are all positive. Now it remains to show that $g^{-1}(x)_{\mathrm{red}} = \sum_{i=1}^r E_i$. Let C' be a curve in an irreducible component of $g^{-1}(x)_{\mathrm{red}}$ of codimension greater than one. We may assume that the curve C' is not contained in $\sum_{i=1}^r E_i$ and intersects it. Then $K_Y \cdot C' = (g^*K_Y - \sum_{i=1}^r a_i E_i) \cdot C' < 0$, since $a_i > 0$ for $i = 1, 2, \dots, r$. This is a contradiction to the fact that K_Y is relatively nef. Therefore $g^{-1}(x)_{\mathrm{red}}$ must coincide with $\sum_{i=1}^r E_i$.

Take a blowing-up $\sigma: \tilde{X} \to Y$ such that the composite $f = g \circ \sigma: \tilde{X} \to X$ becomes a good resolution of the singularity (X, x). Then the proper transform of each component of D is a component of the essential divisor. By Lemma 6, the number of components of D should be less than or equal to one. If $D=0$, then the singularity (X, x) is canonical, a contradiction. Therefore D is irreducible. By Proposition 2, $K_{\tilde{X}} = f^*K_X - [D] +$ (the other components), where $[D]$ denotes the proper transform of D on $\tilde{X}$. Then the coefficient r of D in the equality $K_Y = g^*K_X - rD$ is one. Finally we show the last assertion. Assume there exists a component S of the singular locus of codimension one in D. Denote the multiplicity of D at a general point of S by m (≥ 2). Then, in the expression $K_{\tilde{X}} = f^*K_X - [D] + \sum_{i=1}^{s} m_i E_i$, there exists an exceptional component E_i such that E_i is mapped onto S and $m_i = -(m-1)$, because, at a general point of S, S is non-singular $(n-2)$-fold in a non-singular n-fold Y. By Lemma 6, we have $m_i \geq 0$ for every i, which leads to a contradiction $m \leq 1$.

DEFINITION 8. A normal surface S satisfying the following mutually equivalent conditions (see [U], for example) is called a normal $K3$-surface:

(1) the minimal resolution of S is a $K3$-surface;

(2) K_S is trivial and S is birational to a $K3$-surface;

(3) K_S is trivial, $H^1(S, \mathcal{O}_S) = 0$ and all the singularities on S are rational double points.

THEOREM. *Let (X, x) be a three-dimensional normal isolated singularity. Then, (X, x) is a simple K3-singularity if and only if $D = g^{-1}(X)_{\mathrm{red}}$ is a normal K3-surface for a $\boldsymbol{Q}$-factorial terminal modification $g: Y \to X$ of the singularity.*

PROOF. First of all, note that the singularities on Y is isolated, since 3-dimensional terminal singularities are proved to be isolated (cf. [R]). Assume that (X, x) is a simple $K3$-singularity. By Lemma 7, D is irreducible and $K_Y = g^*K_X - D$. From the exact sequence

$$0 \to \mathcal{O}(K_Y) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0\,,$$

we have exact sequences of local cohomologies

$$H_p^i(Y, \mathcal{O}_Y) \to H_p^i(D, \mathcal{O}_D) \to H_p^{i+1}(Y, \mathcal{O}(K_Y))$$

for every point $p \in D$. Since $\mathcal{O}_Y$ and $\mathcal{O}_Y(K_Y)$ are both Cohen-Macaulay $\mathcal{O}_Y$-modules, we get $H_p^i(\mathcal{O}_Y) = H_p^i(\mathcal{O}_Y(K_Y)) = 0$ for $i = 0, 1, 2$. By the exact sequence, we have $H_p^i(\mathcal{O}_D) = 0$ for $i = 0, 1$ which implies depth $\mathcal{O}_D \geq 2$. Now, a two-dimensional variety D turns out to be a Cohen-Macaulay variety. On the other hand, D is non-singular in codimension one by Lemma 7. By Serre's Criterion, we see that D is normal. Moreover $\mathcal{O}(K_D) \simeq \mathcal{O}_D$. In fact, the equality of Weil divisors $K_Y = g^*K_X - D$ yields the isomorphism $\mathcal{O}(K_D) \simeq \mathcal{O}_D$ on the outside of the singular locus of Y by the adjunction formula. And the isomorphism can be extended to every point of D, since the singularities of Y are isolated.

By [U], such a normal surface is either a normal $K3$-surface, an Abelian surface or birational to a ruled surface. However D is not an Abelian surface, because $H^1(D, \mathcal{O}_D) = R^1 g_* \mathcal{O}_Y = 0$ by the Cohen-Macaulay property of (X, x). And D is not birational to a ruled surface, because a resolution $\tilde{D} \to D$ of D must satisfy $H^2(\tilde{D}, \mathcal{O}_{\tilde{D}}) \simeq \boldsymbol{C}$ by the property of the essential divisor of a purely elliptic singularity (cf. [I1, 3.7]). Thus, D must be a normal $K3$-surface.

Conversely, let D be a normal $K3$-surface. Consider the exact sequence

$$0 \to \mathcal{O}(K_Y) \xrightarrow{\alpha} \mathcal{O}(K_Y + D) \xrightarrow{\beta} \mathcal{O}_D(K_D) .$$

We claim that β is surjective. Denote the cokernel of α by C. Here we have $\operatorname{depth}_p C \geq 2$ for every point p on Y, because $\operatorname{depth}_p \mathcal{O}(K_Y) = 3$ and $\operatorname{depth}_p \mathcal{O}(K_Y + D) \geq 2$. Then the inclusion $C \subset \mathcal{O}(K_D)$ becomes an equality, since both sides coincide with each other on the complement of finite points on Y. Now the claim is proved. Replacing X by a sufficiently small Stein neighbourhood of x, we have an exact sequence:

$$\Gamma(Y, \mathcal{O}(K_Y + D)) \xrightarrow{\varphi} \Gamma(D, \mathcal{O}(K_D)) \to H^1(Y, \mathcal{O}(K_Y)) = 0 ,$$

where the last term is zero by the Grauert-Riemenschneider vanishing theorem. Therefore there exists $\theta \in \Gamma(Y, \mathcal{O}(K_Y + D))$ such that $\varphi(\theta)$ is a nowhere-vanishing holomorphic 2-form on D. Since the singularities on Y are $\boldsymbol{Q}$-factorial, the intersection of every two Weil divisors consists of curves, if they do intersect. Let Z be the zero divisor of θ. Then $Z \cap D$ turns out to be empty, for otherwise $\varphi(\theta)$ would vanish on the curves $Z \cap D$, a contradiction. Therefore the 3-form θ defines an isomorphism $\mathcal{O}_Y(-D) \simeq \mathcal{O}_Y(K_Y)$, which implies that K_X is a Cartier divisor on X. Now we are going to prove that (X, x) is a Cohen-Macaulay singularity. By the above isomorphism we have an exact sequence

$$0 \to \mathcal{O}(K_Y) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0 .$$

Since Y has only rational singularities, $H^i(Y, \mathcal{O}(K_Y)) = 0$ for $i > 0$ by the Grauert-Riemenschneider vanishing theorem. This yields isomorphisms $H^i(Y, \mathcal{O}_Y) \simeq H^i(D, \mathcal{O}_D)$ for $i > 0$. In our situation, both Y and D are Du Bois varieties. By [I1, Proposition 1.4], the singularity (X, x) is a Du Bois singularity. The preceding isomorphism for $i = 1$ implies that (X, x) is a Cohen-Macaulay singularity. Now (X, x) is a Gorenstein Du Bois singularity, which is proved to be either rational or purely elliptic by [I1, 2.3]. The geometric genus $p_g(X, x) = \dim H^2(Y, \mathcal{O}_Y) = \dim H^2(D, \mathcal{O}_D) = 1$ means that (X, x) is purely elliptic. Furthermore, the essential divisor of a good resolution of the singularity has a component $[D]$ with $H^2([D], \mathcal{O}_{[D]}) = 1$, which means that (X, x) is of type $(0, 2)$.

EXAMPLE 2. Every normal $K3$-surface presented as a quartic in $\boldsymbol{P}^3$ can be the exceptional divisor on a $\boldsymbol{Q}$-factorial terminal modification of a simple $K3$-singularity.

Let $D' \subset \boldsymbol{P}^3$ be a normal quartic with only rational singularities and H a general hypersurface of degree d which does not pass through the singular points of D'. Denote the blowing-up of $\boldsymbol{P}^3$ at the non-singular center $D' \cap H$ by $\sigma: \tilde{X} \to \boldsymbol{P}^3$ and the proper transform of D' on $\tilde{X}$ by D. For $d \geq 5$, the divisor D can be contracted in $\tilde{X}$ to a simple $K3$-singularity (X, x). The singularity (X, x) is a hypersurface singularity if and only if $d = 5$. In this case, the defining equation is $\varphi - \psi$ where φ and ψ are the defining equations of D' and H in $\boldsymbol{P}^3$, respectively.

REMARK. We can also define a simple Abelian singularity as follows: a three-dimensional normal isolated singularity (X, x) is called a simple Abelian singularity if it is quasi-Gorenstein, purely elliptic of type $(0, 2)$ and not a Cohen-Macaulay singularity. In this case, the exceptional divisor $D = g^{-1}(x)_{\text{red}}$ of a $\boldsymbol{Q}$-factorial terminal modification $g: Y \to X$ is an Abelian surface.

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