ON THE NONEXISTENCE OF STABLE CURRENTS
IN SUBMANIFOLDS OF A EUCLIDEAN SPACE

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Abstract. In 1973, Lawson and Simons conjectured that there are no stable currents in any compact, simply connected Riemannian manifold M^m which is 1/4-pinched. In this paper, we regard M^m as a submanifold immersed in a Euclidean space and prove the conjecture under some pinched conditions about the sectional curvatures and the principal curvatures of M^m. We also show that there is no stable p-current in a submanifold of M^m and the p-th homology group vanishes when the shape operator of the submanifold satisfies certain conditions.

1. Introduction. For any compact Riemannian manifold M^m, a theorem of Federer and Fleming [1] asserts that any non-trivial integral homology class in $H_p(M, Z)$ corresponds to a stable current. By using techniques from the calculus of variations in geometric measure theory together with the Federer-Fleming theorem, Lawson and Simons [3] showed that there are no stable currents in a sphere S^m, and there is no stable p-current in submanifolds immersed in S^m with sufficiently small second fundamental form. The result of [3] on submanifolds of S^m has been extended to submanifolds of a Euclidean space E^m or of $S^{m_1} \times S^{m_2}$ by Xin [4] and Zhang [5], respectively.

In [3], Lawson and Simons conjectured that there are no stable varifolds, in particular, there are no stable currents in any compact, simply connected Riemannian manifold M^m which is 1/4-pinched. As variants of the above conjecture, there have been many results on stable minimal submanifolds and stable harmonic maps. However, the original conjecture by Lawson and Simons is still open. In 1985, with comparison theorems, Howard [2] proved that there is a constant $\delta(m, p) > 1/4$ such that there is no stable p-current in M^m when M^m is $\delta(m, p)$-pinched. However, the expression of $\delta(m, p)$ is complex and unfortunately $\lim_{m \to \infty} \delta(m, p) = 1$.

In this paper, we regard such a compact Riemannian manifold M^m as a submanifold immersed in a Euclidean space E^{m+k} with essential codimension k. We shall prove that there are no stable currents in M^m if one of the following conditions is satisfied:

1) The sectional curvatures $K_M > (k/4)(\lambda_0 - \mu_0)^2$, where λ_0 and μ_0 are the maximum and the minimum of the principal curvatures of M^m, respectively;

2) $\lambda\mu > (1/4)(\lambda - \mu)^2$, where λ and μ are any two principal curvatures of M^m.

The condition 1) indicates that Lawson-Simons' conjecture is true if $k(\lambda_0 - \mu_0)^2 \leq 1$ on M^m.

With condition 2), we can prove that if any two principal curvatures λ, μ of a compact submanifold $M^m (m > 3)$ in E^{m+k} satisfy the inequality $3 - 2\sqrt{2} < \lambda/\mu < 3 + 2\sqrt{2}$, then M^m is homeomorphic to a sphere.

Let N^n be a compact submanifold immersed in the manifold M^m satisfying the above condition 1) or 2) and S a p-rectifiable set in N^n. By using the shape operators A_λ associated with the immersion $N^n \to M^m$, we introduce a selfadjoint linear operator Q^A on p-space $T_x S \subset T_x N$. We shall show that there is no stable p-current in N^n if, for any $x \in N^n$ and any p-subspace V in $T_x N$, $\operatorname{tr} Q^A \leq 0$.

2. Preliminaries.   We use the same notation as in [3, 5] throughout this paper. In this section, we recall several formulas employed in [5]. Also, we prove three lemmas which will be used in the proof of our main theorems.

Let M^m be an m-dimensional compact Riemannian manifold with metric $\langle \ , \ \rangle$ and Levi-Civita connection ∇. Denote by (S, ξ) an oriented, p-rectifiable set in M^m. The set of rectifiable p-currents is defined by

$$\Re_p(M) = \left\{ \mathcal{S} = \sum_{n=1}^\infty n \mathcal{S}_n \,\Big|\, \mathcal{S}_n = (S_n, \xi_n), \, M(\mathcal{S}) = \sum_{n=1}^\infty n \, \mathcal{H}^p(S_n) < \infty \right\}.$$

$\mathcal{S} \in \Re_p(M)$ is called an integral p-current if $\mathcal{S}$ and the boundary $\partial \mathcal{S}$ are both rectifiable currents. The space of integral p-currents is denoted by $\Im_p(M)$. The direct sum $\Im_*(M) = \bigoplus_p \Im_p(M)$ together with the boundary operator $\partial : \Im_*(M) \to \Im_*(M)$ forms a differential chain complex.

FEDERER-FLEMING THEOREM [1].   *For each $p \geq 0$ there is a natural isomorphism*

$$H_p(\Im_*(M)) \cong H_p(M, Z).$$

Moreover, for each $\alpha \in H_p(\Im_*(M))$ there exists a current $\mathcal{S} \in \alpha$ of "least mass", that is, $M(\mathcal{S}) \leq M(\mathcal{S}')$ for all $\mathcal{S}' \in \alpha$.

Let $\phi_t : M^m \to M^m$ be a 1-parameter group of local diffeomorphisms generated by a smooth vector field $X \in C(TM)$. A current $\mathcal{S} \in \Re_p(M)$ is said to be stable if for each $X \in C(TM)$ there is an $\varepsilon > 0$ such that

$$M(\phi_{t*}\mathcal{S}) \geq M(\mathcal{S})$$

for all $|t| < \varepsilon$. This implies that for any $X \in C(TM)$ we have

$$\frac{d}{dt} M(\phi_{t*}\mathcal{S}) \Big|_{t=0} = 0, \qquad \frac{d^2}{dt^2} M(\phi_{t*}\mathcal{S}) \Big|_{t=0} \geq 0.$$

Let (S, ξ) be an oriented, p-rectifiable set. Then ξ is an $\mathcal{H}^p$-measurable section of $\bigwedge^p TM$ over S with the property that for $\mathcal{H}^p$-almost all $x \in S$, ξ_x is a simple p-vector of unit length representing $T_x S$. For such a pair (S, ξ), let

$$Q_\xi(X) = \frac{d^2}{dt^2} \|\phi_{t*}\xi\| \Big|_{t=0}.$$

Then the second variation formula derived by Lawson and Simons [3] can be given by

$$(2.1) \qquad \frac{d^2}{dt^2} M(\phi_{t*} \mathcal{S}) \Big|_{t=0} = \sum_n n \int_{S_n} Q_{\xi_n}(X) \, d\mathcal{H}^p(x) \, .$$

If $X = \nabla f$ for some $f \in C^3(M)$, and $\{e_i, e_\alpha\}$ $(i = 1, 2, \ldots, p; \; \alpha = p+1, \ldots, m)$ is an orthonormal basis of $T_x M$ with $\xi = e_1 \wedge \cdots \wedge e_p$, then (see [3, p. 436])

$$(2.2) \qquad Q_\xi(X) = \left[\sum_i \langle a^X(e_i), e_i \rangle \right]^2 + 2 \sum_{i,\alpha} \langle a^X(e_i), e_\alpha \rangle^2 + \sum_i \langle \nabla_{X, \, e_i} X, e_i \rangle \, ,$$

where $a^X(e_i) = \nabla_{e_i} X$ and $\nabla_{X, \, e_i} X = \nabla_X \nabla_{e_i} X - \nabla_{\nabla_X e_i} X$.

For a p-rectifiable set S in M^m, it is known that for $\mathcal{H}^p$-almost all points $x \in S$, there exists an approximate p-space $T_x S \subset T_x M$, to S. Now we introduce an operator Q^A on $T_x S$ and establish the relationship between Q^A and Q_ξ.

Let $\varphi : M^m \to E^{m+k}$ be an isometric immersion of M^m into the Euclidean $(n+k)$-space E^{m+k} and D the Levi-Civita connection of E^{m+k}. Denote by $V(E^{m+k}, M)$ the normal bundle of M^m in E^{m+k}. For a smooth section $v \in C(V(E^{m+k}, M))$, let A_v be the shape operator determined by v. At $x \in M^m$, $A_v : T_x M \to T_x M$ is a selfadjoint linear operator, which is defined by

$$A_v X = -(D_X v)^T \, ,$$

where $X \in C(TM)$. Let $\{v_\lambda\}$ be an orthonormal basis of the normal space $V_x(E^{m+k}, M)$ and $A_\lambda = A_{v_\lambda}$. For a given integer $p \in (0, m)$, let V be a p-dimensional subspace in $T_x M$ and $\{e_i\}$ an orthonormal basis of V. Define a selfadjoint linear map $Q^A : V \to V$ associated with the immersion φ by

$$(2.3) \qquad Q^A X = \sum_\lambda \left[2 \left(\sum_i \langle A_\lambda^2 X, e_i \rangle e_i - B_\lambda^2 X \right) - (\mathrm{tr}\, A_\lambda - \mathrm{tr}\, B_\lambda) B_\lambda X \right] ,$$

where $X \in V$ and B_λ is a map on V associated with A_λ defined by

$$(2.4) \qquad B_\lambda X = \sum_i \langle A_\lambda X, e_i \rangle e_i \, .$$

Q^A is independent of the choice of bases of $V_x(E^{m+k}, M)$ and V. Its trace is given by

$$(2.5) \qquad \mathrm{tr}\, Q^A = \sum_i \langle Q^A e_i, e_i \rangle = \sum_\lambda \left[2 \sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 - (\mathrm{tr}\, A_\lambda - \mathrm{tr}\, B_\lambda) \mathrm{tr}\, B_\lambda \right] ,$$

where $\{e_\alpha\}$ is an orthonormal basis of $V^\perp$ which is the orthogonal complement of V in $T_x M$.

Let (S, ξ) be an oriented, p-rectifiable set. At $x \in S$, we associate a tangent p-space $V = T_x S \subset T_x M$. Choose an orthonormal basis $\{e_i, e_\alpha\}$ of $T_x M$ such that $\{e_i\}$ is a basis of V and $\xi = e_1 \wedge \cdots \wedge e_p$. We now consider Q_ξ given by (2.2) as a quadratic form defined on the set

$$(2.6) \qquad \theta = \{v^T \in C(TM) \mid v \in E^{m+k}, \, v^T(x) = \text{orthogonal projection of } v \text{ onto } T_x M\} \, .$$

Note that at $x \in M^m$, $\{e_i, e_\alpha, \nu_\lambda\}$ is an orthonormal basis of E^{m+k}. Therefore,

$$\text{tr } Q_\xi = \sum_i Q_\xi(e_i) + \sum_\alpha Q_\xi(e_\alpha) + \sum_\lambda Q_\xi(\nu_\lambda).$$

Making use of the proof given in [5], we obtain

LEMMA 1 [5]. *Let M^m be a submanifold immersed in E^{m+k}. Then*

$$\text{tr } Q_\xi = \text{tr } Q^A.$$

LEMMA 2. *Let $x_1, x_2, \ldots, x_m$; $y_1, y_2, \ldots, y_m$ be real numbers satisfying*

$$\text{(2.7)} \qquad \sum_a (x_a)^2 = \sum_a (y_a)^2 = 1, \qquad \sum_a x_a y_a = 0.$$

Then for given real numbers $k_1, k_2, \ldots, k_m$, we have

$$\left(\sum_a k_a x_a y_a \right)^2 \le \frac{1}{4}(k_{\max} - k_{\min})^2,$$

where $k_{\max} = \max_{1 \le a \le m}\{k_a\}$, $k_{\min} = \min_{1 \le a \le m}\{k_a\}$.

PROOF. In order to get the maximum of the function $f = (\sum_a k_a x_a y_a)^2$ under the constraints (2.7), partially differentiating

$$F = \left(\sum_a k_a x_a y_a \right)^2 + C_1 \left[\sum_a (x_a)^2 - 1 \right] + C_2 \left[\sum_a (y_a)^2 - 1 \right] + C_3 \sum_a x_a y_a$$

with respect to each variable and equating to zero, we obtain

$$2 k_a y_a \sum_b k_b x_b y_b + 2 C_1 x_a + C_3 y_a = 0,$$

$$2 k_a x_a \sum_b k_b x_b y_b + 2 C_2 y_a + C_3 x_a = 0,$$

where C_1, C_2, C_3 are Lagrange multipliers. Set $u = \sum_a k_a (x_a)^2$, $v = \sum_a k_a (y_a)^2$, $w = \sum_a k_a x_a y_a$. Then the above equations can be rewritten as

$$\text{(2.8)} \qquad\qquad 2 w k_a y_a + 2 C_1 x_a + C_3 y_a = 0,$$

$$\text{(2.9)} \qquad\qquad 2 w k_a x_a + 2 C_2 y_a + C_3 x_a = 0.$$

By using the constraints (2.7), from (2.8) and (2.9) we have

$$2 w^2 + 2 C_1 = 0, \quad 2 w v + C_3 = 0; \quad 2 w u + C_3 = 0, \quad 2 w^2 + 2 C_2 = 0.$$

Hence,

$$\text{(2.10)} \qquad\qquad C_1 = C_2 = -w^2 = -f, \quad C_3 = -2 w v = -2 w u.$$

If $w = 0$, then $f = 0$.

Now assume that $w \neq 0$. Then $C_1 = C_2 \neq 0$, and $C_3 = -2wv = -2wu$ gives $u = v$. From (2.8) and (2.9) we obtain

$$2C_1[(x_a)^2 - (y_a)^2] = 0.$$

Hence, $(x_a)^2 - (y_a)^2 = 0$, $a = 1, 2, \ldots, m$. Without loss of generality, we may suppose

$$x_s = y_s \; (s = 1, \ldots, k); \quad x_t = -y_t \; (t = k+1, \ldots, m).$$

Then (2.8) or (2.9) implies

$$(2wk_s + 2C_1 + C_3)x_s = 0 \quad (s = 1, \ldots, k),$$
$$(2wk_t - 2C_1 + C_3)x_t = 0 \quad (t = k+1, \ldots, m).$$

Now, $\sum_a x_a y_a = \sum_s (x_s)^2 - \sum_t (x_t)^2$; from (2.7) we have $\sum_s (x_s)^2 = \sum_t (x_t)^2 = 1/2$. Hence there are at least an s_0 and a t_0 such that $x_{s_0} \neq 0$ and $x_{t_0} \neq 0$. Therefore we obtain

(2.11) $$2wk_{s_0} + 2C_1 + C_3 = 0, \quad 2wk_{t_0} - 2C_1 + C_3 = 0.$$

From (2.11) and (2.10) we get

$$k_{s_0} = -\frac{2C_1 + C_3}{2w} = w + u, \quad k_{t_0} = \frac{2C_1 - C_3}{2w} = u - w,$$

which imply that $w = (1/2)(k_{s_0} - k_{t_0})$. Consequently, $f = w^2 \leq (1/4)(k_{\max} - k_{\min})^2$.

A similar argument proves the following

LEMMA 3. *Let* $x_1, x_2, \ldots, x_m$; $y_1, y_2, \ldots, y_m$ *be real numbers satisfying*

$$\sum_a (x_a)^2 = \sum_a (y_a)^2 = 1, \quad \sum_a x_a y_a = 0.$$

If given real numbers $k_1, k_2, \ldots, k_m$ *satisfy*

$$k_a k_b > \frac{1}{4}(k_a - k_b)^2 \; (resp., \geq), \quad a, b = 1, 2, \ldots, m,$$

then

$$2\left(\sum_a k_a x_a y_a\right)^2 - \sum_a k_a(x_a)^2 \sum_b k_b(y_b)^2 < 0 \; (resp., \leq).$$

3. Main results. In this section, we assume that M^m is an immersed submanifold in E^{m+k} with essential codimension k, i.e., M^m can be immersed in E^{m+k}, but not in E^{m+l} with $l < k$.

Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_x(E^{m+k}, M)$, and associate the shape operator $A_\lambda = A_{\nu_\lambda}$. From the Gauss equation in M^m, the sectional curvature of the 2-plane $X \wedge Y \subset T_x M$ is

(3.1) $$K(X \wedge Y) = \sum_\lambda [\langle A_\lambda X, X \rangle \langle A_\lambda Y, Y \rangle - \langle A_\lambda X, Y \rangle^2],$$

where X and Y are mutually orthogonal unit vectors in $T_x M$.

THEOREM 1. *Let M^m be a compact submanifold in E^{m+k}. Denote by λ_0 and μ_0 the maximum and the minimum of the principal curvatures of M^m, respectively. If the sectional curvatures of M^m satisfy*

$$(3.2) \qquad K_M > \frac{k}{4}(\lambda_0 - \mu_0)^2 \,,$$

then there are no stable currents in M^m and $H_p(M, Z) = 0$ for any $p \in (0, m)$.

PROOF.   For each fixed index λ, there is an orthonormal basis $\{E_a\}$ in $T_x M$ such that

$$(3.3) \qquad A_\lambda E_a = k_{\lambda a} E_a \,, \quad a = 1, 2, \ldots, m \,.$$

Let (S, ξ) be an oriented p-rectifiable set in M^m. With a point $x \in S$, associate a tangent p-space $V = T_x S \subset T_x M$. Choose an orthonormal basis $\{e_i, e_\alpha\}$ of $T_x M$ such that $\{e_i\}$ is a basis of V. Also, let

$$(3.4) \qquad e_i = \sum_a e_i^a E_a \,, \quad e_\alpha = \sum_a e_\alpha^a E_a \,.$$

Then

$$(3.5) \qquad \sum_a (e_i^a)^2 = 1 \,, \quad \sum_a (e_\alpha^a)^2 = 1 \,, \quad \sum_a e_i^a e_\alpha^a = 0 \,,$$

and

$$A_\lambda e_i = \sum_a e_i^a k_{\lambda a} E_a \,.$$

Hence we have

$$\langle A_\lambda e_i, e_\alpha \rangle = \sum_a e_i^a e_\alpha^a k_{\lambda a} \,.$$

From Lemma 2 we see that $f = (\sum_a e_i^a e_\alpha^a k_{\lambda a})^2$, under the constraints (3.5), has the maximum $(1/4)(k_{\max} - k_{\min})^2$, where $k_{\max} = \max_{1 \le a \le m}\{k_{\lambda a}\}$, $k_{\min} = \min_{1 \le a \le m}\{k_{\lambda a}\}$. Therefore,

$$(3.6) \qquad \langle A_\lambda e_i, e_\alpha \rangle^2 \le \frac{1}{4}(\lambda_0 - \mu_0)^2 \,.$$

In (3.6), λ_0 and μ_0 are the maximum and the minimum of the principal curvatures of M^m, respectively, which are independent of the index λ.

Combining (3.1) and (2.5), we obtain

$$(3.7) \qquad \operatorname{tr} Q^A = -\sum_{i,\alpha} K(e_i \wedge e_\alpha) + \sum_{\lambda,i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 \,.$$

From (3.6) and (3.7), the condition (3.2) implies $\operatorname{tr} Q^A < 0$.

Let θ be the set given by (2.6). If $v^T \in \theta$, v^T is the gradient ∇f of the function $f(x) = \langle v, x \rangle$ on M^m. For $X \in \theta$, let ϕ_t be the flow generated by X and set

$$Q_S(X) = \frac{d^2}{dt^2} M(\phi_{t*} S)\bigg|_{t=0} \,,$$

where $\mathcal{S} \in \Re_p(M)$. Then from (2.1) we have

(3.8)
$$\operatorname{tr} Q_{\mathcal{S}} = \sum_n n \int_{S_n} \operatorname{tr} Q_{\xi_n} d\mathcal{H}^p(x).$$

From Lemma 1, we get $\operatorname{tr} Q_{\xi_n} = \operatorname{tr} Q^A < 0$. Hence, $\operatorname{tr} Q_{\mathcal{S}} < 0$. This implies that there is no stable p-current in M^m for any $p \in (0, m)$. Federer-Fleming's theorem then shows that $H_p(M, Z) = 0$, $p = 1, 2, \ldots, m - 1$.

A similar argument as in Theorem 1 proves the following

THEOREM 2. *Let M^m be a compact submanifold immersed in a simply connected space form $N^{m+k}(c)$ with $c \geq 0$. Denote by λ_0 and μ_0 the maximum and the minimum of the principal curvatures of M^m, respectively. If the sectional curvatures of M^m satisfy*

$$K_M > \frac{k}{4}(\lambda_0 - \mu_0)^2 - c,$$

then there are no stable currents in M^m and $H_p(M, Z) = 0$ for $1 \leq p \leq m - 1$.

THEOREM 3. *Let M^m be a compact submanifold immersed in E^{m+k}. If any two principal curvatures λ, μ of M^m satisfy*

$$\lambda\mu > \frac{1}{4}(\lambda - \mu)^2,$$

then there are no stable currents in M^m and $H_p(M, Z) = 0$ for $p = 1, 2, \ldots, m - 1$.

PROOF. As in the proof of Theorem 1, for each fixed index λ, from (3.3) and (3.4) we have

$$A_\lambda e_i = \sum_a e_i^a k_{\lambda a} E_a, \qquad A_\lambda e_\alpha = \sum_a e_\alpha^a k_{\lambda a} E_a,$$

where e_i^a and e_α^a satisfy the conditions (3.5). Hence we have

$$\langle A_\lambda e_i, e_\alpha \rangle = \sum_a e_i^a e_\alpha^a k_{\lambda a},$$

$$\operatorname{tr} A_\lambda = \sum_i \langle A_\lambda e_i, e_i \rangle + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle = \sum_{i,a} (e_i^a)^2 k_{\lambda a} + \sum_{\alpha,a} (e_\alpha^a)^2 k_{\lambda a},$$

$$\operatorname{tr} B_\lambda = \sum_i \langle A_\lambda e_i, e_i \rangle = \sum_{i,a} (e_i^a)^2 k_{\lambda a}.$$

Therefore

(3.9)
$$2 \sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 - (\operatorname{tr} A_\lambda - \operatorname{tr} B_\lambda) \operatorname{tr} B_\lambda$$

$$= \sum_{i,\alpha} \left[2 \left(\sum_a e_i^a e_\alpha^a k_{\lambda a} \right)^2 - \sum_a (e_i^a)^2 k_{\lambda a} \sum_b (e_\alpha^b)^2 k_{\lambda b} \right].$$

From Lemma 3 and the assumption of Theorem 3, (3.9) then implies that

$$2 \sum_{i,\alpha} \langle A_\lambda e_i, e_\alpha \rangle^2 - (\mathrm{tr} A_\lambda - \mathrm{tr} B_\lambda)\mathrm{tr} B_\lambda < 0 .$$

Hence, from Lemma 1, we obtain $\mathrm{tr} Q_{\xi_n} = \mathrm{tr} Q^A < 0$. Consequently, from (3.8), we have $\mathrm{tr} Q_{\mathcal{S}} < 0$.   □

REMARK 1. The inequality $\lambda\mu > (1/4)(\lambda - \mu)^2$ is equivalent to

$$[\lambda - (3 - 2\sqrt{2})\mu][\lambda - (3 + 2\sqrt{2})\mu] < 0 ,$$

which is also equivalent to

(3.10) $$3 - 2\sqrt{2} < \frac{\lambda}{\mu} < 3 + 2\sqrt{2} .$$

By the same proof as used for the second part of Theorem 1 in [6], we can prove the following

THEOREM 4. *Let M^m ($m > 3$) be a compact connected and orientable submanifold immersed in E^{m+k}. If any two principal curvatures λ, μ of M^m satisfy the inequality (3.10), then M^m is homeomorphic to a sphere.*

THEOREM 5. *Let M^m be a submanifold in E^{m+k}. Suppose that its principal curvatures satisfy one of the following conditions*:

C1. $\lambda\mu \geq (1/4)(\lambda - \mu)^2$,

C2.   *the sectional curvatures $K_M \geq (k/4)(\lambda_0 - \mu_0)^2$,*

where λ_0 and μ_0 are the maximum and the minimum of the principal curvatures of M^m, respectively. Let N^n be a compact submanifold immersed in M^m and $p \in (0, n)$ a given integer. If, for any $x \in N^n$ and any p-subspace V in $T_x N$, the selfadjoint map Q^A on V associated with the immersion $N^n \to M^m$ satisfies

$$\mathrm{tr} Q^A < 0 ,$$

then there is no stable p-current in N^n and $H_p(N, Z) = H_{n-p}(N, Z) = 0$.

PROOF. Let ∇, $\bar{\nabla}$, D be the Levi-Civita connections of N^n, M^m, E^{m+k}, respectively. Associated with the immersion $N^n \to E^{m+k}$, the shape operator A'_ν determined by $\nu \in C(V(E^{m+k}, N))$ is given by

$$A'_\nu Y = -(D_Y \nu)^T ,$$

where $Y \in C(TN)$. In particular, if $\nu \in C(V(M, N))$,

(3.11) $$A'_\nu Y = -(D_Y \nu)^T = -[\bar{\nabla}_Y \nu + \bar{h}(\nu, Y)]^T = -(-A_\nu Y + \nabla^\perp_Y \nu)^T = A_\nu Y ,$$

where $\bar{h}$ is the second fundamental form of the immersion $M^m \to E^{m+k}$. Also, if $\nu \in C(V(E^{m+k}, M))$, then

(3.12) $$A'_\nu Y = -(-\bar{A}_\nu Y + \bar{\nabla}^\perp_Y \nu)^T = (\bar{A}_\nu Y)^T .$$

At $x \in N^n$, we take an orthonormal basis $\{\nu_\lambda, \eta_a\}$ of $V_x(E^{m+k}, N)$ so that $\{\nu_\lambda\}$ and $\{\eta_a\}$ are bases of $V_x(M, N)$ and $V_x(E^{m+k}, M)$, respectively. Let $A_\lambda = A_{\nu_\lambda}$ and $\bar{A}_a = \bar{A}_{\eta_a}$. Denote by

$Q^{A'}$ the selfadjoint linear operator on V associated with the immersion $N^n \to E^{m+k}$ defined by (2.3). It follows from (3.11) and (3.12) that

$$(3.13) \qquad \operatorname{tr} Q^{A'} = \operatorname{tr} Q^A + \bar{A}(V) \,,$$

where

$$\bar{A}(V) = \sum_{a,i,\alpha} [2\langle \bar{A}_a e_i, e_\alpha \rangle^2 - \langle \bar{A}_a e_\alpha, e_\alpha \rangle \langle \bar{A}_a e_i, e_i \rangle] \,.$$

By using Lemma 3 with the condition C1, or Lemma 2 with C2, we then obtain $\bar{A}(V) \le 0$. If $\operatorname{tr} Q^A < 0$, then (3.13) implies $\operatorname{tr} Q^{A'} < 0$, and from (3.8), $\operatorname{tr} Q_{\mathcal{S}} < 0$. The proof is completed.

REMARK 2. If $M^m = E^m$, then $K_M = \lambda_0 = \mu_0 = 0$ in E^m, and Theorem 4 reduces to Theorem 1 of Xin [4].

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