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A REMARK ON MINIMAL FOLIATIONS OF LIE GROUPS

By

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1. Statement of the result.

Let G be a 3-dimensional Lie group, $\mathfrak{g}$ its Lie algebra of left invariant vector fields and $\langle,\rangle$ a left invariant metric on G. A 1 or 2-dimensional subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ gives rise to a foliated riemannian manifold $(G, \langle,\rangle, \mathcal{F}(\mathfrak{l}))$ (cf. [2]). Then we have the following

THEOREM. *Suppose that G is simply connected and nonunimodular. If $(G, \langle,\rangle, \mathcal{F}(\mathfrak{l}))$ is a minimal foliation and the metric $\langle,\rangle$ is bundle like, then, independent of the dimension of $\mathfrak{l}$, G is isomorphic to a semidirect product $S\times_{\tau}\boldsymbol{R}$ and $S(\subset G)$ is of negative constant Gaussian curvature. Here $S=\left\{\begin{pmatrix} a & \xi \\ 0 & 1/a\end{pmatrix}; a>0, \xi\in\boldsymbol{R}\right\}$, $\boldsymbol{R}$ the additive group of real numbers and τ a homomorphism of $\boldsymbol{R}$ into the group of automorphism of S.*

REMARK 1. *If* $\dim\mathfrak{l}=2$ (*resp.* $\dim\mathfrak{l}=1$) *in the above theorem, S (resp. $\boldsymbol{R}$) is the leaf through the identity of G.*

REMARK 2. *Suppose that G is unimodular and $(G, \langle,\rangle, \mathcal{F}(\mathfrak{l}))$ is a minimal foliation with bundle like metric $\langle,\rangle$, then all leaves are flat (cf.* [1]).

2. Definitions.

Let $(M, g, \mathcal{F})$ be an n-dimensional foliated riemannian manifold, that is, an n-dimensional riemannian manifold M with a riemannian metric g admitting a foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is given by an integrable subbundle E of the tangent bundle of M. The maximal connected integral submanifolds of E are called leaves. $(M, g, \mathcal{F})$ is called minimal if all leaves are minimal submanifolds of M, and the metric g is called bundle like metric with respect to $\mathcal{F}$ if for each point $x\in M$ there exists a neighborhood U of x, a $(n-p)$-dimensional ($p=\operatorname{rank} E$) riemannian manifold $(V, \bar{g})$ and a riemannian submersion φ: $(U, g\restriction U)\to(V, \bar{g})$ such that

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$\varphi^{-1}(y)$ is an intersection of U and some leaf.

Let G be an n-dimensional connected Lie group and $\mathfrak{g}$ the Lie algebra of left invariant vector fields on G. Taking a left invariant metric $\langle , \rangle$ on G and a p-dimensional subalgebra $\mathfrak{l}$, we have in a natural manner a foliated riemannian manifold $(G, \langle , \rangle, \mathcal{F}(\mathfrak{l}))$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for $\mathfrak{g}$ with $e_i \in \mathfrak{l}$ $(i=1, \cdots, p)$. If we denote by C_{ij}^k the structure constants of $\mathfrak{g}$ with respect to this basis: $[e_i, e_j]=\sum_{k=1}^n C_{ij}^k e_k$, then the metric $\langle , \rangle$ is bundle like with respect to $\mathcal{F}(\mathfrak{l})$ if and only if

$$C_{ij}^k + C_{ik}^j = 0, \quad 1 \leqq i \leqq p,\ p+1 \leqq j,\ k \leqq n, \tag{2.1}$$

and $(G, \langle , \rangle, \mathcal{F}(\mathfrak{l}))$ is minimal if and only if

$$\sum_{i=1}^p C_{ji}^i = 0, \qquad p+1 \leqq j \leqq n. \tag{2.2}$$

Let $\mathfrak{q}$, $\mathfrak{m}$ be Lie algebras, σ a representation of $\mathfrak{m}$ in $\mathfrak{q}$ such that $\sigma(Y)$ is a derivation of $\mathfrak{q}$ for all $Y \in \mathfrak{m}$. For $X, X' \in \mathfrak{q}$ and $Y, Y' \in \mathfrak{m}$, let

$$[(X, Y), (X', Y')] = ([X, X'] + \sigma(Y)X' - \sigma(Y')X, [Y, Y']).$$

It is then verified that this converts the vector space $\mathfrak{q} \times \mathfrak{m}$ into a Lie algebra. We denote it by $\mathfrak{q} \times_\sigma \mathfrak{m}$ and call it the semidirect product of $\mathfrak{q}$ with $\mathfrak{m}$ relative to σ. Let A and B be connected Lie groups and let $\tau(b \to \tau_b)$ be a homomorphism of B into the group of automorphism of A. We assume that the map $(a, b) \to \tau_b(a)$ is of class C^∞ from $A \times B$ into A. For $a_1, a_2 \in A$ and $b_1, b_2 \in B$, let $(a_1, b_1)(a_2, b_2) = (a_1 \tau_{b_1}(a_2), b_1 b_2)$. Then this converts the set $A \times B$ into a Lie group. We denote this Lie group by $A \times_\tau B$ and call it the semidirect product of A with B relative to τ.

3. Proof of Theorem

We consider first the case of $\dim \mathfrak{l}=2$. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathfrak{g}$ with respect to $\langle , \rangle$ such that $\mathfrak{l}$ is generated by e_2 and e_3. By (2.1) and (2.2) we see that the bundle-likeness of the metric and the minimality of the foliation implies the following relation.

$$\begin{aligned} [e_1, e_2] &= s e_2 + A e_3 \\ [e_1, e_3] &= B e_2 - s e_3 \\ [e_2, e_3] &= a e_2 + b e_3, \end{aligned} \tag{3.1}$$

where a, b, A, B, s are constants. Now we recall that a connected Lie group is called unimodular if the linear transformation $\mathrm{ad}(X)$ has trace zero for every X in the associated Lie algebra. Since G is nonunimodular we see that $[e_2, e_3] \neq 0$, and from the Jacobi identity it follows that $[e_1, [e_2, e_3]]=0$, that is,

$$as+bB=0, \qquad aA-bs=0. \tag{3.2}$$

Without loss of generality we may assume that $b\neq 0$. Then, putting $E_1=e_1$, $E_2=(1/b)e_2$, $E_3=[e_2, e_3]$, we have from (3.2)

$$\begin{aligned} &[E_1, E_2]=kE_3 \ (k=A/b^2) \\ &[E_1, E_3]=0, \qquad [E_2, E_3]=E_3. \end{aligned} \tag{3.3}$$

Let $\mathfrak{q}$ and $\mathfrak{m}$ denote the Lie algebras of S and $\boldsymbol{R}$ respectively. Choose a basis $\{X, Y\}$ for $\mathfrak{q}$ so that $[X, Y]=Y$, and let $\{Z\}$ be a basis for $\mathfrak{m}$. For the representation σ of $\mathfrak{m}$ in $\mathfrak{q}$ defined by $\sigma(Z)=\mathrm{ad}(-kY)$ we construct the semidirect product $\mathfrak{q}\times_\sigma\mathfrak{m}$. Then $X'=(X, 0)$, $Y'=(Y, 0)$ and $Z'=(0, Z)$ form a basis for $\mathfrak{q}\times_\sigma\mathfrak{m}$ and satisfy $[Z', X']=kY'$, $[Z', Y']=0$, $[X', Y']=Y'$,w hich implies together with (3.3) that $\mathfrak{g}$ and $\mathfrak{q}\times_\sigma\mathfrak{m}$ are isomorphic. Now define the homomorphism τ of $\boldsymbol{R}$ into the group of automorphism of S by $\tau_t(g)=a_t g a_t^{-1}$, $g\in S$, where $a_t=\exp t(-kY)$. Since G and $S\times_\tau\boldsymbol{R}$ are simply connected and their Lie algebras are isomorphic, G is isomorphic to $S\times_\tau\boldsymbol{R}$.

Let ∇ denote the riemannian connection associated with $\langle\,,\rangle$, then it holds that for every $X, Y, Z\in\mathfrak{g}$

$$2\langle\nabla_X Y, Z\rangle=\langle[X, Y], Z\rangle+\langle[Z, X], Y\rangle-\langle[Y, Z], X\rangle. \tag{3.4}$$

Let L denote the connected Lie subgroup of G with Lie algebra $\mathfrak{l}$. If we denote by $\bar{\nabla}$ the induced connection on L and by $\bar{R}$ its curvature tensor, then we have by (3.4)

$$\begin{aligned} &\bar{\nabla}_{e_2}e_2=-ae_3, \qquad \bar{\nabla}_{e_3}e_3=eb_2, \\ &\bar{\nabla}_{e_3}e_2=-be_3, \qquad \bar{\nabla}_{e_2}e_3=ae_2, \end{aligned}$$

and therefore

$$\begin{aligned} \langle\bar{R}(e_2, e_3)e_3, e_2\rangle &= -\langle\bar{\nabla}_{e_2}e_2, \bar{\nabla}_{e_3}e_3\rangle+\langle\bar{\nabla}_{e_2}e_3, \bar{\nabla}_{e_3}e_2\rangle-a\langle\bar{\nabla}_{e_2}e_3, e_2\rangle-b\langle\bar{\nabla}_{e_3}e_3, e_2\rangle \\ &= -a^2-b^2. \end{aligned}$$

This shows that the Gaussian curvature of L with respect to the induced connection equals $-|[e_2, e_3]|^2<0$.

Finally, in the case of $\dim\mathfrak{l}=1$, if $\{e_1, e_2, e_3\}$ is an orthonormal basis for $\mathfrak{g}$ with $e_1\in\mathfrak{l}$, then from (2.1), (2.2) it follows that for some constant A

$$[e_1, e_2]=Ae_3, \qquad [e_1, e_3]=-Ae_2.$$

So, putting $[e_2, e_3]=ce_1+ae_2+be_3$ and taking account of the nonunimodularity we have

$$a^2+b^2\neq 0, \qquad 0=[e_1, [e_2, e_3]]=-bAe_2+aAe_3,$$

which implies that $A=0$ and e_1 belongs to the center of $\mathfrak{g}$. Consequently, e_1 is parallel and $c=0$. Hence the bracket relation between e_1, e_2 and e_3 is given by (3.1) with $s=A=B=0$. Therefore the preceeding argument applies also in this case. Actually we have $G=S\times \boldsymbol{R}$ (direct product), and this is also a riemannian product and S is of negative constant Gaussian curvature. Now the proof is completed.

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References

[1] Milnor, J., Curvature of left invariant metrics on Lie groups, Advances in Math. **21** (1976), 293-329.

[2] Takagi, R. and Yorozu, S., Minimal foliations of Lie groups, Tôhoku Math. J. vol. 36, no. **4** (1984), 541-554.

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