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A GLIMPSE AT THE DUNKL–WILLIAMS INEQUALITY

M. S. MOSLEHIAN[1], F. DADIPOUR[2*], R. RAJIĆ[3] AND A. MARIĆ[4]

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ABSTRACT. In this paper we survey the results on the Dunkl–Williams inequality in normed linear spaces. These are related to the geometry of normed linear spaces, the characterizations of inner product spaces, some inequalities regarding operators on Hilbert spaces and elements of Hilbert C^*-modules.

1. DUNKL–WILLIAMS NORM INEQUALITY

In 1964, Dunkl and Williams [10] proved that the inequality

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|} \tag{1.1}$$

holds for all nonzero elements x, y in a (real) normed linear space $\mathscr{X}$. To see it, note that

$$\begin{aligned}\|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\| &\leq \|x\|\left\|\frac{x}{\|x\|}-\frac{y}{\|x\|}\right\|+\|x\|\left\|\frac{y}{\|x\|}-\frac{y}{\|y\|}\right\| \\ &= \|x-y\|+\left\|\frac{(\|y\|-\|x\|)y}{\|y\|}\right\| \\ &= \|x-y\|+\big|\|y\|-\|x\|\big| \\ &\leq 2\|x-y\|. \end{aligned} \tag{1.2}$$

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* Corresponding author.

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Similarly we have

$$\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x-y\|. \tag{1.3}$$

The inequality (1.1) now follows by adding (1.2) and (1.3).
Two years later, Kirk and Smiley [17] showed that the equality holds in (1.1) if and only if $x = y$.

The Dunkl–Williams inequality (1.1) gives the upper bound for the *angular distance*

$$\alpha[x,y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

between nonzero vectors x and y. The angular distance, also called the *Clarkson distance*, was introduced by Clarkson [5], in order to make a detailed analysis of the triangle inequality in uniformly convex spaces.

The Dunkl–Williams inequality has many interesting refinements, reverses and generalizations, which have been obtained over the years. Massera and Schäffer [22] proved that

$$\alpha[x,y] \leq \frac{2\|x-y\|}{\max\{\|x\|,\|y\|\}} \tag{1.4}$$

for all nonzero vectors $x, y \in \mathscr{X}$. This inequality is the strengthening of the Dunkl–Williams inequality and actually precedes it. Kelly [16] proved that for distinct nonzero vectors $x, y \in \mathscr{X}$ the equality holds in (1.4) if and only if x and y span the unit parallelogram with vertices $\pm\|y-x\|^{-1}(y-x)$ and $\pm\|x\|^{-1}x$ in the underlying normed linear space.

The best known refinement of the Dunkl–Williams inequality so far was obtained by Maligranda in [20], where upper and lower bounds for the angular distance between nonzero vectors $x, y \in \mathscr{X}$ were established:

$$\alpha[x,y] \leq \frac{\|x-y\| + |\,\|x\| - \|y\|\,|}{\max\{\|x\|,\|y\|\}}\,, \tag{1.5}$$

$$\alpha[x,y] \geq \frac{\|x-y\| - |\,\|x\| - \|y\|\,|}{\min\{\|x\|,\|y\|\}}\,. \tag{1.6}$$

By paying our attention to the proof of the inequality (1.1), it seems that Dunkl and Williams were implicitly aware of the inequality (1.5). An alternative proof of the inequality (1.6) was given by Mercer in [24].

Maligranda's inequalities (1.5) and (1.6) can be rewritten in the following forms:

$$\|x+y\| \leq \|x\| + \|y\| - \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|\right) \min\{\|x\|,\|y\|\} \tag{1.7}$$

and

$$\|x+y\| \geq \|x\| + \|y\| - \left(2 - \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|\right) \max\{\|x\|,\|y\|\}. \tag{1.8}$$

Note that (1.7) is a refinement of the usual norm triangle inequality.

Another improvement of the Dunkl–Williams inequality was given by Pečarić and Rajić [28] who showed that for nonzero vectors $x, y \in \mathscr{X}$ it holds

$$\alpha[x,y] \leq \frac{(2\|x-y\|^2 + 2(\|x\| - \|y\|)^2)^{\frac{1}{2}}}{\max\{\|x\|, \|y\|\}}. \tag{1.9}$$

The inequality (1.9) is stronger than the Massera–Schäffer inequality (1.4), but is weaker than Maligranda's inequality (1.5).

There is also a generalization of the Dunkl–Williams inequality for nonzero vectors of a normed linear space due to Al-Rashed [1], who proved the following result.

Theorem 1.1. [1] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space, and let $q \in \mathbb{R}$, $q > 0$. For nonzero $x, y \in \mathscr{X}$ the following statements hold.*
(i) *If $0 < q \leq 1$, then $\alpha[x,y] \leq 2^{1+\frac{1}{q}} \frac{\|x-y\|}{(\|x\|^q + \|y\|^q)^{\frac{1}{q}}}$.*
(ii) *If $q \geq 1$, then $\alpha[x,y] \leq 4 \frac{\|x-y\|}{(\|x\|^q + \|y\|^q)^{\frac{1}{q}}}$.*

The notion of angular distance can be generalized by considering the *p-angular distance* ($p \in \mathbb{R}$, $p \geq 0$) between nonzero elements x and y in a normed linear space $\mathscr{X}$ as

$$\alpha_p[x,y] := \|\|x\|^{p-1}x - \|y\|^{p-1}y\|$$

(see [5, 20]). The following estimate for p-angular distance is a generalization of the Massera–Schäffer inequality obtained by Maligranda in [20].

Theorem 1.2. [20] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space, and let $p \in \mathbb{R}$, $p \geq 0$. For nonzero $x, y \in \mathscr{X}$ the following statements hold.*
(i) If $0 \leq p \leq 1$, then $\alpha_p[x,y] \leq (2-p)\frac{\|x-y\|}{\max\{\|x\|,\|y\|\}^{1-p}}$.
(ii) If $p \geq 1$, then $\alpha_p[x,y] \leq p \max\{\|x\|, \|y\|\}^{p-1}\|x-y\|$.

In the case of a normed linear space, we have a generalization of the Dunkl–Williams inequality obtained by Dadipour and Moslehian [8].

Theorem 1.3. [8] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space, $p \in [0,1]$ and $q > 0$. Then the following inequality holds*

$$\alpha_p[x,y] \leq 2^{1+\frac{1}{q}} \frac{\|x-y\|}{(\|x\|^{(1-p)q} + \|y\|^{(1-p)q})^{\frac{1}{q}}}$$

for all nonzero elements x and y in $\mathscr{X}$.

A generalization of the Dunkl–Williams inequality and its reverse for finitely many elements of a normed linear space was established by Pečarić and Rajić in [30]. By modifying the method used in [15], they obtained the following result.

Theorem 1.4. [30] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space and $x_1, \dots, x_n$ nonzero elements of $\mathscr{X}$. Then we have*

$$\text{(i)} \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \leq \min_{1\leq i\leq n} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{j=1}^{n} |\, \|x_j\| - \|x_i\| \,| \right) \right\}, \tag{1.10}$$

$$\text{(ii)} \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \geq \max_{1\leq i\leq n} \left\{ \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{j=1}^{n} |\, \|x_j\| - \|x_i\| \,| \right) \right\}. \tag{1.11}$$

In the same paper they also characterized the case of equality in Theorem 1.4 for the elements of a strictly convex normed linear space as follows.

Theorem 1.5. [30] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space and $x_1, \dots, x_n$ nonzero elements of $\mathscr{X}$.*

(i) *The equality in (1.10) holds if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1, \dots, n\}$ and $v \in \mathscr{X}$ satisfying $\operatorname{sgn}(\|x_i\| - \|x_j\|)\frac{x_j}{\|x_j\|} = v$ for all $j \in \{1, \dots, n\}$ such that $\|x_j\| \neq \|x_i\|$ and $\sum_{j=1}^{n} x_j = \|\sum_{j=1}^{n} x_j\| v$.*

(ii) *The equality in (1.11) holds if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1, \dots, n\}$ and $v \in \mathscr{X}$ satisfying $\operatorname{sgn}(\|x_j\| - \|x_i\|)\frac{x_j}{\|x_j\|} = v$ for all $j \in \{1, \dots, n\}$ such that $\|x_j\| \neq \|x_i\|$ and $\sum_{j=1}^{n} \frac{x_j}{\|x_j\|} = \|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\| v$.*

Remark 1.6. Note that in the case when $n = 2$, by putting $x_1 = x$ and $x_2 = -y$ in Theorem 1.4, we get Maligranda's inequalities (1.5) and (1.6).

Dragomir [9] generalized Theorem 1.4 by providing upper and lower bounds for the norm of linear combination $\sum_{j=1}^{n} \alpha_j x_j$, in which α_j are scalars and $x_j \in \mathscr{X}$ for $j \in \{1, \dots, n\}$. His result was further extended by Zhao et al. in [34].

2. From the Dunkl–Williams inequality to characterization of inner product spaces

There are a lot of significant natural geometric properties, which fail in general normed spaces, such as non Euclidean ones. Some of these interesting properties hold just when the space is an inner product one. This is the most important motivation for studying characterizations of inner product spaces.

The first norm characterization of inner product spaces was given by Fréchet [11] in 1935. He proved that a normed space $(\mathscr{X}, \|\cdot\|)$ is an inner product one if and only if

$$\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 - \|x + y\|^2 - \|y + z\|^2 - \|x + z\|^2 = 0$$

for all $x, y, z \in \mathscr{X}$. In 1936, Jordan and von Neumann [14] showed that a normed space $(\mathscr{X}, \|\cdot\|)$ is an inner product one if and only if the parallelogram law $\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$ holds for all $x, y \in \mathscr{X}$. Since then, the problem of finding necessary and sufficient conditions for a normed space to be an inner product one has been investigated by many mathematicians who considered some geometric aspects of underlying spaces. The interested reader is referred to [2, 31, 27] and references therein.

There are interesting norm inequalities connected with characterizations of inner product spaces. One of the celebrated characterizations of inner product spaces was based on the Dunkl–Williams inequality. First we note that the constant 4 in inequality (1.1) is the best possible choice in normed spaces.
To show this, consider $\mathscr{X} = \mathbb{R}^2$ with the norm of $x = (x_1, x_2)$ given by $\|x\|_1 = |x_1| + |x_2|$. Take $x = (1, \varepsilon)$ and $y = (1, 0)$, where $\varepsilon > 0$ is small. Then

$$\alpha[x, y]\frac{\|x\|_1 + \|y\|_1}{\|x - y\|_1} = \frac{4 + 2\varepsilon}{1 + \varepsilon} \longrightarrow 4 \quad (\text{as } \varepsilon \longrightarrow 0).$$

If the norm of $x = (x_1, x_2)$ is given by $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, one can take the vectors $x = (1, 1)$ and $y = (1 - \varepsilon, 1 + \varepsilon)$, where $\varepsilon > 0$ is small enough, to show that

$$\alpha[x, y]\frac{\|x\|_\infty + \|y\|_\infty}{\|x - y\|_\infty} = \frac{2(2 + \varepsilon)}{1 + \varepsilon} \longrightarrow 4 \quad (\text{as } \varepsilon \longrightarrow 0).$$

Dunkl and Williams proved that the constant 4 can be replaced by 2 if $\mathscr{X}$ is an inner product space. To prove this fact (see, e.g. [10]), first note that

$$\begin{aligned}
\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 &= \left\langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\rangle \\
&= 2 - 2\mathrm{Re}\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \\
&= \frac{1}{\|x\|\,\|y\|}\left(2\|x\|\,\|y\| - 2\mathrm{Re}\,\langle x, y\rangle\right) \\
&= \frac{1}{\|x\|\,\|y\|}\left(\|x - y\|^2 - (\|x\| - \|y\|)^2\right) .
\end{aligned}$$

Hence

$$\begin{aligned}
&\|x - y\|^2 - \frac{1}{4}(\|x\| + \|y\|)^2\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 = \\
&\qquad \frac{(\|x\| - \|y\|)^2}{4\|x\|\,\|y\|}\left((\|x\| + \|y\|)^2 - \|x - y\|^2\right) \geq 0\,.
\end{aligned}$$

In 1964, Kirk and Smiley [17] showed that if the inequality

$$\alpha[x, y] \leq 2\frac{\|x - y\|}{\|x\| + \|y\|} \tag{2.1}$$

holds for all nonzero elements x and y of a normed linear space $\mathscr{X}$, then $\mathscr{X}$ is an inner product space. In the same work they also showed that the equality holds in (2.1) if and only if $\|x\| = \|y\|$ or $\|y\|x + \|x\|y = 0$. To do this, they used Lorch's characterization of inner product spaces (see [19]).

Jiménez–Melado, Llorens–Fuster and Mazcuñán–Navarro [13] introduced the *Dunkl–Williams constant* of a normed linear space $\mathscr{X}$ as

$$\mathrm{DW}(\mathscr{X}) := \sup\left\{\alpha[x, y]\frac{\|x\| + \|y\|}{\|x - y\|} :\ x, y \in \mathscr{X},\, x \neq 0,\, y \neq 0,\, x \neq y\right\}.$$

Observe that $2 \le \mathrm{DW}(\mathscr{X}) \le 4$ for every normed linear space $\mathscr{X}$, and $\mathrm{DW}(\mathscr{X}) = 2$ precisely when $\mathscr{X}$ is an inner product space. We have shown that $\mathrm{DW}((\mathbb{R}, \|\cdot\|_1)) = \mathrm{DW}((\mathbb{R}, \|\cdot\|_\infty)) = 4$, so the extreme value 4 can be achieved as well. In fact, the Dunkl–Williams constant $\mathrm{DW}(\mathscr{X})$ measures "how much" a space $\mathscr{X}$ is close (or far) to be an inner product one (cf. [13]).

It is known that every Hilbert space is uniformly nonsquare. Moreover, among all Banach spaces one can characterize the uniformly nonsquare ones by means of the Dunkl–Williams constant. Namely, a Banach space $\mathscr{X}$ is uniformly nonsquare if and only if $\mathrm{DW}(\mathscr{X}) < 4$. This result was proved by Baronti and Papini [4]. Jiménez–Melado et al. proved in [13] that for every Banach space $\mathscr{X}$, the inequalities

$$\max\{2\varepsilon_0(\mathscr{X}), 4\rho'_{\mathscr{X}}(0), 2\} \le \mathrm{DW}(\mathscr{X}) \le 2 + J(\mathscr{X})$$

hold, where $\varepsilon_0(\mathscr{X})$, $\rho'_{\mathscr{X}}(0)$, and $J(\mathscr{X})$ denote the characteristic of convexity, the characteristic of smoothness, and the James constant of $\mathscr{X}$, respectively, and obtained some geometric properties of Banach spaces in terms of the Dunkl–Williams constant.

In [24], Mercer showed that two independent vectors x and y in an inner product space $\mathscr{X}$ for which $\|x\| \ne \|y\|$ satisfy the following refinement of the inequality (2.1),

$$\alpha[x, y] \le 2\frac{\|x-y\|}{\|x\|+\|y\|} - t,$$

where

$$0 < t = \frac{2\left(\frac{\|x\|-\|y\|}{\|x\|+\|y\|}\right)^2\left(2 - \frac{2\|x-y\|}{\|x\|+\|y\|}\right)}{\frac{2\|x-y\|}{\|x\|+\|y\|} - \left(\frac{\|x\|-\|y\|}{\|x\|+\|y\|}\right)^2 + \sqrt{\frac{4\|x-y\|^2}{(\|x\|+\|y\|)^2} + \frac{(\|x\|-\|y\|)^4}{(\|x\|+\|y\|)^4} - 4\frac{(\|x\|-\|y\|)^2}{(\|x\|+\|y\|)^2}}}.$$

In 1993, Al-Rashed [1] generalized the Kirk–Smiley characterization of inner product spaces. The result can be reformulated as follows.

Theorem 2.1. [1] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space, and $q > 0$. Then the following inequality*

$$\alpha[x, y] \le 2^{\frac{1}{q}} \frac{\|x-y\|}{(\|x\|^q + \|y\|^q)^{\frac{1}{q}}} \quad (x, y \ne 0) \tag{2.2}$$

holds if and only if the given norm is induced by an inner product.

Dadipour and Moslehian [8] extended the Kirk–Smiley characterization by using the notion of p-angular distance ($p \in [0,1)$). They provided a suitable extension of the inequality (2.2), for which the given norm is induced by an inner product.

Theorem 2.2. [8] *Let $(\mathscr{X}, \|\cdot\|)$ be a normed linear space, and $p \in [0,1)$. Then the following statements are mutually equivalent*:

(i) $\alpha_p[x, y] \le 2^{\frac{1}{q}} \frac{\|x-y\|}{(\|x\|^{(1-p)q} + \|y\|^{(1-p)q})^{\frac{1}{q}}}$ *$(x, y \ne 0)$, for all $q \in (0, 1]$;*

(ii) $\alpha_p[x,y] \leq 2^{\frac{1}{q}} \frac{\|x-y\|}{(\|x\|^{(1-p)q}+\|y\|^{(1-p)q})^{\frac{1}{q}}}$ $(x,y\neq 0)$, *for some* $q>0$;

(iii) $(\mathscr{X}, \|\cdot\|)$ *is an inner product space.*

Proof. We shall only give a proof of (ii) $\Rightarrow$ (iii). For this, we need a result obtained by Lorch in [19], which states that a real normed linear space $(\mathscr{X}, \|\cdot\|)$ is an inner product one if and only if for all $x,y\in\mathscr{X}\setminus\{0\}$ satisfying $\|x\|=\|y\|$, the inequality $\|x+y\|\leq\|\gamma x+\gamma^{-1}y\|$ must hold for all real $\gamma\neq 0$.

Let us now take $\gamma\neq 0$, and $x,y\in\mathscr{X}\setminus\{0\}$ satisfying $\|x\|=\|y\|$. By Lorch's characterization, it is enough to prove that $\|x+y\|\leq\|\gamma x+\gamma^{-1}y\|$.

Let $n\in\mathbb{N}\cup\{0\}$. Applying the inequality (ii) to $\gamma^{p^n}x$ and $-\gamma^{-p^n}y$ for x and y, respectively, we obtain

$$\alpha_p[\gamma^{p^n}x, -\gamma^{-p^n}y] \leq 2^{\frac{1}{q}} \frac{\|\gamma^{p^n}x+\gamma^{-p^n}y\|}{(\|\gamma^{p^n}x\|^{(1-p)q}+\|\gamma^{-p^n}y\|^{(1-p)q})^{\frac{1}{q}}}.$$

For $\gamma>0$, it follows from the definition of α_p that

$$\left\|\frac{\gamma^{p^n}x}{\gamma^{p^n(1-p)}\|x\|^{1-p}}+\frac{\gamma^{-p^n}y}{\gamma^{-p^n(1-p)}\|y\|^{1-p}}\right\| \leq 2^{\frac{1}{q}} \frac{\|\gamma^{p^n}x+\gamma^{-p^n}y\|}{\|x\|^{1-p}\big(\gamma^{p^n(1-p)q}+\gamma^{-p^n(1-p)q}\big)^{\frac{1}{q}}},$$

or equivalently

$$\Big(\frac{\gamma^{p^n(1-p)q}+\gamma^{-p^n(1-p)q}}{2}\Big)^{\frac{1}{q}}\|\gamma^{p^{n+1}}x+\gamma^{-p^{n+1}}y\| \leq \|\gamma^{p^n}x+\gamma^{-p^n}y\|;$$

whence $0\leq\|\gamma^{p^{n+1}}x+\gamma^{-p^{n+1}}y\|\leq\|\gamma^{p^n}x+\gamma^{-p^n}y\|$, since $\gamma^{p^n(1-p)q}+\gamma^{-p^n(1-p)q}\geq 2$. Hence $\{\|\gamma^{p^n}x+\gamma^{-p^n}y\|\}_{n=0}^{\infty}$ is a convergent sequence of nonnegative real numbers. Thus we get

$$\|x+y\| = \lim_{n\to\infty}\|\gamma^{p^n}x+\gamma^{-p^n}y\| \leq \|\gamma x+\gamma^{-1}y\|$$

due to $0\leq p<1$.

Now let γ be negative. Put $\mu=-\gamma>0$. From the positive case we get

$$\|x+y\|\leq\|\mu x+\mu^{-1}y\| = \|\gamma x+\gamma^{-1}y\|.$$

□

3. Operator approaches to the Dunkl–Williams inequality

In this section we present several operator–valued versions of the Dunkl–Williams inequality which are related to some known operator–valued inequalities of Bohr's type.

By $\mathbb{B}(\mathscr{H})$ we denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$. The inner product on $\mathscr{H}$ will be denoted by $\langle\cdot,\cdot\rangle$. A self–adjoint operator $A\in\mathbb{B}(\mathscr{H})$ is *positive* if $\langle Ax,x\rangle\geq 0$ for all $x\in\mathscr{H}$. We write $A\geq 0$ if A is positive. If $A,B\in\mathbb{B}(\mathscr{H})$ are self-adjoint operators such that $A-B\geq 0$, we write $A\leq B$. By $|A|$ we denote the *absolute value* of $A\in\mathbb{B}(\mathscr{H})$, that is, $|A|=(A^*A)^{\frac{1}{2}}$, where A^* stands for the adjoint operator of A.

Pečarić and Rajić [28] introduced an operator–valued version of (1.9). They estimated $|\,A|A|^{-1}-B|B|^{-1}\,|$ for operator angular distance, where A and B are Hilbert space operators such that $|A|$ and $|B|$ are invertible. To do this, they used an operator version of the Bohr inequality due to Hirzallah [12], which states that for Hilbert space operators A, B and $r, s > 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$ the operator inequality $|A-B|^2 \le r|A|^2 + s|B|^2$ holds (see also [26, 23]). Moreover, the equality holds if and only if $(1-r)A = B$.

Theorem 3.1. [28] *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible, and let $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|A|A|^{-1} - B|B|^{-1}|^2 \le |A|^{-1}(r|A-B|^2 + s(|A| - |B|)^2)|A|^{-1} \,. \tag{3.1}$$

The equality holds in (3.1) if and only if

$$(r-1)(A-B)|A|^{-1} = B(|A|^{-1} - |B|^{-1}). \tag{3.2}$$

In the same paper, the authors fully described the case of equality in (3.1) when $r \ge 2$. In this case, the equality holds in (3.1) precisely when $A = B$. By adding one more condition on operators A and B, they also got a refinement of the equality condition (3.2) when $1 < r < 2$. They showed that, for invertible $(r-2)A - rB$, the equality holds in (3.1) if and only if $A = B$; while for invertible $|A - B|$ the equality holds precisely when $A = \frac{r}{r-2}B$.

Dadipour, Fujii and Moslehian [6] (see also [7]) presented an operator Dunkl–Williams inequality involving the p-angular distance.

Theorem 3.2. [6, 7] *Let $A, B \in \mathbb{B}(\mathscr{H})$ such that $|A|$ and $|B|$ are invertible, and $p, r, s \in \mathbb{R}$ where $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|A|A|^{p-1} - B|B|^{p-1}|^2 \le |A|^{p-1}(r|A-B|^2 + s|\,|B|^p|A|^{1-p} - |B|\,|^2)|A|^{p-1}.$$

The equality holds if and only if

$$(r-1)(A-B)|A|^{p-1} = B(|A|^{p-1} - |B|^{p-1}).$$

Proof.

$$\begin{aligned}
& |A|A|^{p-1} - B|B|^{p-1}|^2 \\
= \;& |A|A|^{p-1} - B|A|^{p-1} - B|B|^{p-1} + B|A|^{p-1}|^2 \\
= \;& |(A-B)|A|^{p-1} - B(|B|^{p-1} - |A|^{p-1})|^2 \\
\le \;& r|(A-B)|A|^{p-1}|^2 + s|B(|B|^{p-1} - |A|^{p-1})|^2 \\
= \;& r|A|^{p-1}|A-B|^2|A|^{p-1} + s(|B|^{p-1} - |A|^{p-1})|B|^2(|B|^{p-1} - |A|^{p-1}) \\
= \;& r|A|^{p-1}|A-B|^2|A|^{p-1} + s|A|^{p-1}(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|)|A|^{p-1} \\
= \;& |A|^{p-1}[r|A-B|^2 + s(|A|^{1-p}|B|^p - |B|)(|B|^p|A|^{1-p} - |B|)]|A|^{p-1} \\
= \;& |A|^{p-1}(r|A-B|^2 + s|\,|B|^p|A|^{1-p} - |B|\,|^2)|A|^{p-1}.
\end{aligned}$$

In addition, the equality holds if and only if

$$(r-1)(A-B)|A|^{p-1} = B(|A|^{p-1} - |B|^{p-1}).$$

□

Theorem 3.2 was recently improved by Saito and Tominaga [32] for the case when $p = 0$. Using the polar decompositions of operators A and B, they established the following result, in which the invertibility of $|A|$ and $|B|$ are not required anymore.

Theorem 3.3. [32] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be the operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|(U-V)|A||^2 \ \leq \ r|A-B|^2 + s(|A|-|B|)^2. \tag{3.3}$$

The equality holds in (3.3) if and only if

$$(r-1)(A-B) = V(|B|-|A|) \quad \textit{and} \quad U^*U = V^*V. \tag{3.4}$$

The following improvement of equality conditions (3.4) was also obtained in [32].

Theorem 3.4. [32] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be the operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$.*
(i) *If $r \geq 2$, then the equality holds in (3.3) if and only if $A = B$.*
(ii) *If $1 < r < 2$, then the equality holds in (3.3) if and only if*

$$A = B\Big(I - \frac{2}{2-r}W^*W\Big) \ \textit{and} \ |A| = |B|\Big(I + \frac{2r}{(2-r)s}W^*W\Big),$$

where W is the partial isometry which is appeared in the polar decomposition of $A-B$.

A similar type of Theorem 3.3 was obtained in [6].

Theorem 3.5. [6] *Let $A, B \in \mathbb{B}(\mathscr{H})$ be the operators with the polar decompositions $A = U|A|$ and $B = V|B|$, and let $p \in (0,1]$ and $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|(U|A|^p - V|B|^p)|A|^{1-p}|^2 \leq r|A-B|^2 + s|\,|B|^p|A|^{1-p} - |B|\,|^2\,. \tag{3.5}$$

The equality holds if and only if $(r-1)(A-B) = V(|B| - |B|^p|A|^{1-p})$.

Remark 3.6. Observe that (3.5) implies

$$|U|A|^p - V|B|^p|^2 \leq |A|^{p-1}(r|A-B|^2 + s|\,|B|^p|A|^{1-p} - |B|\,|^2)|A|^{p-1}.$$

This shows that in the case $p \in (0,1]$, the inequality from Theorem 3.2 can be expressed in the form in which the invertibility of $|A|$ and $|B|$ are not needed anymore.

In [7], the authors presented some necessary and sufficient conditions for the case of equality in (3.5). More precisely, they proved that if A, B and r, s, p are the same as in Theorem 3.5, for which

$$|(U|A|^p - V|B|^p)|A|^{1-p}|^2 = r|A-B|^2 + s||B|^p|A|^{1-p} - |B||^2, \tag{3.6}$$

then the following statements hold.
(i) $(r-1)|A-B|^2 = \frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2 - |B|^2\,.$
(ii) $|B| \leq (\frac{1}{r}|A|^{1-p}|B|^{2p}|A|^{1-p} + \frac{1}{s}|A|^2)^{\frac{1}{2}}\,.$
(iii) $(r-1)|A-B| = |\,|B|^p|A|^{1-p} - |B|\,|$ and $A-B = -VW|A-B|$, where W is the partial isometry which is appeared in the polar decomposition of $|B|^p|A|^{1-p} - |B|\,.$
Moreover, (3.6) and (iii) are equivalent.

4. The Dunkl–Williams inequality in inner product C^*-modules

The notion of Hilbert C^*-module is a generalization of that of Hilbert space in which the field of scalars $\mathbb{C}$ is replaced by a C^*-algebra. The basic theory of Hilbert C^*-modules can be found in [18, 33].

The formal definition is as follows.

A *right inner product C^*-module $\mathscr{X}$ over a C^*-algebra $\mathscr{A}$* (or a *right inner product $\mathscr{A}$-module*) is a right $\mathscr{A}$-module together with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ satisfying the conditions:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in \mathscr{X}$, $\alpha, \beta \in \mathbb{C}$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in \mathscr{X}$, $a \in \mathscr{A}$,
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in \mathscr{X}$,
(iv) $\langle x, x\rangle \geq 0$ for $x \in \mathscr{X}$,
(v) $\langle x, x\rangle = 0$ if and only if $x = 0$.

We can define a norm on $\mathscr{X}$ by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

An inner product $\mathscr{A}$-module which is complete with respect to its norm is called a *Hilbert C^*-module over $\mathscr{A}$*, or a *Hilbert $\mathscr{A}$-module*.

Clearly, every inner product space is an inner product $\mathbb{C}$-module. Every C^*-algebra can also be regarded as a Hilbert C^*-module over itself via $\langle a, b\rangle = a^*b$ $(a, b \in \mathscr{A})$.

For every $x \in \mathscr{X}$, the absolute value of x is defined as the unique positive square root of $\langle x, x\rangle \in \mathscr{A}$, that is, $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

By using the characterization of the triangle equality for the elements of an inner product C^*-module obtained in [3], Pečarić and Rajić [29] characterized equality attainedness for each of the inequalities (1.10) and (1.11).

Theorem 4.1. [29] *Let $\mathscr{X}$ be an inner product C^*-module over a C^*-algebra $\mathscr{A}$, and $x_1, \ldots, x_n$ nonzero elements of $\mathscr{X}$.*
(i) *If $\sum_{j=1}^n x_j \neq 0$, then*

$$\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| = \min_{1\leq i\leq n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n \big|\,\|x_j\| - \|x_i\|\,\big|\right)\right\}$$

if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1, \ldots, n\}$ and a state φ of $\mathscr{A}$ such that $\operatorname{sgn}(\|x_i\| - \|x_k\|)\sum_{j=1}^n \varphi(\langle x_j, x_k\rangle) = \left\|\sum_{j=1}^n x_j\right\| \|x_k\|$ for all $k \in \{1, \ldots, n\}$ satisfying $\|x_k\| \neq \|x_i\|$.
(ii) *If $\sum_{j=1}^n x_j = 0$, then*

$$\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| = \min_{1\leq i\leq n}\frac{1}{\|x_i\|}\sum_{j=1}^n \big|\,\|x_j\| - \|x_i\|\,\big|$$

if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i, k \in \{1, \ldots, n\}$ satisfying $\|x_i\| \neq \|x_k\|$ and a state φ of $\mathscr{A}$ such that $\operatorname{sgn}(\|x_i\|-\|x_j\|)\operatorname{sgn}(\|x_i\|-\|x_k\|)\varphi(\langle x_j, x_k\rangle) = \|x_j\|\|x_k\|$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$ satisfying $\|x_j\| \neq \|x_i\|$.

Theorem 4.2. [29] *Let $\mathscr{X}$ be an inner product C^*-module over a C^*-algebra $\mathscr{A}$, and $x_1, \dots, x_n$ nonzero elements of $\mathscr{X}$.*
(i) *If $\sum_{j=1}^n \frac{x_j}{\|x_j\|} \neq 0$, then*

$$\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| = \max_{1\leq i\leq n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n \left|\,\|x_j\| - \|x_i\|\,\right|\right)\right\}$$

if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i \in \{1, \dots, n\}$ and a state φ of $\mathscr{A}$ such that $\operatorname{sgn}(\|x_k\| - \|x_i\|)\sum_{j=1}^n \varphi\big(\langle \frac{x_j}{\|x_j\|}, x_k\rangle\big) = \big\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\big\|\|x_k\|$ for all $k \in \{1, \dots, n\}$ satisfying $\|x_k\| \neq \|x_i\|$.
(ii) *If $\sum_{j=1}^n \frac{x_j}{\|x_j\|} = 0$, then*

$$\max_{1\leq i\leq n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n \left|\,\|x_j\| - \|x_i\|\,\right|\right)\right\} = 0$$

if and only if $\|x_1\| = \cdots = \|x_n\|$ or there exist $i, k \in \{1, \dots, n\}$ satisfying $\|x_i\| \neq \|x_k\|$ and a state φ of $\mathscr{A}$ such that $\operatorname{sgn}(\|x_i\|-\|x_j\|)\operatorname{sgn}(\|x_i\|-\|x_k\|)\varphi(\langle x_j, x_k\rangle) = \|x_j\|\|x_k\|$ for all $j \in \{1, \dots, n\} \setminus \{k\}$ satisfying $\|x_j\| \neq \|x_i\|$.

Dadipour and Moslehian in [25] established a generalization of the Dunkl–Williams inequality and its reverse in the framework of inner product C^*-modules as follows.

Theorem 4.3. [25] *Let $\mathscr{X}$ be an inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. If $x_j \in \mathscr{X}$ and $a_j \in \mathscr{A}$ for $j = 1, \dots, n$ such that $a_j, a_j - a_i$ are scalar multiples of coisometries, then*

$$\text{(i)}\quad \left\|\sum_{j=1}^n x_j a_j\right\| \leq \min_{1\leq i\leq n}\left\{\left\|\sum_{j=1}^n x_j\right\| \|a_i\| + \sum_{j=1}^n \|x_j\|\ \|a_j - a_i\|\right\}, \tag{4.1}$$

$$\text{(ii)}\quad \left\|\sum_{j=1}^n x_j a_j\right\| \geq \max_{1\leq i\leq n}\left\{\left\|\sum_{j=1}^n x_j\right\| \|a_i\| - \sum_{j=1}^n \|x_j\|\ \|a_j - a_i\|\right\}. \tag{4.2}$$

Theorem 4.3 generalizes Theorem 1.4 as well as some results due to Dragomir [9] for the elements of inner product C^*-modules. The authors also described the case of equality in the inequality (4.1).

Theorem 4.4. [25] *Let $\mathscr{X}$ be an inner product C^*module over a unital C^*-algebra $\mathscr{A}$. Let $x_1, \dots, x_n$ be nonzero elements of $\mathscr{X}$ and $a_1, \dots, a_n$ nonzero elements of $\mathscr{A}$ such that $a_i \neq a_j$ for some i, j and the elements $a_j, a_j - a_i$ are scalar multiples of coisometries for all i, j.*
(i) *If $\sum_{j=1}^n x_j \neq 0$, then*

$$\left\|\sum_{j=1}^n x_j a_j\right\| = \min_{1\leq k\leq n}\left\{\left\|\sum_{j=1}^n x_j\right\| \|a_k\| + \sum_{j=1}^n \|x_j\|\ \|a_j - a_k\|\right\}$$

if and only if there exist $i \in \{1, \ldots, n\}$ *and a state* φ *of* $\mathscr{A}$ *such that*

$$\sum_{j=1}^{n} \varphi(a_i^* \langle x_j, x_k\rangle (a_k - a_i)) = \left\|\sum_{j=1}^{n} x_j\right\| \|a_i\| \; \|x_k\| \|a_k - a_i\|$$

for all $k \in \{1, \ldots, n\}$ *satisfying* $a_k \neq a_i$.
(ii) *If* $\sum_{j=1}^{n} x_j = 0$, *then*

$$\left\|\sum_{j=1}^{n} x_j a_j\right\| = \min_{1\le k\le n} \left\{\sum_{j=1}^{n} \|x_j\| \; \|a_j - a_k\|\right\}$$

if and only if there exist $i, l \in \{1, \ldots, n\}$ *satisfying* $a_i \neq a_l$ *and a state* φ *of* $\mathscr{A}$ *such that*

$$\varphi((a_l^* - a_i^*)\langle x_l, x_k\rangle (a_k - a_i)) = \|a_l - a_i\| \|a_k - a_i\| \|x_l\| \|x_k\|$$

for all $k \in \{1, \ldots, n\} \setminus \{l\}$ *satisfying* $a_k \neq a_i$.

The following result was obtained by applying Theorem 4.4 to scalar multiples of the identity. It characterizes the equality case in an inequality due to Dragomir [9] in inner product C^*-modules.

Corollary 4.5. [25] *Let* $\mathscr{X}$ *be an inner product* C^**-module over a unital* C^**-algebra* $\mathscr{A}$. *Let* $x_1, \ldots, x_n$ *be nonzero elements of* $\mathscr{X}$ *and* $\alpha_1, \ldots, \alpha_n$ *nonzero scalars satisfying* $\alpha_i \neq \alpha_j$ *for some i,j.*
(i) *If* $\sum_{j=1}^{n} x_j \neq 0$, *then*

$$\left\|\sum_{j=1}^{n} \alpha_j x_j\right\| = \min_{1\le k\le n} \left\{|\alpha_k| \left\|\sum_{j=1}^{n} x_j\right\| + \sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\|\right\}$$

if and only if there exist $i \in \{1, \ldots, n\}$ *and a state* φ *of* $\mathscr{A}$ *such that*

$$\operatorname{cis}\left(\arg \bar{\alpha}_i + \arg(\alpha_k - \alpha_i)\right) \sum_{j=1}^{n} \varphi\langle x_j, x_k\rangle = \left\|\sum_{j=1}^{n} x_j\right\| \|x_k\|$$

for all $k \in \{1, \ldots, n\}$ *satisfying* $\alpha_k \neq \alpha_i$.
(ii) *If* $\sum_{j=1}^{n} x_j = 0$, *then*

$$\left\|\sum_{j=1}^{n} \alpha_j x_j\right\| = \min_{1\le k\le n} \left\{\sum_{j=1}^{n} |\alpha_j - \alpha_k| \|x_j\|\right\}$$

if and only if there exist $i, l \in \{1, \ldots, n\}$ *satisfying* $\alpha_i \neq \alpha_l$ *and a state* φ *of* $\mathscr{A}$ *such that*

$$\operatorname{cis}\left(\arg(\bar{\alpha}_l - \bar{\alpha}_i) + \arg(\alpha_k - \alpha_i)\right) \varphi\langle x_l, x_k\rangle = \|x_l\| \|x_k\|$$

for all $k \in \{1, \ldots, n\} \setminus \{l\}$ *satisfying* $\alpha_k \neq \alpha_i$.

Finally, with connection to the Dunkl–Williams inequality in the framework of Hilbert C^*-modules, we shall mention the following result due to Dadipour and Moslehian [7]. It is also a generalization of Theorem 3.1 and Theorem 3.2.

Theorem 4.6. *Let x, y be elements of a Hilbert C^*-module $\mathscr{X}$ such that $|x|$ and $|y|$ are invertible, and $p, r, s \in \mathbb{R}$ where $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$|x|x|^{p-1} - y|y|^{p-1}|^2 \leq |x|^{p-1}[r|x-y|^2 + s|\,|y|^p|x|^{1-p} - |y|\,|^2]|x|^{p-1}.$$

The equality holds if and only if $(r-1)(x-y)|x|^{p-1} = y(|x|^{p-1} - |y|^{p-1})$.

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[1,2] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.

E-mail address: moslehian@ferdowsi.um.ac.ir, moslehian@ams.org

E-mail address: dadipoor@yahoo.com

URL: http://www.um.ac.ir/~moslehian/

[3] Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.

E-mail address: rajna.rajic@zg.t-com.hr

[4] Sortina 1A, 10000 Zagreb, Croatia.

E-mail address: ana.maric86@gmail.com