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ON A JENSEN–MERCER OPERATOR INEQUALITY

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Abstract. A general formulation of the Jensen–Mercer operator inequality for operator convex functions, continuous fields of operators and unital fields of positive linear mappings is given. As consequences, a global upper bound for Jensen's operator functional and some properties of the quasi-arithmetic operator means and quasi-arithmetic operator means of Mercer's type are obtained.

1. Introduction

Inspired by Mercer's variant of Jensen's inequality [7]

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)\leq f\left(a\right)+f\left(b\right)-\frac{1}{W_n}\sum_{i=1}^{n}w_if\left(x_i\right),$$

for a convex function $f:[a,b]\rightarrow\mathbb{R}$, real numbers $x_1,\ldots,x_n\in[a,b]$ and positive real numbers $w_1,\ldots,w_n$, where $W_n=\sum_{i=1}^{n}w_i$, the following variant of Jensen's operator inequality for a convex function $f\in C\left([m,M]\right)$, selfadjoint operators $A_1,...,A_k\in\mathcal{B}\left(H\right)$ with spectra in $[m,M]$ and positive linear maps $\Phi_1,...,\Phi_k\in\mathbf{P}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ with $\sum_{j=1}^{k}\Phi_j\left(\mathbf{1}\right)=\mathbf{1}$ was proved in [5]

$$f\left(\left(m+M\right)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)\right)\leq\left(f\left(m\right)+f\left(M\right)\right)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(f\left(A_j\right)\right). \qquad (1.1)$$

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Moreover, in the same paper the following series of inequalities was proved

$$\begin{aligned} & f\left((m+M)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)\right) \\ & \leq \frac{M\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}f(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m\mathbf{1}}{M-m}f(m) \\ & \leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(f\left(A_j\right)\right). \end{aligned}$$

We assume that H and K are Hilbert spaces, $\mathcal{B}\left(H\right)$ and $\mathcal{B}\left(K\right)$ are C^*-algebras of all bounded operators on the appropriate Hilbert spaces, $\mathbf{P}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ is the set of all positive linear mappings from $\mathcal{B}\left(H\right)$ to $\mathcal{B}\left(K\right)$ and $C\left([m,M]\right)$ is the set of all real valued continuous functions defined on an interval $[m,M]$.

Inequality (1.1) is called the Jensen–Mercer operator inequality and its refinement for an operator convex function $f\in C\left([m,M]\right)$ is also given in [6]

$$\begin{aligned} f\left((m+M)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)\right) & \leq \sum_{j=1}^{k}\Phi_j\left(f\left((m+M)\mathbf{1}-A_j\right)\right) \\ & \leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(f\left(A_j\right)\right), \end{aligned}$$

or, more precisely, the following series of inequalities was proved

$$\begin{aligned} & f\left((m+M)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)\right) \\ & \leq \sum_{j=1}^{k}\Phi_j\left(f\left((m+M)\mathbf{1}-A_j\right)\right) \\ & \leq \frac{M\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(A_j\right)}{M-m}f(M)+\frac{\sum_{j=1}^{k}\Phi_j\left(A_j\right)-m\mathbf{1}}{M-m}f(m) \\ & \leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1}-\sum_{j=1}^{k}\Phi_j\left(f\left(A_j\right)\right). \end{aligned}$$

In this paper we give a general form of these results for continuous fields of operators and unital fields of positive linear mappings, and some applications.

In Section 2 we give a general form of the Jensen–Mercer operator inequality for convex functions and its refinement for operator convex functions. In Section 3 we give a global upper bound for Jensen's operator functional and some properties of the quasi-arithmetic operator means and quasi-arithmetic operator means of Mercer's type. The obtained global upper bound for Jensen's operator functional is analogous to that for Jensen's functional in the real discrete case, given in [8] and [1].

2. Main result

Let T be a locally compact Hausdorff space, and let $\mathcal{A}$ be a C^*-algebra of operators on a Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \to x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \to \|x_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t \,\mathrm{d}\mu(t)$, which is the unique element in $\mathcal{A}$ such that $\varphi\left(\int_T x_t \,\mathrm{d}\mu(t)\right) = \int_T \varphi(x_t)\,\mathrm{d}\mu(t)$ for every linear functional φ in the norm dual $\mathcal{A}^*$, cf. [2].

Let $(\phi_t)_{t\in T}$ be a field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$ of operators on a Hilbert space K. We say that such a field is continuous if the function $t \to \phi_t(x)$ is continuous for every $x \in \mathcal{A}$. If in addition the C^*-algebras are unital and $\phi_t(\mathbf{1})$ is integrable with integral $\mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is *unital*.

The following general form of Jensen's operator inequality was proved in [3].

Theorem A. *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is an unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \le \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \tag{2.1}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I.

Note here that inequality (2.1) holds for the class of operator convex functions which is a proper subclass of the class of convex function. However, our general form of the Jensen–Mercer operator inequality holds for the larger class of all convex functions.

Theorem 2.1. *Let $f \in C([m, M])$ be an operator convex function, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $(\phi_t)_{t\in T}$ is an unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then*

$$\begin{aligned} & f\left((m+M)\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \\ & \le \int_T \phi_t(f((m+M)\mathbf{1} - x_t))\,\mathrm{d}\mu(t) \\ & \le (f(m)+f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \end{aligned} \tag{2.2}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ *of self-adjoint elements in* $\mathcal{A}$ *with spectra contained in* $[m, M]$*. Moreover, the series of inequalities*

$$\begin{aligned} & f\left((m+M)\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \\ & \leq \int_T \phi_t\left(f\left((m+M)\mathbf{1} - x_t\right)\right)\mathrm{d}\mu(t) \\ & \leq \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(M) + \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} f(m) \\ & \leq (f(m) + f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \end{aligned} \tag{2.3}$$

holds. If $f \in C([m, M])$ *is operator concave, then the inequalities in* (2.2) *and* (2.3) *are reversed.*

Proof. Since f is continuous and operator convex, the same is also true for the function $g : [m, M] \to \mathbb{R}$ defined by $g(z) = f(m + M - z)$. From Theorem A for function g follows the first inequality in (2.2) and (2.3). Since f is operator convex it is also convex. Thus, the inequality

$$f(z) \leq \frac{z-m}{M-m} f(M) + \frac{M-z}{M-m} f(m) \tag{2.4}$$

holds for every $z \in [m, M]$. Using functional calculus and taking $z = x_t$, from (2.4) follows

$$f(x_t) \leq \frac{x_t - m\mathbf{1}}{M-m} f(M) + \frac{M\mathbf{1} - x_t}{M-m} f(m).$$

Applying the unital positive linear mappings ϕ_t and integrating, we obtain

$$\int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \leq \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} f(M) + \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(m). \tag{2.5}$$

Using inequality (2.5) for function g, and then for function f we obtain

$$\begin{aligned} & \int_T \phi_t\left(f\left((m+M)\mathbf{1} - x_t\right)\right)\mathrm{d}\mu(t) \\ & = \int_T \phi_t(g(x_t))\,\mathrm{d}\mu(t) \\ & \leq \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} g(m) + \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} g(M) \\ & = \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(M) + \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} f(m) \\ & = (f(m) + f(M))\mathbf{1} - \frac{M\mathbf{1} - \int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(m) - \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t) - m\mathbf{1}}{M-m} f(M) \\ & \leq (f(m) + f(M))\mathbf{1} - \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t). \end{aligned}$$

The last statement follows immediately from the fact that if f is operator concave then $-f$ is operator convex. □

Remark 2.2. If $f \in C([m,M])$ is convex, then it can be shown that the general form of the Jensen–Mercer operator inequality

$$f\left((m+M)\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \le (f(m)+f(M))\mathbf{1}-\int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t), \tag{2.6}$$

and the series of inequalities

$$\begin{aligned} &f\left((m+M)\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right) \\ &\le \frac{M\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m} f(M) + \frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)-m\mathbf{1}}{M-m} f(m) \\ &\le (f(m)+f(M))\mathbf{1}-\int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) \end{aligned} \tag{2.7}$$

also hold. If $f \in C([m,M])$ is concave, then the inequalities in (2.6) and (2.7) are reversed.

3. Applications

From Theorem A we have

$$\mathbf{0} \le \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) - f\left(\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right)$$

which we can consider as the global (not depending on $(\phi_t)_{t\in T}$ and $(x_t)_{t\in T}$) lower bound zero for Jensen's operator functional

$$\mathcal{J}(f,(\phi_t)_{t\in T},(x_t)_{t\in T}) := \int_T \phi_t(f(x_t))\,\mathrm{d}\mu(t) - f\left(\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right)$$

defined for an operator convex function f, an unital field of positive linear mappings $(\phi_t)_{t\in T}$ and a bounded continuous field $(x_t)_{t\in T}$ as in Theorem A. Using our results from Theorem 2.1 we can get an upper global bound for Jensen's operator functional. In case f is an operator concave function, zero is the upper bound for Jensen's operator functional and from Theorem 2.1 we can get its lower bound.

Theorem 3.1. *Let $f \in C([m,M])$ be an operator convex function, and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Let $(\phi_t)_{t\in T}$ be an unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ and let $(x_t)_{t\in T}$ be a bounded continuous field of self-adjoint elements in $\mathcal{A}$ with spectra contained in $[m,M]$. Then*

$$\mathcal{J}(f,(\phi_t)_{t\in T},(x_t)_{t\in T}) \le (f(m)+f(M))\mathbf{1} - 2f\left(\frac{1}{2}(m+M)\mathbf{1}\right). \tag{3.1}$$

If f is operator concave, then the inequality in (3.1) is reversed.

Proof. From Theorem 2.1 we have

$$\int_T \phi_t\left(f\left(x_t\right)\right) \mathrm{d}\mu\left(t\right) \leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1} - f\left(\left(m+M\right)\mathbf{1} - \int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right). \tag{3.2}$$

Since f is operator convex,

$$\begin{aligned} &\frac{1}{2} f\left(\left(m+M\right)\mathbf{1} - \int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) + \frac{1}{2} f\left(\int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) \\ &\geq f\left(\frac{1}{2}\left[\left(m+M\right)\mathbf{1} - \int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right] + \frac{1}{2}\int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) \\ &= f\left(\frac{1}{2}\left(m+M\right)\mathbf{1}\right). \end{aligned}$$

Hence,

$$f\left(\left(m+M\right)\mathbf{1} - \int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) + f\left(\int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) \geq 2f\left(\frac{1}{2}\left(m+M\right)\mathbf{1}\right). \tag{3.3}$$

Now, combining inequalities (3.2) and (3.3) we have

$$\begin{aligned} &\int_T \phi_t\left(f\left(x_t\right)\right) \mathrm{d}\mu\left(t\right) - f\left(\int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) \\ &\leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1} \\ &\qquad - \left[f\left(\left(m+M\right)\mathbf{1} - \int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right) + f\left(\int_T \phi_t\left(x_t\right) \mathrm{d}\mu\left(t\right)\right)\right] \\ &\leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1} - 2f\left(\frac{1}{2}\left(m+M\right)\mathbf{1}\right). \end{aligned}$$

The last statement follows immediately from the fact that if f is operator concave then $-f$ is operator convex. □

For the discrete case we can conclude the following.

Corollary 3.2. *Let $A_1, ..., A_k \in \mathcal{B}\left(H\right)$ be selfadjoint operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, ..., \Phi_k \in \mathbf{P}\left[\mathcal{B}\left(H\right), \mathcal{B}\left(K\right)\right]$ positive linear maps with $\sum_{j=1}^k \Phi_j\left(\mathbf{1}\right) = \mathbf{1}$. If $f \in C\left([m, M]\right)$ is operator convex on $[m, M]$, then*

$$\mathcal{J}_k\left(f, \boldsymbol{A}, \boldsymbol{\Phi}\right) \leq \left(f\left(m\right)+f\left(M\right)\right)\mathbf{1} - 2f\left(\frac{1}{2}\left(m+M\right)\mathbf{1}\right), \tag{3.4}$$

where $\boldsymbol{A} = \left(A_1, ..., A_k\right)$, $\boldsymbol{\Phi} = \left(\Phi_1, ..., \Phi_k\right)$ and

$$\mathcal{J}_k\left(f, \boldsymbol{A}, \boldsymbol{\Phi}\right) = \sum_{j=1}^k \Phi_j\left(f\left(A_j\right)\right) - f\left(\sum_{j=1}^k \Phi_j\left(A_j\right)\right).$$

If f is operator concave, then the inequality in (3.4) is reversed.

Remark 3.3. It is interesting that analogous result in the real discrete case is proved in [8], although it follows from the series of inequalities given in [4] (see also [1]). In the real case one can also obtain that

$$\sup_{m\leq z\leq M}\left\{\frac{z-m}{M-m}f\left(M\right)+\frac{M-z}{M-m}f\left(m\right)-f\left(z\right)\right\}$$

is another upper bound for Jensen's functional which is better than

$$f\left(m\right)+f\left(M\right)-2f\left(\frac{m+M}{2}\right)$$

(see for example [1, Lemma 2.5]), but the second one is simpler.

Using our results from Theorem 2.1 and Theorem 3.1 we can also get some properties of the quasi-arithmetic operator means and quasi-arithmetic operator means of Mercer's type defined for a strictly monotone function $\varphi\in C\left([m,M]\right)$, an unital field of positive linear mappings $\left(\phi_t\right)_{t\in T}$ and a bounded continuous field $\left(x_t\right)_{t\in T}$, respectively as

$$\begin{aligned}
M_{\varphi}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right)&=\varphi^{-1}\left(\int_T\phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)\right),\\
\widetilde{M}_{\varphi}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right)&=\varphi^{-1}\left(\left(\varphi\left(m\right)+\varphi\left(M\right)\right)\mathbf{1}-\int_T\phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)\right).
\end{aligned}$$

Theorem 3.4. *Let $\varphi,\psi\in C\left([m,M]\right)$ be two strictly monotone functions.*

(i) *If either $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator increasing, or $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator decreasing, then*

$$\begin{aligned}
&\widetilde{M}_{\varphi}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right)\\
&\leq\psi^{-1}\left(\int_T\phi_t\left(\left(\psi\circ\varphi^{-1}\right)\left(\left(\varphi\left(m\right)+\varphi\left(M\right)\right)\mathbf{1}-\varphi\left(x_t\right)\right)\right)\mathrm{d}\mu\left(t\right)\right)\\
&\leq\psi^{-1}\Big(\frac{\varphi\left(M\right)\mathbf{1}-\int_T\phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(M\right)\\
&\qquad+\frac{\int_T\phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)-\varphi\left(m\right)\mathbf{1}}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(m\right)\Big)\\
&\leq\widetilde{M}_{\psi}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right).
\end{aligned}\tag{3.5}$$

(ii) *If either $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing, or $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, then the inequalities in (3.5) are reversed.*

Proof. Suppose that $\psi\circ\varphi^{-1}$ is operator convex. If in Theorem 2.1 we let $f=\psi\circ\varphi^{-1}$ and replace x_t, m and M with $\varphi\left(x_t\right)$, $\varphi\left(m\right)$ and $\varphi\left(M\right)$ respectively, then we obtain

$$
\begin{aligned}
&\psi\left(\varphi^{-1}\left((\varphi(m)+\varphi(M))\,\mathbf{1}-\int_T \phi_t\left(\varphi\left(x_t\right)\right) \mathrm{d}\mu\left(t\right)\right)\right) \\
&\leq \int_T \phi_t\left(\left(\psi\circ\varphi^{-1}\right)\left(\left(\varphi\left(m\right)+\varphi\left(M\right)\right)\mathbf{1}-\varphi\left(x_t\right)\right)\right) \mathrm{d}\mu\left(t\right) \\
&\leq \frac{\varphi\left(M\right)\mathbf{1}-\int_T \phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(M\right)+\frac{\int_T \phi_t\left(\varphi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right)-\varphi\left(m\right)\mathbf{1}}{\varphi\left(M\right)-\varphi\left(m\right)}\psi\left(m\right) \\
&\leq \left(\psi\left(m\right)+\psi\left(M\right)\right)\mathbf{1}-\int_T \phi_t\left(\psi\left(x_t\right)\right)\mathrm{d}\mu\left(t\right).
\end{aligned} \tag{3.6}
$$

If $\psi\circ\varphi^{-1}$ is operator concave then we get reversed inequalities in (3.6).
If ψ^{-1} is operator increasing, then (3.6) implies (3.5). If ψ^{-1} is operator decreasing, then the reverse of (3.6) implies (3.5). Analogously, we get the reverse of (3.5) in the cases when $\psi\circ\varphi^{-1}$ is operator convex and ψ^{-1} is operator decreasing, or $\psi\circ\varphi^{-1}$ is operator concave and ψ^{-1} is operator increasing □

Theorem 3.5. *Let $\varphi,\psi\in C\left([m,M]\right)$ be two strictly monotone functions.*

(i) *If either φ is operator concave and φ^{-1} is operator increasing or φ is operator convex and φ^{-1} is operator decreasing, and either ψ is operator convex and ψ^{-1} is operator increasing or ψ is operator concave and ψ^{-1} is operator decreasing, then*

$$
\begin{aligned}
&\widetilde{M}_\varphi\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right) \\
&\leq \varphi^{-1}\left(\frac{M\mathbf{1}-\int_T \phi_t\left(x_t\right)\mathrm{d}\mu\left(t\right)}{M-m}\varphi(M)+\frac{\int_T \phi_t\left(x_t\right)\mathrm{d}\mu\left(t\right)-m\mathbf{1}}{M-m}\varphi(m)\right) \\
&\leq \varphi^{-1}\left(\int_T \phi_t\left(\varphi\left(\left(m+M\right)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu\left(t\right)\right) \\
&\leq \widetilde{M}_{\mathbf{1}}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right) \\
&\leq \psi^{-1}\left(\int_T \phi_t\left(\psi\left(\left(m+M\right)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu\left(t\right)\right) \\
&\leq \psi^{-1}\left(\frac{M\mathbf{1}-\int_T \phi_t\left(x_t\right)\mathrm{d}\mu\left(t\right)}{M-m}\psi(M)+\frac{\int_T \phi_t\left(x_t\right)\mathrm{d}\mu\left(t\right)-m\mathbf{1}}{M-m}\psi(m)\right) \\
&\leq \widetilde{M}_\psi\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right),
\end{aligned} \tag{3.7}
$$

where

$$
\widetilde{M}_{\mathbf{1}}\left(\left(\phi_t\right)_{t\in T},\left(x_t\right)_{t\in T}\right):=\left(m+M\right)\mathbf{1}-\int_T \phi_t\left(x_t\right)\mathrm{d}\mu\left(t\right).
$$

(ii) *If either φ is operator convex and φ^{-1} is operator increasing or φ is operator concave and φ^{-1} is operator decreasing, and either ψ is operator concave and ψ^{-1} is operator increasing or ψ is operator convex and ψ^{-1} is operator decreasing, then the inequalities in (3.7) are reversed.*

Proof. Suppose that φ is operator concave and φ^{-1} is operator increasing, and ψ is operator convex and ψ^{-1} is operator increasing. By Theorem 2.1, we have

$$\begin{aligned}
&\varphi\left((m+M)\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right)\\
&\geq \int_T \phi_t(x_t)\left(\varphi\left((m+M)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu(t)\\
&\geq \frac{M\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m}\varphi(M)+\frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)-m\mathbf{1}}{M-m}\varphi(m)\\
&\geq \left(\varphi(m)+\varphi(M)\right)\mathbf{1}-\int_T \phi_t\left(\varphi(x_t)\right)\mathrm{d}\mu(t).
\end{aligned}$$

Since φ^{-1} is operator increasing, it follows that

$$\begin{aligned}
&\widetilde{M}_\varphi\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right)\\
&\leq \varphi^{-1}\left(\frac{M\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m}\varphi(M)+\frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)-m\mathbf{1}}{M-m}\varphi(m)\right)\\
&\leq \varphi^{-1}\left(\int_T \phi_t(x_t)\left(\varphi\left((m+M)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu(t)\right)\\
&\leq \widetilde{M}_{\mathbf{1}}\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right).
\end{aligned}$$

Also, by Theorem 2.1, we have

$$\begin{aligned}
&\psi\left((m+M)\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)\right)\\
&\leq \int_T \phi_t(x_t)\left(\psi\left((m+M)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu(t)\\
&\leq \frac{M\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m}\psi(M)+\frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)-m\mathbf{1}}{M-m}\psi(m)\\
&\leq \left(\psi(m)+\psi(M)\right)\mathbf{1}-\int_T \phi_t\left(\psi(x_t)\right)\mathrm{d}\mu(t).
\end{aligned}$$

Since ψ^{-1} is operator increasing, it follows that

$$\begin{aligned}
&\widetilde{M}_{\mathbf{1}}\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right)\\
&\leq \psi^{-1}\left(\int_T \phi_t(x_t)\left(\psi\left((m+M)\mathbf{1}-x_t\right)\right)\mathrm{d}\mu(t)\right)\\
&\leq \psi^{-1}\left(\frac{M\mathbf{1}-\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)}{M-m}\psi(M)+\frac{\int_T \phi_t(x_t)\,\mathrm{d}\mu(t)-m\mathbf{1}}{M-m}\psi(m)\right)\\
&\leq \widetilde{M}_\psi\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right).
\end{aligned}$$

Hence, we have inequalities (3.7). In remaining cases the proof is analogous. $\square$

Theorem 3.6. *Let $\varphi, \psi \in C([m,M])$ be two strictly monotone functions. If $\psi \circ \varphi^{-1}$ is operator convex, then*

$$\begin{aligned} &\psi\left(M_{\psi}\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right)\right)-\psi\left(M_{\varphi}\left((\phi_t)_{t\in T},(x_t)_{t\in T}\right)\right) \\ &\leq \left(\left(\psi\circ\varphi^{-1}\right)(m)+\left(\psi\circ\varphi^{-1}\right)(M)\right)\mathbf{1}-2\left(\psi\circ\varphi^{-1}\right)\left(\frac{1}{2}(m+M)\mathbf{1}\right). \end{aligned} \tag{3.8}$$

If $\psi \circ \varphi^{-1}$ is operator concave, then the inequality in (3.8) is reversed.

Proof. In Theorem 3.1 we let $f = \psi \circ \varphi^{-1}$ and replace x_t with $\varphi(x_t)$. □

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