Banach J. Math. Anal. 1 (2007), no. 1, 117–124

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

REMARKS ON ORTHOGONALITY PRESERVING MAPPINGS IN NORMED SPACES AND SOME STABILITY PROBLEMS

JACEK CHMIELIŃSKI[1]

Dedicated to Themistocles M. Rassias

Submitted by S.-M. Jung

Abstract. We consider the Birkhoff–James orthogonality in normed spaces and classes of linear mappings exactly and approximately preserving this relation. Some related stability problems are posed.

1. Introduction

In a normed space X (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), with the norm not necessarily coming from an inner product, one can consider the Birkhoff–James orthogonality (cf. [2, 13]):

$$x \perp_{\mathrm{B}} y \quad \Longleftrightarrow \quad \forall\, \alpha \in \mathbb{K}: \quad \|x + \alpha y\| \geq \|x\|.$$

One can also consider the semi–orthogonality coming from a semi–inner–product in X. Namely, due to G. Lumer [17] and J.R. Giles [12] (cf. also [11]) there exists a mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ satisfying the following properties:

(s1) $[\lambda x + \mu y|z] = \lambda\, [x|z] + \mu\, [y|z]\,, \quad x, y, z \in X,\ \lambda, \mu \in \mathbb{K};$
(s2) $[x|\lambda y] = \overline{\lambda}\, [x|y]\,, \quad x, y \in X,\ \lambda \in \mathbb{K};$
(s3) $[x|x] = \|x\|^2, \quad x \in X;$
(s4) $|\, [x|y]\, | \leq \|x\| \cdot \|y\|, \quad x, y \in X.$

Date: Received: 13 April 2007; Revised: 16 September 2007; Accepted: 27 October 2007.
2000 *Mathematics Subject Classification.* Primary 46B20, 46C50; Secondary 39B82.
Key words and phrases. Mappings preserving orthogonality, Birkhoff–James orthogonality, semi–inner–product, approximate orthogonality, stability.

We will call each mapping $[\cdot|\cdot]$ satisfying (s1)–(s4) a *semi–inner–product* (s.i.p.) in a (normed) space X. (We assume that a s.i.p. is associated with the given norm in X, i.e., (s3) is satisfied.) Note that there may exist infinitely many different semi–inner–products in X. There is a unique s.i.p. in X if and only if X is smooth (i.e., there is a unique supporting hyperplane at each point of the unit sphere S or, equivalently, the norm is Gâteaux differentiable on S—cf. [9]). If X is an inner product space the only s.i.p. on X is the inner-product itself ([17], Theorem 3). We say that s.i.p. is *continuous* iff $\mathrm{Re}\,[y|x+\lambda y] \to \mathrm{Re}\,[y|x]$ as $\mathbb{R} \ni \lambda \to 0$ for all $x, y \in S$. The continuity of s.i.p. is equivalent to the smoothness of X ([12, Theorem 3]). For a fixed s.i.p. in X we define a related *semi–orthogonality.* For $x, y \in X$

$$x \perp_{\mathrm{s}} y \quad :\Leftrightarrow \quad [y|x] = 0.$$

Note that for an inner product space: $\perp_{\mathrm{B}} = \perp_{\mathrm{s}} = \perp$.

Theorem 1.1 ([12, Theorem 2])**.** *If X is smooth, then $\perp_{\mathrm{B}} = \perp_{\mathrm{s}}$.*

2. Orthogonality preserving mappings

Koehler and Rosenthal [15] showed that a linear operator from a normed space into itself is an isometry if and only if it preserves some semi–inner–product. This can be slightly extended.

Theorem 2.1. *Let X and Y be (real or complex) normed spaces and let $f : X \to Y$ be a linear operator. Then f is a similarity, i.e., for some $\gamma > 0$*

$$\|fx\| = \gamma \|x\|, \qquad x \in X,$$

if and only if there exist semi–inner–products $[\cdot|\cdot]_X$ and $[\cdot|\cdot]_Y$ in X and Y, respectively, such that

$$[fx|fy]_Y = \gamma^2\,[x|y]_X\,, \qquad x, y \in X. \tag{2.1}$$

Moreover, if $X = Y$ (with the same norm), then we get the assertion with the same semi–inner–product.

Proof. The sufficiency is obvious. To prove the necessity let us assume that X and Y are different normed spaces (at least the norms are different). Choose an arbitrary s.i.p. $[\cdot|\cdot]_Y$ in Y. Then it suffices to define

$$[x|y]_X := \frac{1}{\gamma^2}\,[fx|fy]_Y\,, \qquad x, y \in X$$

to obtain a s.i.p. in X such that (2.1) is satisfied. If $X = Y$ and the norm is the same, $[\cdot|\cdot]_X = [\cdot|\cdot]_Y$ is not guaranteed by the above reasoning (unless X is smooth which yields the uniqueness of s.i.p.). In this case one can apply the proof of Koehler and Rosenthal (with a slight modification concerning the constant γ). □

Koldobsky [16] showed that a linear mapping from a real normed space into itself, preserving the Birkhoff–James orthogonality must be a similarity. Blanco and Turnšek [3] extended it to complex spaces.

Theorem 2.2 ([3, Theorem 1.3]). *Let X and Y be (real or complex) normed spaces and let $f : X \to Y$ be a linear operator. Then f preserves the Birkhoff–James orthogonality, i.e.,*

$$x \perp_{\mathrm{B}} y \;\Rightarrow\; fx \perp_{\mathrm{B}} fy, \qquad x, y \in X, \tag{2.2}$$

if and only if, for some $\gamma > 0$, $\|fx\| = \gamma \|x\|$, $x \in X$.

Taking $X = Y$ and the identity mapping as f, we obtain:

Corollary 2.3. *Let X be a vector space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in X and let $\perp_{\mathrm{B},1}$ and $\perp_{\mathrm{B},2}$ denote the corresponding Birkhoff–James orthogonality relations. If $\perp_{\mathrm{B},1} \subset \perp_{\mathrm{B},2}$, then $\|x\|_2 = \gamma \|x\|_1$ for all $x \in X$, with some $\gamma > 0$ and, consequently, $\perp_{\mathrm{B},1} = \perp_{\mathrm{B},2}$.*

Blanco and Turnšek remarked also that their proof of Theorem 2.2 can be easily adapted to the case where the Birkhoff–James orthogonality is replaced by a semi-orthogonality. Namely, we have the following result.

Theorem 2.4 (cf. [3, Remark 3.2]). *Let X and Y be (real or complex) normed spaces and let $f : X \to Y$ be a linear operator preserving the semi-orthogonality related to some s.i.p. $[\cdot|\cdot]_X$ and $[\cdot|\cdot]_Y$ in X and Y, respectively, i.e.,*

$$x \perp_{\mathrm{s}} y \;\Rightarrow\; fx \perp_{\mathrm{s}} fy, \qquad x, y \in X. \tag{2.3}$$

Then, for some $\gamma > 0$, $\|fx\| = \gamma \|x\|$, $x \in X$.

All the above results enable us to list the following collection of equivalent conditions.

Theorem 2.5. *Let X and Y be normed spaces. For a linear operator $f : X \to Y$ the following conditions are equivalent:*

(a) $\exists\, \gamma > 0\ \forall\, x \in X \quad \|fx\| = \gamma \|x\|$;
(b) $\exists\, \gamma > 0\ \forall\, x, y \in X \quad [fx|fy]_Y = \gamma^2\, [x|y]_X$;
(c) $\exists\, \gamma > 0\ \forall\, x, y \in X \quad |\,[fx|fy]_Y\,| = \gamma^2 |\,[x|y]_X\,|$;
(d) $\forall\, x, y \in X \quad x \perp_{\mathrm{s}} y \;\Leftrightarrow\; fx \perp_{\mathrm{s}} fy$;
(e) $\forall\, x, y \in X \quad x \perp_{\mathrm{s}} y \;\Rightarrow\; fx \perp_{\mathrm{s}} fy$;
(f) $\forall\, x, y \in X \quad x \perp_{\mathrm{B}} y \;\Rightarrow\; fx \perp_{\mathrm{B}} fy$;
(g) $\forall\, x, y \in X \quad x \perp_{\mathrm{B}} y \;\Leftrightarrow\; fx \perp_{\mathrm{B}} fy$.

The conditions (b)–(e) *should be understood that they are satisfied with respect to some semi–inner-products $[\cdot|\cdot]_X$ and $[\cdot|\cdot]_Y$ in X and Y, respectively.*

Proof. (a) $\Rightarrow$ (b) follows from Theorem 2.1; implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are trivial; (e) $\Rightarrow$ (a) from Theorem 2.4. This proves equivalency of (a)-(e). Moreover, it is easy to show (a) $\Rightarrow$ (g), (g) $\Rightarrow$ (f) is trivial and (f) $\Rightarrow$ (a) follows from Theorem 2.2, which proves equivalency of (a), (f) and (g). □

Remark 2.6. Note that, in particular, the property that a linear mapping preserves the Birkhof-James orthogonality is equivalent to that it preserves the semi-orthogonality (although $\perp_{\mathrm{B}}$ and $\perp_{\mathrm{s}}$ need not be equivalent unless we assume the smoothness of the norm).

Remark 2.7. For the case $X = Y$ the results are also true with the same semi-inner product applied for arguments and values (cf. remarks in the proof of Theorem 2.1).

Taking $X = Y$ and the identity mapping we obtain:

Corollary 2.8. *Let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two norms in a linear space* X *(with some corresponding semi–inner–products* $[\cdot|\cdot]_1$ *and* $[\cdot|\cdot]_2$*, semi–orthogonalities* $\perp_{\mathrm{s},1}, \perp_{\mathrm{s},2}$ *and the Birkhoff–James orthogonalities* $\perp_{\mathrm{B},1}, \perp_{\mathrm{B},2}$*). Then the following conditions are equivalent:*

(a) $\exists\, \gamma > 0\ \forall\, x \in X \quad \|x\|_2 = \gamma\|x\|_1$;
(b) $\exists\, \gamma > 0\ \forall\, x, y \in X \quad [x|y]_2 = \gamma^2\, [x|y]_1$;
(c) $\exists\, \gamma > 0\ \forall\, x, y \in X \quad |\, [x|y]_2\, | = \gamma^2 |\, [x|y]_1\, |$;
(d) $\perp_{\mathrm{s},1} = \perp_{\mathrm{s},2}$;
(e) $\perp_{\mathrm{s},1} \subset \perp_{\mathrm{s},2}$;
(f) $\perp_{\mathrm{B},1} \subset \perp_{\mathrm{B},2}$;
(g) $\perp_{\mathrm{B},1} = \perp_{\mathrm{B},2}$.

Theorem 2.9. *Let* X *be a normed space. Suppose that there exists an inner product space* Y *and a linear mapping* f *from* X *into* Y *or from* Y *onto* X *such that* f *preserves the Birkhoff–James orthogonality. Then* X *is an inner product space (the norm in* X *comes from an inner product).*

Proof. 1. Suppose that $f : X \to Y$ is linear and $x \perp_{\mathrm{B}} y \;\Rightarrow\; fx \perp fy$ for all $x, y \in X$. From Theorem 2.2, there exists $\gamma > 0$ such that $\|fx\| = \gamma\|x\|$ for $x \in X$. Therefore, for all $x, y \in X$

$$\begin{aligned} &\|fx + fy\|^2 + \|fx - fy\|^2 - 2\|fx\|^2 - 2\|fy\|^2 \\ &= \gamma^2 \left(\|x + y\|^2 + \|x - y\|^2 - 2\|x\|^2 - 2\|y\|^2\right). \end{aligned} \tag{2.4}$$

Since the norm in Y satisfies the parallelogram identity, so does the norm in X whence X is an inner product. 2. Supposing that $f : Y \to X$ is linear, surjective and $x \perp y \;\Rightarrow\; fx \perp_{\mathrm{B}} fy$ for all $x, y \in Y$, using again Theorem 2.2 and (2.4), we get the assertion. □

We follow Kestelman (cf. [19]) in saying that $f : X \to Y$ *preserves right-angles* iff

$$x - z \perp_{\mathrm{B}} y - z \;\Rightarrow\; f(x) - f(z) \perp_{\mathrm{B}} f(y) - f(z), \qquad x, y, z \in X. \tag{2.5}$$

Obviously, provided $f(0) = 0$, it is a stronger condition than (2.3) whence a linear solution of (2.5) has to be a similarity. However, Tissier [19] has proved that for a real inner product space X (with $\dim X \geq 2$) no linearity assumption is needed to prove that (2.5) yields similarity of f. One can ask if it is also true in normed spaces, with the Birkhoff–James orthogonality.

3. Approximate orthogonality and approximately orthogonality preserving mappings

Let $\varepsilon \in [0,1)$. The natural way to define an ε-orthogonality of vectors x, y in an inner product space is the following one:

$$x \perp^{\varepsilon} y \quad \Leftrightarrow \quad |\langle x|y\rangle| \leq \varepsilon \|x\|\,\|y\|.$$

In normed spaces, the following notion of the ε-Birkhoff–James orthogonality was introduced by Dragomir [10].

$$x \underset{\varepsilon}{\perp}_{\mathrm{B}} y \;:\Leftrightarrow\; \forall\, \lambda \in \mathbb{K}: \;\|x + \lambda y\| \geq (1-\varepsilon)\|x\|. \tag{3.1}$$

Obviously, this relation generalizes the Birkhoff–James one. For inner product spaces, it can be shown that $x \underset{\varepsilon}{\perp}_{\mathrm{B}} y \;\Leftrightarrow\; x \perp^{\delta} y$ with $\delta := \sqrt{(2-\varepsilon)\varepsilon}$ (see [10, Proposition 1]). In order to have the latter equivalence with $\delta = \varepsilon$, one can consider (cf. [4]) a slight modification of (3.1)

$$x \perp_{\mathrm{D}}^{\varepsilon} y \;:\Leftrightarrow\; \forall\, \lambda \in \mathbb{K}: \;\|x + \lambda y\| \geq \sqrt{1-\varepsilon^2}\|x\|. \tag{3.2}$$

Suppose that there are two equivalent norms in X, i.e.,

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1, \qquad x \in X$$

with some $0 < m \leq M$. Note that for $x, y \in X$ such that $x \perp_{\mathrm{B},1} y$ we have

$$\|x + \lambda y\|_2 \geq \frac{m}{M}\|x\|_2 \quad \text{for all } \lambda \in \mathbb{K}.$$

Therefore $x \underset{\varepsilon}{\perp}_{\mathrm{B},2} y$ with $\varepsilon = 1 - \frac{m}{M}$.

An alternative definition of the ε-Birkhoff–James orthogonality (not equivalent to (3.2) in general) was given by the author in [4].

$$x \perp_{\mathrm{B}}^{\varepsilon} y \;:\Leftrightarrow\; \forall\, \lambda \in \mathbb{K}: \;\|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|. \tag{3.3}$$

For a given semi–inner–product one can define the *approximate semi-orthogonality* (ε-semi–orthogonality):

$$x \perp_{\mathrm{s}}^{\varepsilon} y \quad :\Leftrightarrow \quad |\,[y|x]\,| \leq \varepsilon\|x\|\cdot\|y\|.$$

Note that for an inner product space: $\perp_{\mathrm{s}}^{\varepsilon} = \perp_{\mathrm{B}}^{\varepsilon} = \perp_{\mathrm{D}}^{\varepsilon} = \perp^{\varepsilon}$. The author has proved also the following generalization of Theorem 1.1.

Theorem 3.1 ([4, Theorem 3.3]). *If X is a smooth normed space, then $\perp_{\mathrm{B}}^{\varepsilon} = \perp_{\mathrm{s}}^{\varepsilon}$.*

Now, we can deal with mappings which approximately preserve the Birkhoff–James orthogonality. For $\varepsilon \in [0,1)$, $f : X \to Y$ can be called an ε-orthogonality preserving mapping if it satisfies

$$x \perp_{\mathrm{B}} y \;\Rightarrow\; f(x) \underset{\varepsilon}{\perp}_{\mathrm{B}} f(y), \qquad x, y \in X$$

or, in an alternative sense,

$$x \perp_{\mathrm{B}} y \;\Rightarrow\; f(x) \perp_{\mathrm{B}}^{\varepsilon} f(y), \qquad x, y \in X. \tag{3.4}$$

Similarly, for given semi–inner–products in X and Y, one can consider mappings preserving *approximately* semi-orthogonality, i.e., satisfying:

$$x \perp_{\mathrm{s}} y \;\Rightarrow\; f(x) \perp_{\mathrm{s}}^{\varepsilon} f(y), \qquad x, y \in X. \tag{3.5}$$

Note that, in view of Theorem 3.1, for smooth spaces X and Y the conditions (3.4) and (3.5) are equivalent.

In the realm of inner product spaces the class of linear approximately orthogonality preserving mappings has been characterized in [5, Theorem 2]. Recently Turnšek [20] has made some quantitative improvements so the result finally reads as follows.

Theorem 3.2. *Let X and Y be inner product spaces and let $f : X \to Y$ be a nontrivial linear mapping satisfying*

$$x \perp y \ \Rightarrow \ fx \perp^{\varepsilon} fy, \qquad x, y \in X.$$

Then, with $\gamma = \|f\|$,

$$|\ \langle fx|fy\rangle - \gamma^2 \langle x|y\rangle\ | \leq \frac{4\varepsilon}{1+\varepsilon}\|fx\|\,\|fy\|, \qquad x, y \in X.$$

Problem 3.3. In the realm of normed spaces, characterize the classes of linear mappings approximately preserving the Birkhoff–James orthogonality and the semi–orthogonality.

Now, let us consider a linear mapping which is close to a linear and orthogonality preserving one.

Theorem 3.4. *Let X and Y be normed spaces and let $f : X \to Y$ be a linear Birkhoff–James orthogonality preserving mapping (i.e., f satisfies (2.3)). Assume that $g : X \to Y$ is linear and, with some $\varepsilon \in [0, 1)$,*

$$\|f - g\| \leq \frac{\varepsilon}{2-\varepsilon}\|f\|. \tag{3.6}$$

Then g is an ε-orthogonality preserving mapping in the sense of Dragomir.

Proof. Setting $\gamma := \|f\|$ and $\delta := \frac{\varepsilon\gamma}{2-\varepsilon} < \gamma$ we have from (3.6):

$$\|fx - gx\| \leq \delta\|x\|, \qquad x \in X.$$

Since we have from Theorem 2.2, $\|fx\| = \gamma\|x\|$, we get

$$|\,\gamma\|x\| - \|gx\|\,| = |\,\|fx\| - \|gx\|\,| \leq \|fx - gx\| \leq \delta\|x\|, \qquad x \in X.$$

Hence

$$(\gamma - \delta)\|x\| \leq \|gx\| \leq (\gamma + \delta)\|x\|, \qquad x \in X$$

and

$$\frac{\|gx\|}{\gamma+\delta} \leq \|x\| \leq \frac{\|gx\|}{\gamma-\delta}, \qquad x \in X.$$

Let $x \perp_{\mathrm{B}} y$. Then, for arbitrary $\lambda \in \mathbb{K}$, $\|x + \lambda y\| \geq \|x\|$, and thus

$$\begin{aligned} \|gx + \lambda gy\| &= \|g(x + \lambda y)\| \geq (\gamma - \delta)\|x + \lambda y\| \\ &\geq (\gamma - \delta)\|x\| \geq \frac{\gamma-\delta}{\gamma+\delta}\|gx\| \\ &= (1 - \varepsilon)\|gx\|. \end{aligned}$$

□

The problem arises whether the reverse is true. Namely, whether each ε-orthogonality preserving linear mapping g can be approximated by a linear orthogonality preserving one. In [5] and [6] author considered this stability problem in the realm of inner product spaces obtaining a positive answer under the assumption that the domain is finite-dimensional. It has been extended to the general case by Turnšek [20].

Theorem 3.5 ([20, Theorem 2.3], cf. also [6, Theorem 4])**.** *Let X and Y be Hilbert spaces and let $f : X \to Y$ be a linear mapping satisfying*

$$x \perp y \quad \Rightarrow \quad fx \perp^{\varepsilon} fy, \qquad x, y \in X. \tag{3.7}$$

Then there exists a linear orthogonality preserving mapping $T : X \to Y$ such that

$$\|f - T\| \leq \left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right) \min\{\|f\|, \|T\|\}. \tag{3.8}$$

It has been also proved by Turnšek [20, Example 2.4] that the approximation (3.8) is sharp.

Problem 3.6. Verify the stability of the orthogonality preserving property with respect to the Birkhoff–James orthogonality and the semi–orthogonality.

For Hilbert spaces X and Y, a mapping $f : X \to Y$ satisfying

$$x - z \perp y - z \quad \Rightarrow \quad f(x) - f(z) \perp^{\varepsilon} f(y) - f(z), \qquad x, y, z \in X \tag{3.9}$$

and $f(0) = 0$ satisfies also (3.7). Thus using Theorem 3.5 we get that for each linear mapping f satisfying (3.9), there exists a linear orthogonality preserving (whence also right-angle preserving) mapping T such that the approximation (3.8) holds.

Problem 3.7. In normed spaces consider the stability question for the Birkhoff–James right-angle preserving property.

For inner product spaces, strong relationships has been shown between the stability of the orthogonality preserving property and the stability of the *orthogonality equation*

$$\langle f(x) | f(y) \rangle = \langle x | y \rangle .$$

Various kinds of stability of this equation has been studied by the author (see [1, 7]) and by other authors ([14, 18]), also in more general settings ([8]). It seems that the following problem can be related with previously mentioned ones.

Problem 3.8. Consider the stability of the equation

$$[f(x) | f(y)] = [x | y] , \qquad x, y \in X$$

with the class of approximate solutions defined by the inequality

$$|\, [f(x) | f(y)] - [x | y] \,| \leq \varepsilon \|x\|^p \, \|y\|^p, \qquad x, y \in X$$

where $p \in \mathbb{R}$ is given (in particular with $p = 1$).

References

1. R. Badora, J. Chmieliński, *Decomposition of mappings approximately inner product preserving*, Nonlinear Anal. **62** (2005), 1015–1023.
2. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169-172.
3. A. Blanco, A. Turnšek, *On maps that preserve orthogonality in normed spaces*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 709–716.
4. J. Chmieliński, *On an ε-Birkhoff orthogonality*, J. Inequal. Pure and Appl. Math., **6**(3) (2005), Art. 79.
5. J. Chmieliński, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl., **304** (2005), 158–169.
6. J. Chmieliński, *Stability of the orthogonality preserving property in finite-dimensional inner product spaces*, J. Math. Anal. Appl. **318** (2006), 433–443.
7. J. Chmieliński, *Stability of the Wigner equation and related topics*, Nonlinear Funct. Anal. Appl., **11** (2006), 859–879.
8. J. Chmielinski, M.S. Moslehian, *Approximately C^*-inner product preserving mappings*, Bull. Korean. Math. Soc. (to appear).
9. M.M. Day, *Normed Linear Spaces*, Springer-Verlag, Berlin – Heidelberg – New York, 1973.
10. S.S. Dragomir, *On approximation of continuous linear functionals in normed linear spaces*, An. Univ. Timişoara Ser. Ştiinţ. Mat. **29** (1991), 51-58.
11. S.S. Dragomir, *Semi-Inner Products and Applications*, Nova Science Publishers, Inc., Hauppauge, NY, 2004.
12. J.R. Giles, *Classes of semi–inner–product spaces*, Trans. Amer. Math. Soc. **129** (1967), 436-446.
13. R.C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265-292.
14. S.-M. Jung, *Stability of the orthogonality equation on bounded domain*, Nonlinear Anal. **47** (2001), 2655–2666.
15. D. Koehler, P. Rosenthal, *On isometries of normed linear spaces*, Studia Math. **36** (1970), 213-216.
16. A. Koldobsky, *Operators preserving orthogonality are isometries*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 835-837.
17. G. Lumer, *Semi–inner–product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29-43.
18. Th.M. Rassias, *A new generalization of a theorem of Jung for the orthogonality equation*, Applicable Analysis **81** (2002), 163–177.
19. A. Tissier, *A right-angle preserving mapping* (a solution of a problem proposed in 1983 by H. Kestelman), Advanced Problem 6436, Amer. Math. Monthly **92** (1985), 291-292.
20. A. Turnšek, *On mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **336** (2007), 625–631.

[1] Instytut Matematyki, Akademia Pedagogiczna w Krakowie, Podchorążych 2, 30-084 Kraków, Poland.

E-mail address: jacek@ap.krakow.pl