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SPECTRAL PROPERTIES OF k-QUASI-$*$-$A(n)$ OPERATORS

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ABSTRACT. In this paper, we prove the following assertions: (1) If T is a k-quasi-$*$-$A(n)$ operator, then T is polaroid. (2) If T is a k-quasi-$*$-$A(n)$ operator, then the spectrum σ is continuous. (3) If T or T^* is a k-quasi-$*$-$A(n)$ operator, then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. (4) If T^* is a k-quasi-$*$-$A(n)$ operator, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Finally, the finiteness of a quasi-$*$-$A(n)$ operator is also studied.

1. Introduction

Let H be an infinite dimensional separable Hilbert space, denote $B(H)$ the algebra of all bounded linear operators on H. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and range space of T, respectively; $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ and $\pi(T)$ for the spectrum of T, the approximate point spectrum of T, the point spectrum of T and the set of poles of the resolvent of T. Let $p = p(T)$ be the ascent of T; i.e., the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$, if such an integer does not exist, then we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T; i.e., the smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such an integer does not exist, then we put $q(T) = \infty$.

As natural extensions of hyponormal operators, some operator classes have been introduced in recent years. Let n, k be positive integers and $T \in B(H)$.

(1) An operator T is said to be quasi-$*$-A if $T^*|T^2|T \geq T^*|T^*|^2T$.
(2) An operator T is said to be k-quasi-$*$-A if $T^{*k}|T^2|T^k \geq T^{*k}|T^*|^2T^k$.
(3) An operator T is said to be $*$-$A(n)$ if $|T^{1+n}|^{\frac{2}{1+n}} \geq |T^*|^2$.

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(4) An operator T is said to be quasi-$*$-$A(n)$ if $T^*|T^{1+n}|^{\frac{2}{1+n}}T \geq T^*|T^*|^2T$.

(5) An operator T is said to be k-quasi-$*$-$A(n)$ if $T^{*k}|T^{1+n}|^{\frac{2}{1+n}}T^k \geq T^{*k}|T^*|^2T^k$.

Shen, Zuo and Yang[23] introduced quasi-$*$-A operators. As an extension of quasi-$*$-A operators, Mecheri[17] introduced k-quasi-$*$-A operators. Recently, Zuo and Shen[26] introduced k-quasi-$*$-$A(n)$ operators which are generalization of k-quasi-$*$-A operators.

By definition, $*$-$A(n) \Rightarrow$ quasi-$*$-$A(n) \Rightarrow k$-quasi-$*$-$A(n)$.

Let $K = \oplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H, define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & A & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By straightforward computations, the following assertions hold:

i). $T_{A,B}$ belongs to $*$-$A(n)$ if and only if $B^2 \geq A^2$ and $(AB^{2n}A)^{\frac{1}{n+1}} \geq A^2$.

ii). $T_{A,B}$ belongs to 2-quasi-$*$-$A(n)$ if and only if $A^2B^2A^2 \geq A^6$.

Now we provide an operator which is 2-quasi-$*$-$A(n)$ but not $*$-$A(n)$ operator as follows.

Example 1.1. A non-$*$-$A(n)$ and 2-quasi-$*$-$A(n)$ operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$B^2 - A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \ngeq 0.$$

Thus $T_{A,B}$ is a non-$*$-$A(n)$ operator.

On the other hand, we have

$$A^2(B^2 - A^2)A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

Hence $T_{A,B}$ is a 2-quasi-$*$-$A(n)$ operator.

2. Some properties of k-quasi-$*$-$A(n)$ operators

To study non-normal operator T, it is important to know that T has the single valued extension property(abbrev. SVEP). T has SVEP, if for every open set U of $\mathbb{C}$, the only analytic solution $f : U \to H$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$, called the local resolvent set of T at x, is defined to consist of all $\lambda_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of λ_0, with values in H, which satisfies $(T - \lambda)f(\lambda) = x$. We define the complement of $\rho_T(x)$ by

$\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspaces $H_T(F) := \{x \in F : \sigma_T(x) \subseteq F\}$ for each set $F \subseteq \mathbb{C}$.

Let iso $\sigma(T)$ denote the set of all isolated points of $\sigma(T)$. The operator T is called isoloid if iso $\sigma(T) \subset \sigma_p(T)$ and polaroid if iso $\sigma(T) \subset \pi(T)$. In general, if T is polaroid then it is isoloid. However, the converse is not true.

For every $T \in B(H)$, σ is a compact subset of $\mathbb{C}$. The function σ viewed as a function from each T into its spectrum $\sigma(T)$, equipped with the Hausdorff metric, is well known to be upper semi-continuous, but fails to be continuous in general. Conway and Morrel[10] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators[14]. And this result has been extended to p-hyponormal operators[15], to (p, k)-quasihyponormal, M-hyponormal, $*$-paranormal and paranormal operators[12]. In the following, we extend this result to k-quasi-$*$-$A(n)$ operators.

Before we state our main theorem, we need several preliminary results.

Lemma 2.1. [26] i). *If T is a k-quasi-$*$-$A(n)$ operator, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

where T_1 is $*$-$A(n)$ on $\overline{R(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

ii). *If T is a k-quasi-$*$-$A(n)$ operator and $\lambda \neq 0$, then $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.*

Lemma 2.2. [22] i). *If T is a $*$-$A(n)$ operator, then T has SVEP.*

ii). *If T is a $*$-$A(n)$ operator, then T is simply polaroid.*

Theorem 2.3. *If T is a k-quasi-$*$-$A(n)$ operator, then T has SVEP.*

Proof. If the range of T^k is dense, then T is a $*$-$A(n)$ operator. Hence T has SVEP by Lemma 2.2. Thus we can assume that the range of T^k is not dense. By Lemma 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}).$$

Assume $(T - z)f(z) = 0$. Put $f(z) = f_1(z) \oplus f_2(z)$ on $H = \overline{R(T^k)} \oplus N(T^{*k})$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2 f_2(z) \\ (T_3 - z)f_2(z) \end{pmatrix} = 0.$$

Since T_3 is nilpotent, T_3 has SVEP. Hence $f_2(z) = 0$. Then $(T_1 - z)f_1(z) = 0$. Since T_1 is a $*$-$A(n)$ operator, T_1 has SVEP by Lemma 2.2. Hence $f_1(z) = 0$. Consequently, T has SVEP. $\square$

Theorem 2.4. *If T is a k-quasi-$*$-$A(n)$ operator and $||T^m|| = ||T||^m$ for some positive integer $m \geq k$, then T is normaloid.*

Proof. If T is a k-quasi-$*$-$A(n)$ operator, then $T^{*k}|T^{1+n}|^{\frac{2}{1+n}}T^k \geq T^{*k}|T^*|^2T^k$. We have $||T^{1+n+k}x|| \parallel T^kx \parallel^n \geq ||T^*T^kx||^{1+n}$ for every $x \in H$. Since $m \geq k$, and hence

$$||T^{1+n+m}x|| \parallel T^mx \parallel^n \geq ||T^*T^mx||^{1+n}$$

for every $x \in H$, which implies that

$$||T^{1+n+m}|| \parallel T^m \parallel^n \geq ||T^*T^m||^{1+n}.$$

Then, by the above inequality and $||T^m|| = ||T||^m$

$$||T||^{(m-1)(1+n)}||T^{1+n+m}|| \parallel T \parallel^{mn} \geq ||T^{*(m-1)}||^{1+n}||T^{1+n+m}|| \parallel T^m \parallel^n$$

$$\geq ||T^{*(m-1)}||^{1+n}||T^*T^m||^{1+n} \geq ||T^{*m}T^m||^{1+n} = ||T^m||^{2(1+n)} = ||T||^{2m(1+n)},$$

and therefore

$$||T^{1+n+m}|| = ||T||^{1+n+m}.$$

Thus, by induction, $||T^{(1+n)j+m}|| = ||T||^{(1+n)j+m}$ for every $j \geq 1$. This yields a subsequence $\{T^{n_j}\}$ of $\{T^n\}$, say $T^{n_j} = T^{(1+n)j+m}$, such that $\lim_j ||T^{n_j}||^{\frac{1}{n_j}} = \lim_j(||T||^{n_j})^{\frac{1}{n_j}} = ||T||$. Since $\{||T^n||^{\frac{1}{n}}\}$ is a convergent sequence that converges to the spectral radius of T, and since it has a subsequence that converges to $||T||$, it follows that $r(T) = ||T||$, where $r(T)$ is the spectral radius of T. Hence T is normaloid. □

Corollary 2.5. *If T is a quasi-$*$-$A(n)$ operator, then T is normaloid.*

Corollary 2.6. *If T is a $*$-$A(n)$ operator, then T is normaloid.*

Proposition 2.7. *If T is a k-quasi-$*$-$A(n)$ operator and $\sigma(T) = \{0\}$, then $T^{k+1} = 0$.*

Proof. If the range of T^k is dense, then T is a $*$-$A(n)$ operator, which leads to that T is normaloid, hence $T = 0$. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k})$$

where T_1 is a $*$-$A(n)$ operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 2.1. If $\sigma(T_1) = \{0\}$, then $T_1 = 0$. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

□

Theorem 2.8. *If T is a k-quasi-$*$-$A(n)$ operator and λ is a non-zero isolated point of $\sigma(T)$, then λ is a simple pole of the resolvent of T. Furthermore, T is polaroid.*

Proof. If $\lambda \neq 0$, assume that $R(T^k)$ is dense. Then T is $*$-$A(n)$ and λ is a simple pole of the resolvent of T by Lemma 2.2. So we may assume that T^k does not have dense range. Then by Lemma 2.1 the operator T can be decomposed as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{R(T^k)} \oplus N(T^{*k}),$$

where T_1 is a $*$-$A(n)$ operator on $\overline{R(T^k)}$. Now if λ is a non-zero isolated point of $\sigma(T)$, then $\lambda \in \text{iso}\ \sigma(T_1)$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. Therefore λ is a simple pole of the resolvent of T_1 and the $*$-$A(n)$ operator T_1 can be written as follows:

$$T_1 = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{12} \end{pmatrix} \text{ on } \overline{R(T^k)} = N(T_1 - \lambda) \oplus \overline{R(T_1 - \lambda)},$$

where $\sigma(T_{11}) = \{\lambda\}$. Therefore

$$T - \lambda = \begin{pmatrix} 0 & 0 & T_{21} \\ 0 & T_{12} - \lambda & T_{22} \\ 0 & 0 & T_3 - \lambda \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix}$$

$$\text{on } H = N(T_1 - \lambda) \oplus \overline{R(T_1 - \lambda)} \oplus N(T^{*k}),$$

where

$$F = \begin{pmatrix} T_{12} - \lambda & T_{22} \\ 0 & T_3 - \lambda \end{pmatrix}.$$

Now, we claim that F is an invertible operator on $\overline{R(T_1 - \lambda)} \oplus N(T^{*k})$. First we verify that $T_{12} - \lambda$ is invertible. If not, then λ will be an isolated point in $\sigma(T_{12})$. Since T_{12} is $*$-$A(n)$, λ is an eigenvalue of T_{12} and thus $T_{12}x = \lambda x$ for some non-zero vector x in $\overline{R(T_1 - \lambda)}$. On the other hand, $T_1 x = T_{12} x$ implying x is in $N(T_1 - \lambda)$. Hence x must be a zero vector. This contradiction shows that $T_{12} - \lambda$ is invertible. Since $T_3 - \lambda$ is also invertible, it follows that F is invertible. It is easy to show that $p(T - \lambda) = q(T - \lambda) = 1$. Hence λ is a simple pole of the resolvent of T.

If $\lambda = 0$, by Theorem 2.3 T has SVEP. Define the quasinilpotent part $H_0(T - \lambda) = \{x \in H : \lim\limits_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$, $H_0(T - \lambda) = H_T(\{\lambda\})$ [1, Theorem 2.20], then $H_0(T - \lambda)$ is closed and $\sigma(T|_{H_0(T-\lambda)}) \subseteq \{\lambda\}$ by [16, proposition 1.2.19]. Let $S = T|_{H_0(T-\lambda)}$. Then S is a k-quasi-$*$-$A(n)$ operator. Since $\sigma(S) = \{0\}$, $S^{k+1} = 0$ by Proposition 2.7, and $H_0(T) \subseteq N(T^{k+1})$. Hence in either case $H_0(T) = N(T^{k+1})$. Consequently, T is polaroid. □

Corollary 2.9. *If T is a k-quasi-$*$-$A(n)$ operator, then T is isoloid.*

In the following, we prove that the spectrum σ is continuous on the set of k-quasi-$*$-$A(n)$ operators, the key lemma due to Berberian[4].

Lemma 2.10. [4] *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$ and a map $\varphi : B(H) \to B(K)$ such that*

i). *φ is a faithful $*$-representation of the algebra $B(H)$ on K;*
ii). *$\varphi(A) \geq 0$ for any $A \geq 0$ in $B(H)$;*
iii). *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in B(H)$.*

Theorem 2.11. *The spectrum σ is continuous on the set of k-quasi-$*$-$A(n)$ operators.*

Proof. Suppose T is a k-quasi-$*$-$A(n)$ operator. Let $\varphi : B(H) \to B(K)$ be Berberian's faithful φ-representation of Lemma 2.10. In the following, we shall show that $\varphi(T)$ is also a k-quasi-$*$-$A(n)$ operator. In fact, since T is a k-quasi-$*$-$A(n)$ operator, by Lemma 2.10, we have

$$\begin{aligned}&(\varphi(T))^{*k}(|(\varphi(T))^{1+n}|^{\frac{2}{1+n}}-|(\varphi(T))^*|^2)(\varphi(T))^k\\&=\varphi(T^{*k}(|T^{1+n}|^{\frac{2}{1+n}}-|T^*|^2)T^k)\\&\geq 0.\end{aligned}$$

Hence we have that its Berberian extension $\varphi(T)$ is also a k-quasi-$*$-$A(n)$ operator, By Lemma 2.1 and Proposition 2.7 we have that T belongs to the set $C(i)$ (see definition in [12]). So we have that the spectrum σ is continuous on the set of k-quasi-$*$-$A(n)$ by [12, Theorem 1.1]. This completes the proof. $\square$

3. Weyl type theorems

An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T)=\dim N(T)<\infty$ and $\beta(T)=\dim H/R(T)<\infty$. Moreover if $i(T)=\alpha(T)-\beta(T)=0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T)=\{\lambda\in\mathbb{C}:\ T-\lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(T)=\{\lambda\in\mathbb{C}:\ T-\lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T)\subset\sigma_W(T)\subset\sigma_e(T)\cup\operatorname{acc}\sigma(T)$ where we write $\operatorname{acc} K$ for the set of all accumulation points of $K\subset\mathbb{C}$.

Let

$$\pi_{00}(T)=\{\lambda\in\operatorname{iso}\sigma(T):\ 0<\alpha(T-\lambda)<\infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T)\setminus\sigma_W(T)=\pi_{00}(T).$$

The operator T is called Browder if it is Fredholm of finite ascent and descent. The Browder spectrum of T is given by $\sigma_b(T)=\{\lambda\in\mathbb{C}:\ T-\lambda\,\text{is not Browder}\}$. We say that Browder's theorem holds for T if

$$\sigma_W(T)=\sigma_b(T).$$

More generally, Berkani investigated B-Fredholm theory as follows (see [5, 6, 7]). An operator T is called B-Fredholm if there exists $n\in\mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]}:R(T^n)\ni x\to Tx\in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]})=R(T^{n+1})$ is closed, $\alpha(T_{[n]})=\dim N(T_{[n]})<\infty$ and $\beta(T_{[n]})=\dim R(T^n)/R(T_{[n]})<\infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $i(T_{[n]})=0$. The following result is due to Berkani and Sarih [7].

Proposition 3.1. *Let $T\in B(H)$.*

i).*If $R(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m\geq n$. Moreover, $i(T_{[m]})=i(T_{[n]})(=i(T))$.*

ii).*An operator T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $T=T|\mathcal{M}\oplus T|\mathcal{N}$ where $T|\mathcal{M}$ is Fredholm (Weyl) and $T|\mathcal{N}$ is nilpotent.*

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{−Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [6]. Recently Berkani and Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T in [5].

We define $T \in SF_+^-(H)$ if $R(T)$ is closed, $\alpha(T) < \infty$ and $i(T) \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \alpha(T - \lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(H)\} \subset \sigma_W(T)$. We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

Rakočević [20, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

We define $T \in SBF_+^-(H)$ if there exists a positive integer n such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $0 \geq i(T_{[n]})(= i(T))$ [7]. We define $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(T)\} \subset \sigma_{SF_+^-}(H)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 < \alpha(T - \lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

An operator $T \in B(H)$ satisfies a-Browder's theorem if $\sigma_{ea}(T) = \sigma_{ab}(T)$ (where $\sigma_{ab}(T) = \{\sigma_a(T + K) : TK = KT \text{ and } K \text{ is a compact operator}\}$) and T satisfies generalized a-Browder's theorem if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \pi^a(T)$.

It's known from [6, 11, 13, 21] that if $T \in B(H)$ then we have
generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Weyl's theorem;
generalized a-Weyl's theorem $\Rightarrow$ generalized a-Browder's theorem $\Rightarrow$ Browder's theorem.

Weyl[24] discovered that Weyl's theorem holds for hermitian operators, which has been extended from hermitian operators to hyponormal operators[9], to analytically class A operators by Cao[8], and to quasi-$*$-A operators[27]. In this paper, we extend it to k-quasi-$*$-$A(n)$ operators.

In the following theorem, $H(\sigma(T))$ denotes the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.2. *Let T or T^* be a k-quasi-$*$-$A(n)$ operator. Then Weyl's theorem holds for $f(T)$, where $f \in H(\sigma(T))$.*

Proof. From [2, Theorem 2.11], we have that T is polaroid if and only if T^* is polaroid. We use the fact that if T is polaroid and T or T^* has SVEP then both T and T^*satisfy Weyl's theorem, which can be seen in [2, Theorem 3.3]. Suppose that T or T^* is a k-quasi-$*$-$A(n)$ operator. By Theorem 2.3 and Theorem 2.8 we have that T satisfies Weyl's theorem. Now we show that Weyl's theorem holds for $f(T)$. Since T is polaroid and has SVEP, then $f(T)$ is polaroid by [2, Lemma 3.11] and has SVEP by [1, Theorem 2.40], consequently, Weyl's theorem holds for $f(T)$. $\square$

Corollary 3.3. *Let T or T^* be a k-quasi-$*$-$A(n)$ operator. If F is an operator commuting with T and F^n has a finite rank for some $n \in \mathbb{N}$, then Weyl's theorem holds for $f(T) + F$ for each $f \in H(\sigma(T))$.*

Proof. Suppose T or T^* is a k-quasi-$*$-$A(n)$ operator. By Theorem 2.8 and Theorem 3.2, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Notice that T is isoloid then $f(T)$ is isoloid. The result stems from [19, Theorem 2.4]. $\square$

If a Banach space operator T has SVEP (everywhere), then T and T^* satisfy Browder's (equivalently, generalized Browder's) theorem and a-Browder's (equivalently, generalized a-Browder's) theorem. A sufficient condition for an operator T satisfying Browder's (generalized Browder's) theorem to satisfy Weyl's (resp., generalized Weyl's) theorem is that T is polaroid. Observe that if $T \in B(H)$ has SVEP, then $\sigma(T) = \overline{\sigma_a(T^*)}$. Hence, if T has SVEP and is polaroid, then T^* satisfies generalized a-Weyl's (so also, a-Weyl's) theorem [2, Theorem 2.14, Theorem 2.6].

Theorem 3.4. *Let $T \in B(H)$.*

i). *If T^* is a k-quasi-$*$-$A(n)$ operator, then generalized a-Weyl's theorem holds for T.*

ii). *If T is a k-quasi-$*$-$A(n)$ operator, then generalized a-Weyl's theorem holds for T^*.*

Proof. i) It is well known that T is polaroid if and only if T^* is polaroid [2, Theorem 2.11]. Now since a k-quasi-$*$-$A(n)$ operator is polaroid and has SVEP, [2, Theorem 3.10] gives us the result of the theorem. For ii) we can also apply [2, Theorem 3.10]. $\square$

Since the polaroid condition entails $E(T) = \pi(T)$ and the SVEP for T entails that generalized Browder's theorem holds for T [3, Theorem 3.2], i.e. $\sigma_{BW}(T) = \sigma_D(T)$, where $\sigma_D(T)$ denotes the Drazin spectrum. Therefore, $E(T) = \pi(T) = \sigma(T) \setminus \sigma_D(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Thus we have the following Corollary.

Corollary 3.5. *If T is k-quasi-$*$-$A(n)$, then also T satisfies generalized Weyl's theorem.*

Remark 3.6. 1. Recall [2] that if T is polaroid, then T satisfies generalized Weyl's theorem (resp. generalized a-Weyl's) theorem if and only if T satisfies Weyl's theorem (resp. a-Weyl's theorem). Hence if T is a k-quasi-$*$-$A(n)$ operator, the above equivalences hold.

2. Let $f(z)$ be an analytic function on $\sigma(T)$. If T is polaroid, then $f(T)$ is also polaroid[2].

i). If T^* is k-quasi-$*$-$A(n)$, then $f(T)$ satisfies generalized a-Weyl's theorem. Indeed, since T^* is polaroid, the result holds by [2, Theorem 3.12] ii). If T is k-quasi-$*$-$A(n)$, then $f(T^*)$ satisfies generalized a-Weyl's theorem. Indeed, since T is polaroid, the result holds by [2, Theorem 3.12].

Theorem 3.7. *Let T be a k-quasi-$*$-$A(n)$ operator. If S is an operator quasi-similar to T, then a-Browder's theorem holds for $f(S)$ for each $f \in H(\sigma(S))$.*

Proof. Since T is a k-quasi-$*$-$A(n)$ operator, T has SVEP. Let U be any open set and $f : U \to H$ be any analytic function such that $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since S is an operator quasi-similar to T, there exists an injective operator A with dense range such that $AS = TA$. Thus $A(S-\lambda) = (T-\lambda)A$ for all $\lambda \in U$. If $(S-\lambda)f(\lambda) = 0$ for all $\lambda \in U$, then $A(S-\lambda)f(\lambda) = (T-\lambda)Af(\lambda) = 0$ for all $\lambda \in U$. But T has SVEP; hence $Af(\lambda) = 0$ for all $\lambda \in U$. Since A is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore S has SVEP. Then it follows from [1] that $\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S))$. Hence a-Browder's theorem holds for $f(S)$. □

4. Finite operators

Let $A, B \in B(H)$. We define the generalized derivation $\delta_{A,B} : B(H) \mapsto B(H)$ by $\delta_{A,B}(X) = AX - XB$, we note $\delta_{A,A} = \delta_A$. If the inequality $||T-(AX-XA)|| \geq ||T||$ holds for all $X \in B(H)$ and for all $T \in N(\delta_A)$, then we say that the range of δ_A is orthogonal to the kernel of δ_A in the sense of Birkhoff. The operator $A \in B(H)$ is said to be finite [25] if $||I - (AX - XA)|| \geq 1$ for all $X \in B(H)$, where I is the identity operator. Williams [25] has shown that the class of finite operators contains every normal, hyponormal operators. In [18], Williams results are generalized to a more classes of operators containing the classes of normal and hyponormal operators.

Let $A \in B(H)$, the approximate reduced spectrum of A, $\sigma_{ar}(A)$, is the set of scalars λ for which there exists a normed sequence $\{x_n\}$ in H satisfying

$$(A-\lambda I)x_n \to 0,\ (A-\lambda I)^* x_n \to 0.$$

In this section we present some new classes of finite operators. Recall that an operator $A \in B(H)$ is said to be spectraloid if $\omega(A) = r(A)$, where $\omega(A)$ is the numerical radius of A.

Lemma 4.1. [18] *Let $A \in B(H)$. Then $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$, where $W(A)$ is the numerical range of the operator A.*

Lemma 4.2. [18] *If $\sigma_{ar}(A) \neq \phi$, then A is finite.*

Theorem 4.3. *Let $A \in B(H)$ be spectraloid. Then A is finite.*

Proof. Since A is spectraloid, we have $\omega(A) = r(A)$. Then there exists $\lambda \in \sigma(A) \subset \overline{W(A)}$ such that $|\lambda| = \omega(A)$. Thus $\lambda \in \partial W(A)$. This implies that $\partial W(A) \cap \sigma(A) \neq \emptyset$. Now by applying Lemma 4.2, we get the result. □

Corollary 4.4. *Let $A \in B(H)$. If A is a quasi-$*$-$A(n)$ operator, then A is finite.*

Proof. Since A is a quasi-$*$-$A(n)$ operator, it is normaloid and so is spectraloid, it suffices to apply Theorem 4.3. □

Corollary 4.5. *The following operators are finite:*
i). *class* $*$- *A operators.* ii). *quasi-*$*$*-A operators.*

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