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# A curve of genus 5 having 24 Weierstrass points of weight 5

Takao KATO

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**Abstract.** In this paper, we shall prove that if an irreducible curve $X$ of genus 5 over $\mathbb{C}$ has 24 Weierstrass points of weight 5, then it has exactly three bielliptic involutions.

*Key words*: algebraic curves, Weierstrass points, bielliptic involutions.

## 1. Introduction

Let $X$ be a non-hyperelliptic curve of genus $g \geq 5$ over $\mathbb{C}$. $X$ is called bielliptic if $X$ is a two-sheeted covering of an elliptic curve $E$. Then, there exists an automorphism $\sigma$ of $X$, called a bielliptic involution, such that $X/\langle\sigma\rangle$ is equivalent to $E$. By the Riemann-Hurwitz relation, it is easy to see that there exist $2g-2$ fixed points of $\sigma$. For a fixed point $P$ of $\sigma$, by the pull-back of meromorphic functions on $E$, it is shown that $2k$ $(k = 2, 3, 4, \dots)$ are Weierstrass non-gaps at $P$. More precisely there are two types Weierstrass gap sequences for fixed points of bielliptic involutions. For convenience, we list up them by their order sequences (of holomorphic differentials), instead of the gap sequences (cf. [3, Proposition 1.2]):

$$\text{Type } \mathrm{I}_g : \{0, 1, 2, 4, \dots, 2j, \dots, 2g-6, 2g-4\},$$

$$\text{Type } \mathrm{II}_g : \{0, 1, 2, 4, \dots, 2j, \dots, 2g-6, 2g-2\}.$$

If $g \geq 6$, by the Castelnuovo-Severi inequality (cf. [1]), there exists at most one bielliptic involution. Thus, the sum of the number of points of Type $\mathrm{I}_g$ and Type $\mathrm{II}_g$ is $2g-2$ as far as it is positive. Moreover, Park [8] and Ballico - Del Centina [3] proved the existence of a bielliptic curve of genus $g$ which has exactly $s$ Weierstrass points of Type $\mathrm{II}_g$ for $0 \leq s \leq 2g-2$, $s \neq 2g-3$. On the other hand, in case $g = 5$, bielliptic involutions are not unique. For

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$k = 1, 2, 3, 5$, there exists a curve of genus 5 which has exactly $k$ bielliptic involutions. In particular, curves of genus 5 with 5 bielliptic involutions, called the Humbert curves, have 40 Weierstrass points of Type $\mathrm{I}_5$. For the maximal number of points of Type $\mathrm{II}_5$, there exists a curve having 3 bielliptic involutions all of whose fixed points are 24 Weierstrass points of Type $\mathrm{II}_5$.

Conversely, if $g \geq 8$, then the existence of a Weierstrass point $P \in X$ of Type $\mathrm{I}_g$ or Type $\mathrm{II}_g$ implies the existence of a bielliptic involution of $X$ and $P$ is a fixed point of the bielliptic involution. For $g \geq 11$ or $g = 8$, it was proved by Kato [6], while one knows that the same proof can be applied for $g \geq 8$. Even in case $g = 7$, the existence of a Weierstrass point $P \in X$ of Type $\mathrm{I}_g$ implies the same result as above. These are also proved by the genus formula of plane curves with singular points using [4, part 8.4, Lemma 6 and Theorem 8].

On the other hand, for $g \leq 7$ this result does not hold any more, i.e. there exists a point $P \in X$ such that non-gaps at $P$ begin with $4, 6$ but $|6P|$ is a simple net (cf. Coppens [5, Corollary 21]). For $g = 5$, as mentioned above, there exists a curve having 3 bielliptic involutions all of whose fixed points are 24 Weierstrass points of Type $\mathrm{II}_5$. The purpose of this paper is to prove that the converse holds, i.e. if $X$ of genus 5 has 24 Weierstrass points of Type $\mathrm{II}_5$, then there exist three bielliptic involutions of $X$ (cf. Theorem 2).

## 2. Analysis of Weierstrass points of weight 5

In this section, we shall give a criterion of a curve $X$ to have a bielliptic involution. We denote $D_1 \equiv D_2$ if two divisors $D_1$ and $D_2$ are equivalent.

**Theorem 1** *Let $X$ be a curve of genus 5. Assume $P_i$ $(i = 1, 2, 3, 4)$ are Weierstrass points of Type $II_5$ on $X$, i.e. the gap sequence of $P_i$ is $\{1, 2, 3, 5, 9\}$. If $4P_1 \equiv 4P_2 \equiv 4P_3 \equiv 4P_4$, then $P_i$ $(i = 1, 2, 3, 4)$ are fixed points of a bielliptic involution of $X$.*

*Proof.* It is obvious that $|6P_i|$ is a base point free net. If $|6P_i|$ is composite, since $|4P_i|$ is a base point free pencil, there exists a two-sheeted covering $\pi' : X \to X'$. Since $2, 3$ are non-gaps at $\pi'(P_i)$, while 1 is a gap, $X'$ is an elliptic curve. Hence, we have the desired result. Thus, assume $L = |6P_1|$ is a base point free simple net, i.e. contrary to the conclusion. Let $Y$ be the plane model of degree 6 induced by $L$ and $\pi : X \to Y$ be the morphism.

Then, $p_1 = \pi(P_1) \in Y$ is a singular point of multiplicity 2 with the 6-fold tangent line $l$. Thus, we have a defining equation of $Y$ so that

$$f(x,y) = \sum_{i+j\le 6} a_{ij}x^i y^j = 0,$$

$$a_{00} = a_{10} = a_{01} = a_{11} = a_{02} = \cdots = a_{05} = 0, \tag{2.1}$$

$$a_{20} \neq 0, \quad a_{06} \neq 0.$$

$p_1$ corresponds to $(x,y) = (0,0)$. Moreover, we may assume $a_{20} = 1$ and we shall show that $a_{12} = 0$, later (see the equation (2.5)).

It is well known that a holomorphic differential on $Y$ is given by $(h(x,y)dx)/(f_y(x,y))$, where $h(x,y)$ is a special adjoint polynomial of $f(x,y)$, a suitable polynomial of degree at most $\deg f - 3 = 3$. To see the behavior of $h(x,y)$ at a singular point of $Y$, we shall estimate the order $\mu$ of the pole of $dx/(f_y(x,y))$ at $p_1$.

By the projection $\varpi : Y \to \mathbb{P}^1; (x,y) \mapsto x$, we consider $Y$ as a 6-sheeted covering of the Riemann sphere. By the assumption the point $x = 0$, which corresponds to $P_1$, is a total ramification point of this covering. Hence, we may take a local parameter $t$ at $P_1$ such that $x = x(t) = t^6$. Let $y = y(t) = t^\lambda(\beta_0 + \beta_1 t + \cdots)$, for some $\beta_j$ $(j = 0,1,\ldots), \beta_0 \neq 0$, at $P_1$. It follows that $x^i y^j = \beta_0^j t^{6i+\lambda j} + o(t^{6i+\lambda j})$. Note that by (2), we have

$$6i + \lambda j \geq \begin{cases} 6\lambda, & \text{if } i = 0, \\ 6 + 2\lambda, & \text{if } i = 1, \\ 12, & \text{if } i \geq 2. \end{cases} \tag{2.2}$$

Assume $\lambda = 1$. Then, $6i + \lambda j \geq 6$ and equality holds only if $(i,j) = (0,6)$, whence $f(x(t), y(t)) = \beta_0^6 a_{06} t^6 + \cdots \equiv 0$ which contradicts $a_{06} \neq 0$. Assume $\lambda \geq 4$. Then, $6i + \lambda j \geq 12$ and equality holds only if $(i,j) = (2,0)$, $f(x(t), y(t)) = a_{20}t^{12} + \cdots$ which is a contradiction. Assume $\lambda = 3$. Then $f(x(t), y(t)) = (a_{20} + \beta_0^2 a_{12})t^{12} + \cdots$, whence $a_{20} + \beta_0^2 a_{12} = 0$. This implies that $a_{12} \neq 0$ and $f_y(x(t), y(t)) = 2a_{12}\beta_0 t^9 + \cdots$. Since $dx = 6t^5 dt$, $dx/(f_y(x,y))$ has a pole of order 4 at $t = 0$. On the other hand $h(x,y)$ cannot have a zero of order 4 at $t = 0$. This is a contradiction. Hence, $\lambda = 2$.

Letting $s = t(\beta_0 + \beta_1 t + \cdots)^{1/2}$ (taking some branch, of course), we have a local parameter $s$ at $P_1$ such that $x = x(s) = s^6(1+\alpha_1 s+\alpha_2 s^2+\cdots)$ and $y = y(s) = s^2$. Using the local parameter $s$, we denote

$$f(x(s), y(s)) = \sum_{n=0}^{\infty} A_n s^n \equiv 0 \tag{2.3}$$

$$f_y(x(s), y(s)) = \sum_{n=0}^{\infty} B_n s^n. \tag{2.4}$$

Of course, $A_n = 0$ for all $n$. By (2), we have

$$A_{10} = \alpha_4 a_{10} + \alpha_2 a_{11} + a_{12} + a_{05} = 0, \text{ whence } a_{12} = 0. \tag{2.5}$$

Thus, by $x^2 = s^{12}(1+2\alpha_1 s+(\alpha_1^2+2\alpha_2)s^2+2(\alpha_1\alpha_2+\alpha_3)s^3+\cdots)$, we have

$$A_{12} = \alpha_6 a_{10} + a_{20} + \alpha_4 a_{11} + \alpha_2 a_{12} + a_{13} + a_{06} = 0,$$
$$A_{13} = \alpha_7 a_{10} + 2\alpha_1 a_{20} + \alpha_5 a_{11} + \alpha_3 a_{12} + \alpha_1 a_{13} = 0.$$

Hence,

$$a_{20} + a_{13} + a_{06} = 0, \tag{2.6}$$

$$2\alpha_1 a_{20} + \alpha_1 a_{13} = 0. \tag{2.7}$$

Let $\tau$ be the order of the zero of $f_y(x(s), y(s))$ at $s = 0$. It is obvious that $\tau = \mu + 5$. Note that $f_y(x, y) = \sum a_{ij} j x^i y^{j-1}$ and $x^i y^{j-1} = s^{6i+2j-2} + \cdots$. If $6i + 2j - 2 \leq 9$, then $i = 0, 0 \leq j \leq 5$ or $i = 1, 0 \leq j \leq 2$, whence by (2) and $a_{12} = 0$, we have $B_0 = \cdots = B_9 = 0$. Hence, $\tau \geq 10$ i.e. $\mu \geq 5$. However, since $x^i y^j = s^{6i+2j} + \cdots$, if $\mu = 5$, then $h(x, y)$ contains terms $a + by + cy^2 = a + bs^2 + cs^4$, i.e. $\mu \leq 4$. Hence, we have $\mu \geq 6$, i.e. $\tau \geq 11$. On the other hand, since the degree of a canonical divisor is $2g - 2$ and $\deg h \leq 3$, we have $2g - 2 + \mu \leq 3 \cdot 6 = 18$, whence $\mu \leq 20 - 2g \leq 10$. By a straightforward computation, we have

$$B_{10} = 3a_{13} + 6a_{06} = 0, \tag{2.8}$$

$$B_{11} = 3\alpha_1 a_{13}, \tag{2.9}$$

$$B_{12} = 3\alpha_2 a_{13} + a_{21} + 4a_{14}, \tag{2.10}$$

$$B_{13} = 3\alpha_3 a_{13} + 2\alpha_1 a_{21} + 4\alpha_1 a_{14}. \tag{2.11}$$

By $a_{20} = 1$, (2.6) and (2.8), we have $a_{13} = -2$, $a_{06} = 1$. Hence, $\mu = 6$ if and only if $\alpha_1 \neq 0$. If $\alpha_1 = 0$, then, $h(x,y) = x - y^3 + \cdots = \alpha_1 s^7 + o(s^7)$. Hence, $\mu \neq 7$. By (2.11), $\mu = 8$ if and only if $\alpha_1 = 0$ and $\alpha_3 \neq 0$. Similarly, we have $\mu \neq 9$ and $\mu = 10$ if and only if $\alpha_1 = \alpha_3 = 0$ and $\alpha_5 \neq 0$.

In case $\mu = 6$, there are two singular points $q_1, q_2$ (including the case $q_2$ is infinitely near to $q_1$) of multiplicity 2 beside $p_1$. Let $l$ be the line joining $q_1$ and $q_2$. Then $lx^2dx/(f_y(x,y))$ has a zero of order 8 (resp. 6) at $p_1$ if $p_1 \in l$ (resp. $p_1 \notin l$). Since the order sequence at $P_1$ is $\{0,1,2,4,8\}$, we obatin that $p_1, q_1, q_2$ are collinear and we may assume $l$ is defined by $y = 0$.

In case $\mu = 8$, there is one singular point $q_1$ of multiplicity 2 beside $p_1$. Let $l$ be the line joining $p_1$ and $q_1$. Then $lx^2dx/(f_y(x,y))$ has a zero of order 6 at $p_1$, whence this case does not occur.

In case $\mu = 10$, there is no singular point beside $p_1$ and $x^3dx/(f_y(x,y))$ has a zero of order 8 at $p_1$. Of course, this case is a special case of the case $\mu = 6$ so that $q_1, q_2$ are infinitly near points of $p_1$.

Applying a birational transformation $\xi = y/x$, $\eta = 1/x$ to $f(x,y)$, we have

$$\begin{aligned} &a_{20}\eta^4 + (a_{21}\xi + a_{30})\eta^3 + (a_{13}\xi^3 + a_{22}\xi^2 + a_{31}\xi + a_{40})\eta^2 \\ &\quad + (a_{14}\xi^4 + a_{23}\xi^3 + a_{32}\xi^2 + a_{41}\xi + a_{50})\eta \\ &\quad + (a_{06}\xi^6 + a_{15}\xi^5 + a_{24}\xi^4 + a_{33}\xi^3 + a_{42}\xi^2 + a_{51}\xi + a_{60}) = 0. \end{aligned}$$

As usual, replacing $\eta$ by $\eta - (a_{21}\xi + a_{30})/4$, we have

$$\begin{aligned} F(\xi,\eta) &= a_{20}\eta^4 + (a_{13}\xi^3 + a_{22}\xi^2 + a_{31}\xi + a_{40})\eta^2 \\ &\quad + (a_{14}\xi^4 + a_{23}\xi^3 + a_{32}\xi^2 + a_{41}\xi + a_{50})\eta \\ &\quad + (a_{06}\xi^6 + a_{15}\xi^5 + a_{24}\xi^4 + a_{33}\xi^3 + a_{42}\xi^2 + a_{51}\xi + a_{60}) = 0, \end{aligned}$$

i.e. we may assume $a_{21} = a_{30} = 0$.

**Case $\mu = 6$ and $q_1, q_2$ lie on the line $\xi = 0$.** Let $l_1$ be the line containing $p_1, q_1, q_2$. Since $\eta = \infty$ at $p_1$, $l_1$ is given by $\xi = c$ for a constant $c$. Replacing $\xi$ by $\xi - c$, we may assume $q_1$ and $q_2$ lie on the line $\xi = 0$. Let $q_1$ (resp. $q_2$) be given by $(\xi,\eta) = (0,\beta)$ (resp. $(0,\gamma)$). First, we assume

$\beta \neq \gamma$. Since $q_1$ and $q_2$ are singular points of multiplicity 2, $F(0,\eta)$ has zeros of order 2 at $\eta = \beta$ and $\eta = \gamma$. Hence, noting $a_{20} = 1$, we have $F(0,\eta) = (\eta-\beta)^2(\eta-\gamma)^2$. Therefore, $\gamma = -\beta$ and $a_{50} = 0$. Moreover we have $F_\xi(0,\pm\beta) = 0$, whence $a_{41} = 0$, too. The case $\beta = \gamma = 0$ is a special case of the above case. Thus, $(\xi,\eta) = (0,0)$ is a singular point of multiplicity 2 with the 4-fold tangent line $\xi = 0$. Hence, we have $a_{41} = a_{50} = 0$ (Of course, we have $a_{60} = a_{51} = a_{40} = 0$, but it is not a matter in our discussion).

Let $p_2 = \pi(P_2)$ be given by $(\xi,\eta) = (\beta_2,\gamma_2)$. By the assumption, since $4P_1 \equiv 4P_2$, $F(\beta_2,\eta)$ has a zero of order 4 at $\eta = \gamma_2$, whence $\gamma_2 = 0$ and the coefficients of $\eta^j$ $(j = 0,1,2)$ are zero, in particular, $a_{14}\beta_2^4 + a_{23}\beta_2^3 + a_{32}\beta_2^2 = 0$. In the same reason, we have $a_{14}\beta_3^4 + a_{23}\beta_3^3 + a_{32}\beta_3^2 = a_{14}\beta_4^4 + a_{23}\beta_4^3 + a_{32}\beta_4^2 = 0$, where $p_i = \pi(P_i)$ $(i = 3,4)$ are given by $(\xi,\eta) = (\beta_i, 0)$. Since $\beta_2,\beta_3,\beta_4$ are mutually distinct non-zero numbers, we have $a_{14} = a_{23} = a_{32} = 0$. Thus

$$F(\xi,\eta) = \eta^4 + (a_{13}\xi^3 + \cdots + a_{40})\eta^2 + (a_{06}\xi^6 + \cdots + a_{60}),$$

whence there exists an automorpism $(\xi,\eta) \mapsto (\xi,-\eta)$ of Y. Hence, $|6P_1|$ is composite which contradicts the assumption. Thus there exists a bielliptic involution of $X$ and $P_i$ $(i = 1,2,3,4)$ are its fixed points.

**Case $\boldsymbol{\mu = 10}$.** Since $\alpha_1 = \alpha_3 = 0$, $\alpha_5 \neq 0$, by computation we have

$$\begin{aligned} x^2 = s^{12} &+ 2\alpha_2 s^{14} + (\alpha_2^2 + 2\alpha_4)s^{16} + 2\alpha_5 s^{17} \\ &+ (2\alpha_2\alpha_4 + 2\alpha_6)s^{18} + (2\alpha_2\alpha_5 + 2\alpha_7)s^{19} + \cdots . \end{aligned} \tag{2.12}$$

Since

$$A_{14} = -2\alpha_2 + 2\alpha_2 + a_{14} + a_{21} = 0 \text{ and } a_{21} = 0,$$

we have $a_{14} = 0$. By

$$A_{19} = -2\alpha_7 + 2\alpha_2\alpha_5 + 2\alpha_7 + (a_{14} + 2a_{21})\alpha_5 = 2\alpha_2\alpha_5 = 0$$

and $\alpha_5 \neq 0$ we have $\alpha_2 = 0$. Thus, we have

$$\begin{aligned} A_{18} = &-2\alpha_6 + 2\alpha_2\alpha_4 + 2\alpha_6 + (a_{14} + 2a_{21})\alpha_4 \\ &+ a_{21}\alpha_2^2 + (a_{15} + 2a_{22})\alpha_2 + a_{23} + a_{30} = a_{23} = 0. \end{aligned}$$

Namely, in $F(\xi,\eta)$ the coefficient of $\eta$ is $a_{32}\xi^2 + a_{41}\xi + a_{50}$. Thus, by the same discussion as in the case $\mu = 6$, we have a contradiction.

This completes the proof. $\square$

## 3. Main Theorem

In this section, we shall give the main theorem of this paper mentioned in Section 1.

**Theorem 2** *Let $X$ be a curve of genus 5 having 24 Weierstrass points $P_1, \ldots, P_{24}$ whose gap sequences are $\{1,2,3,5,9\}$, respectively. Then, $X$ has exactly 3 bielliptic involutions $\sigma_i$ $(i = 1,2,3)$ and each of $P_j$ $(j = 1,\ldots,24)$ is a fixed point of some $\sigma_i$.*

To prove this theorem, we shall give a quick review of a sequence of Exercises in [2, VI, F] and its corollaries.

Let $\Gamma$ be the locus of quadrics of rank $\leq 4$ in $\mathbb{P}^4$ containing the canonical model of $X$. Since $X$ is neither hyperelliptic nor trigonal, by [2, VI, F3–12], we have the following fact:

**Lemma 3.1**

i) *$\Gamma$ is a plane quintic which has at most ordinary nodes as its singular points.*
ii) *These nodes correspond to half canonical pencils on $X$.*
iii) *There exists a correspondence from $W_4^1(X)$ to $\Gamma$.*
iv) *There exists a line component $l$ of $\Gamma$ if and only if there exists a two-sheeted covering $X \to E$, where $E$ is an elliptic curve, i.e. a bielliptic covering. In this case, $l$ corresponds to the pull-back of the $g_2^1$'s on $E$.*

By this fact, we have:

**Corollary 3.2**

i) *If $X$ has no bielliptic involution, then there are at most 7 half canonical pencils on $X$.*
ii) *If $X$ has at most 2 bielliptic involutions, then there are at most 8 half canonical pencils on $X$.*

*Proof.* For i), since there is no bielliptic involution of $X$, there is no line component of $\Gamma$. If $\Gamma$ is irreducible of degree 5, then there are at most 6 nodes on $\Gamma$. In case $\Gamma$ has 2 irreducible components $\Gamma_2$ and $\Gamma_3$ of degree 2 and 3, respectively, then there are 6 common points of $\Gamma_2$ and $\Gamma_3$ (they should be nodes on $\Gamma$) and at most one singular point of $\Gamma_3$. Thus there are at most 7 half canonical pencils on $X$. For ii), assume there is exactly one bielliptic involution of $X$. Then, beside one line component $l_1$ of $\Gamma$, there is one irreducible component $\Gamma_4$ of degree 4 or there are 2 irreducible conics $\Gamma_2$ and $\Gamma_2'$. Thus, there are at most 8 nodes on $\Gamma$. In case there are exactly 2 bielliptic involutions on $X$, $\Gamma$ consists of 2 line componets $l_1$, $l_2$ and one irreducible cubic $\Gamma_3$. Thus, there are at most 8 nodes on $\Gamma$. $\square$

**Corollary 3.3** *Assume $X$ has a bielliptic involution $\sigma$ and $l$ is a line component of $\Gamma$ which corresponds to $\sigma$. Let $P$ be a Weierstrass point of Type $II_5$ on $X$. Then $|4P|$ corresponds to a point on $l$ if and only if $P$ is a fixed point of $\sigma$.*

*Proof.* It is obvious by Lemma 3.1 iii). $\square$

*Proof of Theorem 2.* First note that all the $|4P_i|$ $(i = 1, \dots 24)$ are half canonical pencils and that $4P_i \equiv 4P_j$ is equivalent to $|4P_i| = |4P_j|$.

Assume there exists no bielliptic involution of $X$. Then, all the $|6P_i|$ $(i = 1, \dots 24)$ are simple nets. Hence, by Theorem 1, each $4P_i$ is equivalent to at most two $4P_j$'s. Therefore, there are at least $24/3 = 8$ different $|4P_i|$'s, i.e. at least 8 half canonical pencils on $X$. On the other hand, by Corollary 3.2, there are at most 7 half canonical pencils on $X$. This is a contradiction. Thus, there exists at least one bielliptic involution $\sigma_1$ of $X$.

Assume $X$ has exactly one bielliptic involution $\sigma_1$. Then, $\sigma_1$ has exactly 8 fixed points among $P_i$'s, say $P_1, \dots, P_8$. Since $P_i$ $(i = 9, \dots, 24)$ are not fixed by $\sigma_1$, again by Theorem 1, each $4P_i$ is equivalent to at most two $4P_j$'s. Therefore, there are at least 6 different $|4P_i|$'s among $i = 9, \dots 24$. Let $l_1$ be the corresponding line component of $\Gamma$ corresponding to $\sigma_1$. Then, by Corollary 3.3 and its proof, $|4P_i|$ $(i = 9, \dots, 24)$ correspond to nodes outside $l_1$, but by Corollary 3.2, there are at most 4 such nodes. A contradiction.

Assume $X$ has exactly two bielliptic involutions $\sigma_1, \sigma_2$ and $P_i$ $(i = 17, \dots, 24)$ are not fixed by these involutions. Then, by a similar argument as above, there are at least 3 different $|4P_i|$'s among $i = 17, \dots 24$, while there is at most one node outside the lines $l_1, l_2$ corresponding to $\sigma_1, \sigma_2$. Thus, there exist exactly three bielliptic involutions of $X$. This completes the proof. $\square$

**Remark** It is shown in [7] that a curve $X$ having 24 Weierstrass points with three bielliptic involution has to be equivalent to the curve defined by $y^4 = x^2(x^4 - 1)$. It is equivalent to the Wiman curve given in [10] which has the automorphism group of order 192, the maximal order among curves of genus 5.

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Department of Mathematical Sciences
Faculty of Science, Yamaguchi University
Yamaguchi, 753-8512, Japan
E-mail: kato@yamaguchi-u.ac.jp