On Beurling's theorem

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Introduction

Let R, R' be hyperbolic Riemann surfaces and ϕ be an analytic mapping of R into R'. Let K_0 be a closed disk in R and let $R_0=R-K_0$. Let $\tilde{C}$ be the Kuramochi capacity on $R_0 \cup \Delta_N$ and Δ_1 be the set of all minimal Kuramochi boundary points of R. For a metrizable compactification R'^* of R', we denote by $\mathscr{F}(\phi)$ the set of all points in Δ_1 at which ϕ has a fine limit in R'^*. There are two typical extensions of Beurling's theorem [1] to analytic mappings of a Riemann surface to another one, i. e., Z. Kuramochi's [5, 6, 7] and C. Constantinescu and A. Cornea's theorems [3, 4]. The former result states that if ϕ is an almost finitely sheeted mapping and R'^* is H. D. separative, then $\tilde{C}(\Delta_1-\mathscr{F}(\phi))=0$. The latter one states that if ϕ is a Dirichlet mapping and R'^* is a quotient space of the Royden compactification of R', then $\tilde{C}(\Delta_1-\mathscr{F}(\phi))=0$. The present author [9] proved that these two results are independent. In this paper we shall give an another extension of Beurling's theorem such that it contains the above two results: If ϕ is a Dirichlet mapping and R'^* is H. D. separative, then Beurling's theorem is valid.

Notation and terminology

Let R be a hyperbolic Riemann surface. For a subset A of R, we denote by ∂A and A^i the (relative) boundary and the interior of A respectively. We call a closed or open subset A of R is *regular* if ∂A is non-empty and consists of at most a countable number of analytic arcs clustering nowhere in R. We fix a closed disk K_0 in R once for all and let $R_0=R-K_0$.

1. Function spaces and compactifications (cf. [4]).

We denote by $BC=BC(R)$ the space of all bounded continuous (real-valued) functions on R. Let $BCW=BCW(R)$ be (resp. $BCD=BCD(R)$) the family of all bounded continuous Wiener functions (resp. bounded continuous Dirichlet functions) on R. It is known ([4]) that both BCW and BCD are vector sublattices of BC with respect to the maximum and minimum opera-

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tions and that $BCD \subset BCW$. Let $D^{\infty}=D^{\infty}(R)$ be the family of all C^{∞}-functions in BCD.

We refer to [4] for the definitions and properties of Q-compactification R_Q^* of R, the Kuramochi compactification R_N^*, the Royden compactification R_D^*, the Wiener compactification R_W^*. For a subset A of R, we denote by $\bar{A}^Q$ the closure of A in R_Q^* ($Q=N$, D, or W). Let R_1^* and R_2^* be two compactifications of R. If there exists a continuous mapping π of R_1^* onto R_2^* whose restriction to R is the identity and $\pi^{-1}(R)=R$, then we shall say that π is a *cononical mapping* of R_1^* onto R_2^* and that R_2^* is a quotient space of R_1^*. We note that R_N^* is a quotient space of R_D^* and R_D^* is a quotient space of R_W^* ([4]).

2. Dirichlet principle and capacitary potentials

We follow C. Constantinescu and A. Cornea [4] for the definition and properties of Dirichlet functions and the operation $f \to f^F$. For a Dirichlet function f and an open set G in R, we denote by $\|f\|=\|f\|_G$ the Dirichlet norm of f on G. Let G be a regular open set in R with $G \neq R$ in the rest of this section. Let F be a regular closed set in R such that $\bar{F}^D \cap \overline{R-G}^D =\emptyset$. Then there is a function f in BCD such that $f=0$ on $R-F$ and $=1$ on F. Since $f^{(R-G)\cup F}(z)$ does not depend on the choice of such an f, it is denoted by $\omega(\partial F, z, G-F)$ in [5, 7] and by $\overset{G}{1}_F(z)$ in [9]. Let $\{F_n\}_{n=1}^{\infty}$ be a decreasing sequence of regular closed sets in R such that $\bigcap_{n=1}^{\infty} F_n=\emptyset$ and $\bar{F}_1^D \cap \overline{R-G}^D=\emptyset$. Then $\omega(\partial F_n, z, G-F_n)=\overset{G}{1}_{F_n}(z)$ converges locally uniformly and in Dirichlet norm to a harmonic function on G as $n \to \infty$. The limit function is denoted by $\omega(\{F_n\}, z, G)$ in [5, 7].

Let E be a closed subset of Δ_N and F be a regular closed set in R such that $\bar{F}^D \cap \overline{R-G}^D=\emptyset$. We set $E_n=\{z \in R;\, d(z, E) \leqq 1/n\}$ where d is a metric on R_N^*. For each n, we can find a regular closed set F_n in R such that

$$(*) \qquad E_{n+1} \subset F_n \subset E_n - \partial E_n .$$

The function $\omega(\{F_n \cap F\}, z, G)$ is denoted by $\omega(E \cap B(F), z, G)$ in [7].

For a closed subset A of Δ_D, we consider ([8, 9]) the following function:

$$\tilde{\omega}(A)=\tilde{\omega}_a(A)=\inf\left\{\begin{matrix} s(a);\ \text{full-superharmonic}^{1)} \geqq 0 \text{ on } R_0,\ s \geqq 1 \text{ on} \\ U \cap R_0 \text{ for some neighborhood } U \text{ of } A \text{ in } R_D^* \end{matrix}\right\} (a \in R_0).$$

Then $a \to \tilde{\omega}_a(A)$ is harmonic on R_0 and vanishes on ∂K_0. Furthermore $\|\tilde{\omega}(A)\|<\infty$. The capacity $C(A)$ (with respect to K_0) is defined ([8, 9]) by

1) This is called voll-superharmonisch in [4].

$$C(A)=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial\tilde{\omega}(A)}{\partial\nu}ds=\frac{1}{2\pi}\|\tilde{\omega}(A)\|^2.$$

We denote by $\tilde{C}$ the Kuramochi capacity (with respect to K_0) on $R_0\cup\Delta_N$ ([4]). Let π be the canonical mapping of R_D^* onto R_N^* and X be a closed subset of Δ_N. Then $C(\pi^{-1}(X))=\tilde{C}(X)$.

LEMMA 1. *Let E be a closed subset of Δ_N and F be a regular closed set in R. Then $\omega(E\cap B(F), z, R_0)=\tilde{\omega}_z(\pi^{-1}(E)\cap\bar{F}^D)$.*

PROOF. Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed sets in R which satisfies (*). Since $\lim_{n\to\infty}1_{\widetilde{F_n\cap F}}=\tilde{\omega}(\bigcap_{n=1}^\infty\overline{F_n\cap F^D})$ by Corollary 1 to Lemma 11 in [8] and $\bigcap_{n=1}^\infty\overline{F_n\cap F^D}=\pi^{-1}(E)\cap\bar{F}^D$, we obtain that

$$\omega(E\cap B(F), z, R_0)=\omega\left(\{F_n\cap F\}, z, R_0\right)=\lim_{n\to\infty}1_{\widetilde{F_n\cap F}}(z)$$
$$=\tilde{\omega}_z\left(\bigcap_{n=1}^\infty\overline{F_n\cap F^D}\right)=\tilde{\omega}_z(\pi^{-1}(E)\cap\bar{F}^D).$$

PROPOSITION 1. *Let K be a closed subset of Δ_D. Then $C(K)=0$ if and only if there exist a sequence $\{F_n\}_{n=1}^\infty$ of regular closed sets in R and a function f in $BCD(R)$ such that*

(a) $\bar{F}_n^D$ *is a neighborhood of K in R_D^*,*

(b) $\overline{R-F_n^i}^D\cap\bar{F}_{n+1}^D=\emptyset$ $\quad(n=1,2,\cdots)$,

(c) $\bigcap_{n=1}^\infty F_n=\emptyset$,

(d) $f(z)=0$ *for* $z\in\partial F_{2k-1}$ *and* $=1$ *for* $z\in\partial F_{2k}$ $\quad(k=1,2,\cdots)$.

PROOF. The proof of "if" part: Suppose there exist a sequence $\{F_n\}_{n=1}^\infty$ and a function f in $BCD(R)$ which satisfy (a)-(d). We may assume that $F_1\cap K_0=\emptyset$. We set

$$g_{2k-1}=\begin{cases}0 & \text{on } R-F_{2k-1}^i\\ f & \text{on } F_{2k-1}^i-F_{2k},\\ 1 & \text{on } F_{2k}\end{cases}\qquad g_{2k}=\begin{cases}0 & \text{on } R-F_{2k}^i\\ 1-f & \text{on } F_{2k}^i-F_{2k+1}\\ 1 & \text{on } F_{2k+1}\end{cases}$$

$(k=1,2,\cdots)$. Then $g_n\in BCD(R)$ for each n. Since $g_n=0$ on K_0 and $=1$ on F_{n+1}, by the aid of Satz 15.3, (g), in [4], we have $\|1_{\tilde{F}_n}\|\leqq\|g_n\|$ for each n. Since $\|g_{n+1}\|\leqq\|f\|_{F_n^i-F_{n+1}}$, we obtain that $\|1_{\tilde{F}_{n+1}}\|\leqq\|f\|_{F_n^i-F_{n+1}}$ for each n. Since $\|\tilde{\omega}(K)\|\leqq\|1_{\tilde{F}_{n+1}}\|$ by Lemma 9 in [9], we have $\|\tilde{\omega}(K)\|\leqq\|1_{\tilde{F}_{n+1}}\|\leqq\|f\|_{F_n^i-F_{n+1}}$ for each n. Hence we obtain that

$$0\leqq n\|\tilde{\omega}(K)\|^2\leqq\sum_{k=1}^n\|f\|^2_{F_k^i-F_{k+1}}\leqq\|f\|^2_{R-F_{n+1}^i}\leqq\|f\|^2<\infty$$

($n=1, 2, \cdots$). By letting $n \to \infty$, we obtain that $\tilde{\alpha}(K)=0$ and $C(K)=0$.

The "only if" part follows from a discussion similar to that in the proof of Theorem 2 in [9].

3. Dirichlet mappings

Let R and R' be hyperbolic Riemann surfaces and ϕ be an analytic mapping of R into R'. For any $a' \in R'$, let $n_\phi(a')$ be the number of points in $\phi^{-1}(a')$ counting its multiplicity. If $\sup_{a' \in R'} n_\phi(a') < +\infty$, then ϕ is said to be finitely sheeted.

DEFINITION 1 ([4]). ϕ is said to be a *Dirichlet mapping* if there exists a continuous extension of ϕ from R_D^* to $R_D'^*$. In this case we denote by ϕ the continuous extension of ϕ again.

DEFINITION 2 ([5, 7]). ϕ is said to be an *almost finitely sheeted mapping* if it satisfies the following two conditions:

(a) There exists a compact set K' in R' such that $\sup_{a' \in R' - K'} n_\phi(a') < \infty$.

(b) For each $a' \in R'$, there exists a neighborhood $U(a')$ of a' mapped onto a disk in a complex plane by a local parameter at a' such that the covering surface lying over $U(a')$ by ϕ has a finite total area measured with respect to the local parameter.

THEOREM 1. *If ϕ is a Dirichlet mapping and K' is a compact subset of the Royden boundary Δ_D' of R', then $C'(K')=0$ implies $C(\phi^{-1}(K'))=0$, where C' is the capacity on Δ_D' with respect to $K_0' = \phi(K_0)$.*

PROOF. Let K' be a compact subset of Δ_D' with $C'(K')=0$. Then it follows from Proposition 1 that there exist a decreasing sequence $\{F_n'\}_{n=1}^\infty$ of regular closed sets in R' and a function f' in $BCD(R')$ which satisfy (a)–(d) in Proposition 1. Since ϕ is a Dirichlet mapping, $f' \circ \phi$ can be continuously extended over R_D^*. Then it follows from the definition of R_D^* that there is a function g in $BCD(R)$ such that $|f' \circ \phi - g| < 1/3$ on R. We set $f = 3 \min(\max(g, 1/3), 2/3) - 1$. Then $f \in BCD(R)$. If we set $F_n = \phi^{-1}(F_n') \cap R$ ($n=1, 2, \cdots$), then f and $\{F_n\}_{n=1}^\infty$ satisfy (a)–(d) in Proposition 1 for $K = \phi^{-1}(K')$. Thus $C(\phi^{-1}(K'))=0$.

THEOREM 2. *ϕ is an almost finitely sheeted mapping if and only if $\xi' \in D^\infty(R')$ implies $\xi' \circ \phi \in D^\infty(R)$.*

PROOF. Suppose ϕ is almost finitely sheeted. Then there is a compact set K' in R' and a positive constant t such that $n_\phi(a') \leqq t$ for all $a' \in R' - K'$. Let ξ' be any function in $D^\infty(R')$. Then we see that $\|\xi' \circ \phi\|^2_{\phi^{-1}(R'-K')} \leqq t \|\xi'\|^2_{R'-K'}$. On the other hand, since K' is compact, there exists a finite family of open disks $\{U(a_i')\}_{i=1}^n$ in R' such that each $U(a_i')$ satisfies (b) in Definition 2 and $\bigcup_{i=1}^n U(a_i')\supset K'$. Since

$$\iint_{\phi^{-1}(U(a_i'))}|\phi'|^2 dxdy<\infty \qquad (i=1,2,\cdots,n),$$

for the above ξ', we have

$$\begin{aligned}\|\xi'\circ\phi\|^2_{\phi^{-1}(U(a_i'))}&=\iint_{\phi^{-1}(U(a_i'))}|\text{grad}\,\xi'|^2|\phi'|^2 dxdy\\ &\leqq \max_{U(a_i')}|\text{grad}\,\xi'|^2\iint_{\phi^{-1}(U(a_i'))}|\phi'|^2 dxdy<\infty\end{aligned}$$

$(i=1,2,\cdots,n)$. Thus we have

$$\|\xi'\circ\phi\|^2_{\phi^{-1}(K')}\leqq\sum_{i=1}^n\|\xi'\circ\phi\|^2_{\phi^{-1}(U(a_i'))}<\infty\,.$$

Hence $\xi'\circ\phi\in D^\infty(R)$.

Conversely suppose $\xi'\in D^\infty(R')$ implies $\xi'\circ\phi\in D^\infty(R)$. For any relatively compact open disk U in R', there exists a function ξ' in $D^\infty(R')$ such that $\inf_U|\text{grad}\,\xi'|>0$ where grad ξ' is calculated with respect to a local parameter on U. Then we have

$$\begin{aligned}0&\leqq\inf_U|\text{grad}\,\xi'|^2\iint_{\phi^{-1}(U)}|\phi'|^2 dxdy\leqq\|\xi'\circ\phi\|^2_{\phi^{-1}(U)}\\ &\leqq\|\xi'\circ\phi\|^2_R<\infty\,.\end{aligned}$$

Hence we have

$$0\leqq\iint_{\phi^{-1}(U)}|\phi'|^2 dxdy\leqq\frac{\|\xi'\circ\phi\|^2}{\inf_U|\text{grad}\,\xi'|^2}<\infty\,.$$

Thus ϕ satisfies (b) in Definition 2. Next suppose $\sup_{a'\in R'-K'} n_\phi(a')=+\infty$ for any compact set K' in R'. Then there exists a sequence $\{a_n'\}_{n=1}^\infty$ of points in R' tending to the ideal boundary of R' such that $n_\phi(a_n')\geqq 2^n$ and $\phi^{-1}(a_n')$ contains at least distinct 2^n points. There exists a family of mutually disjoint neighborhoods $\{V(a_n')\}_{n=1}^\infty$. For each n, there are 2^n distinct points $\{a_n^i\}_{i=1}^{2^n}$ in $\phi^{-1}(a_n')$ and neighborhoods $U(a_n')$ of a_n' and $U(a_n^i)$ $(i=1,2,\cdots,2^n)$ such that $U(a_n')\subset V(a_n')$ and each $U(a_n^i)$ is conformally equivalent to $U(a_n')$ by ϕ. For each n, we can find $\xi_n'\in D^\infty(R')$ such that $0\leqq\xi_n'\leqq 1$ on R', $\xi_n'=0$ on $R'-U(a_n')$ and $\|\xi_n'\|^2=1/2^n$. If we set $\xi'=\sum_{n=1}^\infty\xi_n'$, then $\xi'\in D^\infty(R')$. Since

$\|\xi'\circ\phi\|^2_{U(a_n^i)}=\|\xi'\|^2_{U(a_n')}=1/2^n$, we have $\|\xi'\circ\phi\|^2\geqq\|\xi'\circ\phi\|^2_{\phi^{-1}(U(a_n'))}\geqq\sum_{i=1}^{2^n}\|\xi'\circ\phi\|^2_{U(a_n^i)}=2^n$. Hence we have $\|\xi'\circ\phi\|=\infty$, which is a contradiction. This completes the proof.

COROLLARY 1. *ϕ is finitely sheeted if and only if $\xi'\in BCD(R')$ implies $\xi'\circ\phi\in BCD(R)$.*

PROOF. The "only if" part is obvious. We shall prove the "if" part. Since $\xi'\in D^\infty(R')$ implies $\xi'\circ\phi\in D^\infty(R)$, it follows from Theorem 2 that ϕ is almost finitely sheeted. Hence there is a compact set K_0' in R' such that $\sup_{a'\in R'-K_0'} n_\phi(a')<\infty$. Suppose $\sup_{a'\in K'} n_\phi(a')=+\infty$ for some compact set K' in R'. Then there exists a sequence $\{a_n'\}_{n=1}^\infty$ of points in R' such that a_n' tends to a point a_0' in K' as $n\to\infty$ and $n_\phi(a_n')\geqq 2^n$ for each n. Let $\{U(a_n')\}_{n=1}^\infty$ and $\{U(a_n^i); (i=1,2,\cdots,2^n; n=1,2,\cdots)\}$ as in the proof of Theorem 1. For each n, we can find $\xi_n'\in BCD(R')$ such that $0\leqq\xi_n'\leqq 1/n$ on R', $\xi_n'=0$ on $R'-U(a_n')$, $\|\xi_n'\|=1/2^n$. We set $\xi'=\sum_{n=1}^\infty\xi_n'$. Since $\sum_{n=1}^m\xi_n'$ converges to ξ' uniformly on R' as $m\to\infty$ and is a Cauchy sequence in Dirichlet norm, it can be seen that $\xi'\in BCD(R')$. Since $\|\xi'\circ\phi\|^2_{\phi^{-1}(U(a_n'))}\geqq\sum_{i=1}^{2^n}\|\xi'\circ\phi\|^2_{U(a_n^i)}=1$ $(n=1,2,\cdots)$, we have $\|\xi'\circ\phi\|=\infty$, which is a contradiction. Therefore we complete the proof.

COROLLARY 2. *If ϕ is an almost finitely sheeted mapping, then it is a Dirichlet mapping.*

4. Suclass W_{HD} of BCW

DEFINITION 3. A function f in $BC(R)$ is said to be *H.D. separative* if $C(\overline{\{f\leqq\alpha\}}^D\cap\overline{\{f\geqq\beta\}}^D)=0$ for any α and β ($\inf f<\alpha<\beta<\sup f$).

We denote by $W_{HD}=W_{HD}(R)$ the family of all bounded continuous H.D. separative functions on R. By the proof of Theorem 5 in [9], we see that W_{HD} is a vector sublattice of BC with respect to the maximum and minimum operations and that W_{HD} contains BCD. Furthermore W_{HD} is closed with respect to the sup norm.

We refer to [8, 9] for the definition of H.D. separative compactifications. It follows from the Corollary to Proposition 8 in [9] that $f\in BC$ is H.D. separative if and only if $R^*_{\{f\}}$ is H.D. separative.

PROPSOITION 2. (a) *$BCD\subset W_{HD}\subset BCW$. These inclusion relations are both strict.*

(b) *Let f be any function in W_{HD} and ξ be any point in Δ_D with $C(\{\xi\})>0$. Then $\lim_{z\to\xi}f(z)$ exists.*

(c) *A compactification R^* of R is H.D. separative if and only if there is a non-empty subfamily Q of W_{HD} such that $R^*=R_Q^*$.*

(d) *Let R^* be H.D. separative. Then $\{f|R; f\in C(R^*)\}\subset W_{HD}$, where $f|R$ is the restriction of f to R.*

PROOF. By Theorem 7 and examples 1, 2, and 3 in [9] (see Diagram 1 in [9]), we have (a). The proof of (b) follows immediately from the definition of H.D. separativeness. By the aid of Theorem 5 in [9], we have (c) and (d).

LEMMA 2. *If ϕ is a Dirichlet mapping, then*

$$\left\{f\circ\phi; f\in W_{HD}(R')\right\}\subset W_{HD}(R).$$

PROOF. Let f be any non-constant function in $W_{HD}(R')$. For any α and β ($\inf f<\alpha<\beta<\sup f$), let $A=\{z\in R; (f\circ\phi)(z)\leqq\alpha\}$, $B=\{z\in R; (f\circ\phi)(z)\geqq\beta\}$, $A'=\{z'\in R'; f(z')\leqq\alpha\}$ and $B'=\{z'\in R'; f(z')\geqq\beta\}$. Then $A=\phi^{-1}(A')$ and $B=\phi^{-1}(B')$. Since ϕ is a Dirichlet mapping, $\overline{A}^D=\overline{\phi^{-1}(A')}^D\subset\phi^{-1}(\overline{A'}^D)$ and $\overline{B}^D=\overline{\phi^{-1}(B')}^D\subset\phi^{-1}(\overline{B'}^D)$. Hence we have $\overline{A}^D\cap\overline{B}^D\subset\phi^{-1}(\overline{A'}^D\cap\overline{B'}^D)$. Since $f\in W_{HD}(R')$, $C(\overline{A'}^D\cap\overline{B'}^D)=0$. Thus it follows from Theorem 1 that $0\leqq C(\overline{A}^D\cap\overline{B}^D)\leqq C(\phi^{-1}(\overline{A'}^D\cap\overline{B'}^D))=0$. Hence $f\circ\phi\in W_{HD}(R)$.

5. Beurlings, theorem

For each $b\in\Delta_1(\subset\Delta_N)$, let $\mathscr{G}_b=\{G; G$ is open in R and $R-G$ is thin at $b\}$[2]. Let X be a compact Hausdorff space and ϕ be a continuous mapping of R into X. For any $b\in\Delta_1$, we set $\phi^\vee(b)=\bigcap_{G\in\mathscr{G}_b}\overline{\phi(G)}$, where $\overline{\phi(G)}$ is the closure of $\phi(G)$ in X. It is known ([4]) that $\phi^\vee(b)$ is a single point or a continuum. Let $\mathscr{I}(\phi)=\{b\in\Delta_1; \phi^\vee(b)$ is a single point$\}$. Then it is known ([4]) that $\mathscr{I}(\phi)$ is a Borel set. In this section we shall denote by π the canonical mapping of R_D^* onto R_N^*.

LEMMA 3 ([7; Lemma 4]). *Let G be a regular open set in R with $G\neq R$ and E be a closed subset of Δ_N with $\tilde{C}(E)>0$. If there is a closed subset A of Δ_D such that $A\cap\overline{R-G}^D=\emptyset$ and $C(\pi^{-1}(E)\cap A)>0$, then there is $b\in E\cap\Delta_1$ with $G\in\mathscr{G}_b$.*

PROOF. By assumption, we can find a regular closed set F in R such that $\overline{F}^D\cap\overline{R-G}^D=\emptyset$ and $\overline{F}^D$ is a neighborhood of A in R_D^*. Then we have $C(\pi^{-1}(E)\cap\overline{F}^D)\geqq C(\pi^{-1}(E)\cap A)>0$. Let $\{F_n\}_{n=1}^\infty$ be a sequence for E which

2) This is denoted by $\tilde{\mathscr{G}}_b$ in [4] (p. 221).

satisfies (*). Since $\lim_{n\to\infty} 1_{\widetilde{F_n \cap F}} = \tilde{\alpha}(\pi^{-1}(E)\cap \overline{F^D})$ by Lemma 1, we have $\lim_{n\to\infty} 1_{\widetilde{F_n \cap F}}>0$. It follows from Lemma 7 in [9] that $\lim_{n\to\infty} 1^G_{\widetilde{F_n\cap F}}\neq 0$. Hence $\omega(E\cap B(F), z, G)=\omega(\{F_n\cap F\}, z, G)=\lim_{n\to\infty} 1^G_{\widetilde{F_n\cap F}}\neq 0$. By Lemma 4 in [7], we obtain the conclusion of this lemma.

By a modification of the proof of Theorem 1 in [7], we have

PROPOSITION 3. *If f is a function in $W_{HD}(R)$, then $\tilde{C}(\varDelta_1-\mathscr{I}(f))=0$.*

PROOF. We may assume that $\inf f=0$ and $\sup f=1$. For any $r>0$ and s $(0<s<1)$, let $D(s, r)=\{z\in R;\ |f(z)-s|<r\}$. It is known ([3, 5, 7]) that $\varDelta_1-\mathscr{I}(f)=\bigcup_{n=1}^{\infty}\bigcap_{i=0}^{2^n}\{b\in\varDelta_1;\ D(i/2^n, 2/n)\notin\mathscr{G}_b\}$. Suppose $\tilde{C}(\varDelta_1-\mathscr{I}(f))>0$. Then there exist n_0 and a compact subset E of $\bigcap_{i=0}^{2^{n_0}}\{b\in\varDelta_1;\ D(i/2^{n_0};\ 2/n_0)\notin\mathscr{G}_b\}$ such that $\tilde{C}(E)>0$. Let r and r' be real numbers such that $1/n_0<r<r'<2/n_0$. For each i, we can find a regular closed sets F_i and F_i' in R such that $D(i/2^{n_0}, 1/n_0)\subset F_i\subset D(i/2^{n_0}, r)\subset D(i/2^{n_0}, r')\subset R-F_i'\subset D(i/2^{n_0}, 2/n_0)$. Since $\overline{F_i^*}\cap\overline{F_i'^*}=\emptyset$ ($\overline{F_i^*}$ and $\overline{F_i'^*}$ are closures of F_i and F_i' in $R^*_{\{f\}}$ respectively) and $R^*_{\{f\}}$ is H.D. separative, it follows from Theorem 2 in [8] that $C(\overline{F_i^D}\cap\overline{F_i'^D})=0$. Snce $\bigcup_{i=0}^{2^{n_0}} D(i/2^{n_0}, 1/n_0)=R$ and $D(i/2^{n_0}, 1/n_0)\subset F_i$ for each i, $\bigcup_{i=0}^{2^{n_0}} F_i=R$. Hence there is i_0 such that $C(\pi^{-1}(E)\cap\overline{F_{i_0}^D})>0$. Since $C(\overline{F_{i_0}^D}\cap\overline{F_{i_0}'^D})=0$, for any ε $(0<\varepsilon<C(\pi^{-1}(E)\cap\overline{F_{i_0}^D}))$, we can find a relatively open subset α of $\varDelta_D$ such that α is a neighborhood of $\overline{F_{i_0}^D}\cap\overline{F_{i_0}'^D}$ in $\varDelta_D$ and $C(\alpha)<\varepsilon$. Then $(\overline{F_{i_0}^D}-\alpha)\cap\overline{F_{i_0}'^D}=\emptyset$ and

$$\varepsilon<C(\pi^{-1}(E)\cap\overline{F_{i_0}^D})\leqq C(\pi^{-1}(E)\cap(\overline{F_{i_0}^D}-\alpha))+C(\alpha)\,.$$

Hence we have $C(\pi^{-1}(E)\cap(\overline{F_{i_0}^D}-\alpha))>0$. Thus, by Lemma 3, there exists $b\in E\cap\varDelta_1$ with $R-F_{i_0}'\in\mathscr{G}_b$. This shows that $D(i_0/2^{n_0}, 2/n_0)\in\mathscr{G}_b$. This is a contradiction. Therefore $\tilde{C}(\varDelta_1-\mathscr{I}(f))=0$.

COROLLARY ([3, 4]). *If f is a function in $BCD(R)$, then $\tilde{C}(\varDelta_1-\mathscr{I}(f))=0$.*

THEOREM 3. *If ϕ is a Dirichlet mapping of R into R' and R'^* is a metrizable H.D. separative compactification of R', then $\tilde{C}(\varDelta_1-\mathscr{I}(\phi))=0$.*

PROOF. Since $C(R'^*)$ is separable with respect to the sup norm, there exists a dense countable subset A of $C(R'^*)$ in $C(R'^*)$. We set $Q=\{f|R';\ f\in A\}$ where $f|R'$ is the restriction of f to R'. Then $R'^*=R_Q'^*$ and $Q\subset W_{HD}(R')$ by (d) in Proposition 2. Since $\varDelta_1-\mathscr{I}(\phi)\subset\bigcup_{f\in Q}(\varDelta_1-\mathscr{I}(f\circ\phi))$ and $\tilde{C}(\varDelta_1-\mathscr{I}(f\circ\phi))=0$ for any $f\in Q$ by Lemma 2 and the above proposition, we have $\tilde{C}(\varDelta_1-\mathscr{I}(\phi))\leqq\sum_{f\in Q}\tilde{C}(\varDelta_1-\mathscr{I}(f\circ\phi))=0$.

Since any almost finitely sheeted mappng is a Dirichlet mapping (Corollary 2 to Theorem 1) and any quotient space of the Royden compactification is H.D. separative (Theorem 3 in [8]), we have the following corollaries:

COROLLARY 1 (Z. Kuramochi [5, 6, 7]). *If ϕ is an almost finitely sheeted mapping and R'^* is a metrizable H.D. separative compactification, then* $\widetilde{C}(\Delta_1 - \mathscr{F}(\phi))=0$.

COROLLARY 2 (C. Constantinescu and A. Cornea [3, 4]). *If ϕ is a Dirichlet mapping and R'^* is a quotient space of R'^*, then* $\widehat{C}(\Delta_1 - \mathscr{F}(\phi))=0$.

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