On the index theorem of Ambrose

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1. Introduction

The index theorem for geodesics under the general boundary condition (two variable end points) has been given by W. Ambrose ([1], see also T. Takahashi [5]). But his proof is very complicated. M. Klingmann ([4]) proved the somewhat more general index theorem using the theory of quadratic forms on Hilbert space. Recently W. Klingenberg ([2], [3]) has obtained the index theorem for closed geodesics from the geodesic flow view point. The purpose of the present note is to give another simple proof of the Ambrose index theorem via Klingenberg's view point. In fact, we need only the fundamental properties of Jacobi fields. Since the concept of conjugate point defined in [1] is not so familiar, we shall give the explicit statement of the Ambrose index theorem for completeness.

Let $(M, \langle\,,\rangle)$ be a riemannian manifold and K, L be submanifolds of M. Let $c:[a, b]\to M$ be a normal geodesic such that $c(a)\in K$, $c(b)\in L$, $\dot{c}(a)\perp T_{c(a)}K$, $\dot{c}(b)\perp T_{c(b)}L$, where $T_{c(a)}K$ etc. denotes the tangent space to K at $c(a)$. We will be concerned with the "number of essentially different curves connecting K and L which are shorter than c". First we shall give some preliminaries.

1.1. Boundary conditions. A boundary condition at $t(a\leqq t\leqq b)$ is, by definition, a pair $\mathscr{S}=(S, A_S)$ where S is a subspace of $\perp\dot{c}(t)$ (the orthogonal complement of $\dot{c}(t)$ in $T_{c(t)}M$) and $A_S: S\to S$ is a self-adjoint linear mapping of S.

EXAMPLE 1. Let P be a submanifold of M which is perpendicular to c at $c(t)$. Then we have the boundary condition (S, A_S) at t by $S:=T_{c(t)}P$, $\langle A_SX, Y\rangle:=H_{\dot{c}(t)}(X, Y)$, where $H_{\dot{c}(t)}$ denotes the second fundamental form of P relative to the normal $\dot{c}(t)$.

Let $\mathscr{J}$ be a vector space of Jacobi fields along c which is perpendicular to c. We shall denote the covariant differentiation with respect to $\dot{c}(t)$ by ∇. If the boundary condition $\mathscr{S}$ at t is given, we define

$$\mathscr{J}_S^*:=\left\{Y\in\mathscr{J}\,|\,Y(t)\in S,\quad \nabla Y(t)-A_SY(t)\perp S\right\}.\quad \dim\mathscr{J}_S^*=\dim M-1.$$

$$\mathscr{J}_S:=\left\{Y\in\mathscr{J}\,|\,Y(t)\in S,\quad \nabla Y(t)=A_SY(t)\right\}.\quad \dim\mathscr{J}_S=\dim S.$$

EXAMPLE 2. Let $\mathscr{S}=(S, A_S)$ (resp. $\mathscr{T}=(T, A_T)$) be a boundary condition at a (resp. b). Then we define the boundary condition $\mathscr{S}^*(t)=(S^*(t), A_{S^*(t)})$ (resp. $\mathscr{T}(t)=(T(t), A_{T(t)})$) at t as follows.

(i) $S^*(t):=\{Y(t)| Y\in \mathscr{J}_S^*\}$ (resp. $T(t):=\{X(t)|X\in \mathscr{J}_T\}$).

(ii) $A_{S^*(t)}Y(t):=pr_{S^*(t)}\nabla Y(t)$ (resp. $A_{T(t)}X(t):=pr_{T(t)}\nabla X(t)$),

where $pr_{S^*(t)}:\perp\dot{c}(t)\to S^*(t)$ etc. denotes the orthogonal projection. Note that (ii) is well-defined. Then it is easy to see that $\mathscr{J}_S^*=\mathscr{J}_{S^*(t)}$ does hold, but $\mathscr{J}_{T(t)}$ is different from $\mathscr{J}_T$ in general.

1.2. Conjugate points. Let $\mathscr{S}$, $\mathscr{T}$ be an ordered pair of boundary conditions at a and b respectively. Let

$C(t_0):=$ vector space of vector fields $Z(t)$ along c for which there exist $Y\in\mathscr{J}_S^*$ and $X\in\mathscr{J}_T$ such that

$Z(u)=Y(u)$ for $u\leqq t_0$, $Z(u)=X(u)$ for $u\geqq t_0$ and

$\nabla Y(t_0)-\nabla X(t_0)\perp T(t_0)$.

Clearly $\mathscr{J}_S^*\cap\mathscr{J}_T\subset C(t_0)$. If $\bar{n}(t_0):=\dim(C(t_0)/\mathscr{J}_S^*\cap\mathscr{J}_T)$ is positive, then we say that t_0 is a conjugate point of the ordered pair $\mathscr{S}$, $\mathscr{T}$ and $\bar{n}(t_0)$ will be called the order of the conjugate point t_0.

1.3. Index theorem. Let c, K, L be as above and $\mathscr{S}$ (resp. $\mathscr{T}$) be the boundary condition defined from K (resp. L) as in Example 1. Let

$\Xi:=$ vector space of H'-vector fields $\xi(t)$ along c such that $\xi(a)\in S$, $\xi(b)\in T$, $\xi(t)\perp\dot{c}(t)$.

We put $RX(t)=R(\dot{c}(t), X(t))\dot{c}(t)$, where $R(X, Y)Z$ denotes the curvature tensor of ∇, and define the index form I_{ST} on Ξ by

$$I_{ST}(X, Y):=\int_a^b\left\{\langle\nabla X(t), \nabla Y(t)\rangle+\langle RX(t), Y(t)\rangle\right\}dt+\langle A_SX(a), Y(a)\rangle -\langle A_TX(b), Y(b)\rangle$$

$$\Big[=\int_a^b\langle RX(t)-\nabla\nabla X(t), Y(t)\rangle dt+\sum\langle\nabla X(t_i-0)-\nabla X(t_i+0), Y(t_i)\rangle +\langle\nabla X(b)-A_TX(b), Y(b)\rangle-\langle\nabla X(a)-A_SX(a), Y(a)\rangle\Big].$$

This is a symmetric bilinear form on Ξ and the "number of essentially different curves connecting K and L which are shorter than c" can be defined as the index of I_{ST} on Ξ, i.e., the dimension of the maximal subspace of Ξ on which I_{ST} is negative definite. Now the Ambrose index theorem asserts that this number may be expressed as the sum of the orders of conjugate points plus "convexity".

Ambrose Index Theorem. *Index of* $I_{ST}=\sum_{a<t<b}\bar{n}(t)+$ *Convexity.*

Convexity is defined as follows. We put $\mathfrak{N}=\{X\in\mathscr{J}_T|X(a)\in S\}$. On $\mathfrak{N}$, we have $I_{ST}(X, X')=\langle A_S X(a)-\nabla X(a), X'(a)\rangle$. Clearly $\mathscr{J}_S^*\cap\mathscr{J}_T\subset$ Null space of $I_{ST|\mathfrak{N}}$. we define

Convexity $:=\dim((\text{Null space of } I_{ST|\mathfrak{N}})/(\mathscr{J}_S^*\cap\mathscr{J}_T))+\text{index } I_{ST|\mathfrak{N}}$.

REMARK. The definition of convexity given in [1] has a different expression. But they are equivalent. See § 2.

2. Proof of the theorem.

2.1 (See [2], [3]). We shall assume dim $M=n+1$. Let $\tau: T^{2n}T_1M\to T_1M$ be the subbundle of the tangent bundle of T_1M (unit tangent bundle of M) consisting of the vectors orthogonal to the geodesic spray. Then for $X_0\in T_1M$, we have the splitting $T^{2n}_{X_0}T_1M=T^n_{X_0h}\oplus T^n_{X_0v}$ of $T^{2n}_{X_0}T_1M=\tau^{-1}(X_0)$ into the horizontal and vertical subspaces. If a normal geodesic $c(t)$, $a\leqq t\leqq b$ is given, from the immersion $\dot{c}:[a, b]\to T_1M$, we have an induced bundle $\tau^{2n}: V^{2n}\to[a, b]$ of τ. Let $\tau^n_h\oplus\tau^n_v$ be the corresponding decomposition of τ^{2n} into its horizontal and vertical subbundles over $[a, b]$. Now there is a natural symplectic structure α on τ^{2n} defined by

$$2\alpha((X_h, X_v), (Y_h, Y_v)):=\langle X_h, Y_v\rangle-\langle Y_h, X_v\rangle\,.$$

Let ϕ_t be the geodesic flow. Then for $(A, B)\in V^{2n}(t_0)=(\tau^{2n})^{-1}(t_0)$ we have $d\phi_t(A, B)=(Y(t), \nabla Y(t))$, where $Y(t)$ is a Jacobi field along c such that $Y(t_0)=A$ and $\nabla Y(t_0)=B$. In the following we shall put $\tilde{Y}(t):=(Y(t), \nabla Y(t))$. It is well known that $d\phi_t$ preserves the symplectic form α. A subspace W of $V^{2n}(t)$ will be called isotropic if $\alpha_{|W}\equiv 0$.

2.2. Now we shall construct an n-dimensional isotropic subspace $V^n=V^n(b)$ from the boundary conditions $\mathscr{S}$, $\mathscr{T}$. This will play an essential role in the following proof. Put $N:=S^*(b)\cap T$ and let $S^*(b):=S_1\oplus N$, $T=T_1\oplus N$, $\perp\dot{c}(b):=S_1\oplus T_1\oplus N\oplus A$ be the orthogonal decompositions of $S^*(b)$, T and $\perp\dot{c}(b)$ respectively. We shall consider the following subspaces of $V^{2n}(b)$.

$$\text{a)}\quad V_1:=\left\{\tilde{Y}(b)=(Y(b), \nabla Y(b))|Y\in\mathscr{J}^*_{S^*(b)}=\mathscr{J}^*_S, \nabla Y(b)-A_T pr_T Y(b)\perp T\right\}.$$

Let Φ: $S^*(b)\to N$ be the linear mapping defined by $\Phi(x):=pr_N A_{S^*(b)}x-pr_N A_T pr_N x$. Then it is easy to see the following lemma.

LEMMA 1. $\tilde{Y}(b)\in V_1$ *if and only if* $Y(b)\in$ *Ker* Φ *and*

$$pr_{S_1}\nabla Y(b)=pr_{S_1}A_{S^*(b)}Y(b), \qquad pr_{T_1}\nabla Y(b)=pr_{T_1}A_T pr_N Y(b),$$
$$pr_N\nabla Y(b)=pr_N A_T pr_N Y(b) \quad (=pr_N A_{S^*(b)}Y(b)).$$

b) Let $\Psi: N\to S^*(b)$ be the linear mapping which assingns to $x\in N$, $pr_{S^*(b)}(\nabla Y(b)-\nabla x(b))$, where we have chosen $X\in\mathscr{J}_T$ such that $X(b)=x$, and $Y\in\mathscr{J}^*_{S^*(b)}$ such that $Y(b)=x$. Note that this definition does not depend on the choice of $Y\in\mathscr{J}^*_{S^*(b)}$ such that $Y(b)=x$. Let $\dim\Psi(N)=d$ and take $x_1,\cdots,x_d\in N$ such that the corresponding $\Psi(x_1),\cdots,\Psi(x_d)$ form a basis of $\Psi(N)$. Next take $X_i\in\mathscr{J}_T$ such that $X_i(b)=x_i$ $(i=1,\cdots,d)$. Now we define

$V_2:=$ subspace of $V^{2n}(b)$ which is spanned by $\widetilde{X}_i(b)=(X_i(b),\nabla X_i(b))$ $i=1,\cdots,d$.

Clearly $\dim V_2=d$. Finally we put

c) $V_3:=\{\widetilde{X}(b)=(X(b),\nabla X(b))|X\in\mathscr{J}_T,\ X(b)\in T_1\}$. $\dim V_3=\dim T_1$.

Then we have

LEMMA 2. *$V^n:=V_1\oplus V_2\oplus V_3$ is an n-dimensional isotropic subspace of $V^{2n}(b)$.*

Proof. First we shall show that $\dim V_1$ is equal to $\dim S^*(b)+\dim A-d$. In fact, since $\Psi: N\to S^*(b)$ may be expressed in the form $\Psi=A_{S^*(b)}-pr_N A_T$, Ψ is the adjoint linear mapping of Φ. So we have $\dim\operatorname{Ker}\Phi=\dim S^*(b)-d$, and by lemma 1 $\dim V_1=\dim S^*(b)+\dim A-d$. Next it is easy to see $V_1\cap V_2=\{0\}$ and $V_3\cap(V_1\oplus V_2)=\{0\}$. So we get $\dim V^n=n$. Finally we shall show that V^n is isotropic. Since elements of V_1, $V_2\oplus V_3$ satisfy the boundary conditions $\mathscr{S}^*(b)$, $\mathscr{T}$ respectively, we have $\alpha_{|V_1}\equiv 0$ and $\alpha_{|V_2\oplus V_3}\equiv 0$. So we must show $\alpha(V_1, V_2\oplus V_3)=0$. If $\widetilde{Y}(b)\in V_1$ and $\widetilde{X}(b)\in V_2\oplus V_3$, then we have

$$\begin{aligned}\alpha(\widetilde{X},\widetilde{Y})&=\langle\nabla Y(b),X(b)\rangle-\langle Y(b),\nabla X(b)\rangle\\&=\langle A_T pr_T Y(b),X(b)\rangle-\langle pr_T Y(b),A_T X(b)\rangle=0.\end{aligned}$$ q.e.d.

2.3. We put $V^n_v(t):=\{\widetilde{Y}(t)=(Y(t),\nabla Y(t))|Y\in\mathscr{J},\ Y(t)=0\}$, $V^n(t):=d\phi_{t-b}V^n(b)=\{\widetilde{U}(t)=(U(t),\nabla U(t))|U\in\mathscr{J},\ \widetilde{U}(b)\in V^n(b)\}$ and $W(t):=V^n(t)\cap V^n_v(t)$. Now we shall describe the conjugate points of the ordered pair $\mathscr{S}$, $\mathscr{T}$ in terms of $W(t)$. We define a linear mapping $\chi_{t_0}: C(t_0)\to W(t_0)$ as follows. If $Z\in C(t_0)$, then by definition, there exist $X\in\mathscr{J}_T$ and $Y\in\mathscr{J}^*_S(=\mathscr{J}^*_{S^*(b)})$ such that

$Z(u)=Y(u)$ $u\leq t_0$, $Z(u)=X(u)$ $u\geq t_0$, and $\nabla Y(t_0)-\nabla X(t_0)\perp T(t_0)$. Then $\widetilde{Y}(b)=(Y(b),\nabla Y(b))$ belongs to V_1. In fact for any $X_0\in\mathscr{J}_T$, we have

$$\begin{aligned}&\langle\nabla Y(b)-A_T pr_T Y(b),X_0(b)\rangle=\langle\nabla Y(b),X_0(b)\rangle-\langle Y(b),\nabla X_0(b)\rangle\\&=\alpha(\widetilde{X}(b),\widetilde{X}_0(b))-\alpha(\widetilde{Y}(b),\widetilde{X}_0(b))=\alpha(\widetilde{X}(t_0),\widetilde{X}_0(t_0))-\alpha(\widetilde{Y}(t_0),\widetilde{X}_0(t_0))\\&=\langle\nabla Y(t_0)-\nabla X(t_0),X_0(t_0)\rangle=0.\end{aligned}$$

Next note that $X(t)$ can be decomposed into Jacobi fields $X(t)=X_1(t)+X_2(t)+X_3(t)$, where $\widetilde{X}_2(b)\in V_2$, $\widetilde{X}_3(b)\in V_3$, and $X_1(b)\in \operatorname{Ker}\Psi$. Then we have $(pr_{S^*(b)}A_T - pr_{S^*(b)}A_{S^*(b)})\,X_1(b)=0$, and consequently $\widetilde{X}_1(b)\in V_1$. Now we put

$$\chi_{t_0}(Z) := \widetilde{Y}(t_0) - \widetilde{X}(t_0) \in W(t_0).$$

Then it is easy to show χ_{t_0} is surjective and $\operatorname{Ker}\chi_{t_0} = \mathscr{J}_T \cap \mathscr{J}_S^*$. Thus we have the following lemma.

LEMMA 3. *We have* $\dim W(t) = \bar{n}(t)$ *for* $a<t<b$. *Thus* $t_0\in(a,b)$ *is a conjugate point of the ordered pair* $\mathscr{S}$, $\mathscr{T}$ *if and only if* $\dim W(t_0)$ *is positive.*

Now the following is standard. For the proof see [3], Proposition 3. 1.

LEMMA 4. *Let* $n_0=\bar{n}(t_0)=\dim W(t_0)$ *be positive. Choose a basis* $\{\widetilde{U}_i(t_0)\}$ $1\leqq i\leqq n$ of $d\phi_{t_0-b}V^n(b)$ *such that* $\widetilde{U}_i(t_0)$ $1\leqq i\leqq n_0$ *form a basis of* $W(t_0)$. *Then*

(i) $\nabla U_i(t_0)$, $1\leqq i\leqq n_0$, $U_j(t_0)$, $n_0+1\leqq j\leqq n$ *form a basis of* $\perp\dot{c}(t_0)$.

(ii) *For all* $t\neq t_0$, *sufficiently near* t_0, $U_i(t)$, $1\leqq i\leqq n$ *form a basis of* $\perp\dot{c}(t)$. *Thus* $\bar{n}(t)=0$ *except for a finite number of value.*

2. 4. By lemma 3, Ambrose index theorem takes the form

$$\text{Index}\quad I_{ST} = \sum_{a<t<b} \dim W(t) + \text{Convexity}.$$

Now for each conjugate point $t_0\in(a,b)$ of the ordered pair $\mathscr{S}$, $\mathscr{T}$, we shall assign a subspace $\zeta W(t_0)$ which is complementary to $\mathscr{J}_S^*\cap\mathscr{J}_T$ in $C(t_0)$. Then since $\zeta W(t_0)$ consists of once broken Jacobi fields of $C(t_0)$, $I_{ST}(\zeta W(t_0), \zeta W(t_0'))=0$ holds and if $t_0\neq t_0'$ they are lineary independent. Next let ζ_0 be the maximal subspace of $\mathfrak{N}=\{X\in\mathscr{J}_T \mid X(a)\in S\}$ over which $I_{ST|\mathfrak{N}}$ is negative definite, and ζ_1 be a subspace of the null space of $I_{ST|\mathfrak{N}}$ which is complementary to $\mathscr{J}_S^*\cap\mathscr{J}_T$. Then clearly ζ_0, ζ_1, $\zeta W=\bigoplus_{a<t<b}\zeta W(t)$ are linearly independent and $I_{ST}(\zeta_0, \zeta_1\oplus\zeta W)=0$. We put $\zeta=\zeta_0\oplus\zeta_1\oplus\zeta W$. Since $I_{ST|\zeta_1\oplus\zeta W}\equiv 0$ holds and the element of $\zeta_1\oplus\zeta W$ don't belong to the null space $\mathscr{J}_S^*\cap\mathscr{J}_T^*$ of I_{ST}, we have

$$\text{index } I_{ST} \geqq \dim\zeta = \sum_{a<t<b} \dim W(t) + \text{Convexity}.$$

In the above note that "negative part" of $\zeta_1\oplus\zeta W$ is linearly independent to ζ_0.

To prove that actually the equality does hold it suffices to show that any $\xi\in\Xi$ such that

$$I_{ST}(\xi,\eta)=0 \text{ for all } \eta\in\zeta, \quad\text{and}\quad I_{ST}(\xi,\xi)\leqq 0$$

belongs already to ζ or null space of I_{ST}.

From the first condition, for any $\widetilde{U}(t_0)\in W(t_0)$ $(a<t_0<b)$, we have $\langle\xi(t_0), \nabla U(t_0)\rangle=I_{ST}(\xi, Z)=0$, with $Z=(\chi_{t_0|\zeta W(t_0)})^{-1}\widetilde{U}(t_0)$. So by lemma 4, $\xi(t)$ may be written in the form $\xi(t)=\sum_{i=1}^{n}w^i(t)\,U_i(t)$, where we can choose a basis $\{\widetilde{U}_i(t)\}$ $i=1, \cdots, n$ of $d\phi_{t-b}V^n(b)$ so that $U_i(b)$, $1\leqq i\leqq t=\dim T$ form a basis of T. In fact, take $\{\widetilde{U}_i(b)\}$, $1\leqq i\leqq t_1=\dim T_1$ which is a basis of V_3 and $\{\widetilde{U}_j(b)\}$, $t_1+1\leqq j\leqq t_1+d$, which is a basis of V_2. Next let $x_k\in N$, $1\leqq k\leqq\dim N-d$ form a basis of $\operatorname{Ker}\Psi$. Choose $U_k\in\mathscr{J}_T$, $t_1+d+1\leqq k\leqq t$, such that $U_k(b)=x_{k-t_1-d}$. Then $\widetilde{U}_k(b)\in V_1$ (see the proof of lemma 3). Then $\{U_i(b)\}$, $1\leqq i\leqq t$ is a basis of T. So we may assume $w^i(b)=0$ for $i>t$.

Then from the second condition we get

$$\begin{aligned}0\geqq I_{ST}(\xi, \xi) &=\int_a^b \|\textstyle\sum\dot{w}^i(t)\,U_i(t)\|^2dt-\langle\sum w^i(b)\,(A_TU_i(b)-\nabla U_i(b)), \sum w^j(b)\,U_j(b)\rangle\\ &\quad+\langle\textstyle\sum w^i(a)\,(A_SU_i(a)-\nabla U_i(a), \sum w^j(a)\,U_j(a)\rangle\\ &\geqq\langle\textstyle\sum w^i(a)\,(A_SU_i(a)-\nabla U_i(a)), \sum w^j(a)\,U_j(a)\rangle\geqq 0\,.\end{aligned}$$

In the above, $\langle\sum w^i(b)\,(A_TU_i(b)-\nabla U_i(b)), \sum w^j(b)\,U_j(b)\rangle=0$ since $\xi(b)\in T$. The last inequality comes from the following. Since $\xi(a)\in S$, the Jacobi field $\sum w^i(a)\,U_i(t)$ may be written in the form $U_S(t)+U_T(t)$, where $\widetilde{U}_S(t)\in d\phi_{t-b}V_1$ and $U_T\in\mathfrak{N}$. Then we have $I_{ST}(U_T, U_T)\geqq 0$, because $I_{ST}(U_T, \zeta_0\oplus\zeta_1)=I_{ST}(\xi, \zeta_0\oplus\zeta_1)=0$ does hold by the first condition. Now

$$\begin{aligned}&\langle\textstyle\sum w^i(a)\,(A_SU_i(a)-\nabla U_i(a)), \sum w^j(a)\,U_j(a)\rangle=\langle A_SU_T(a)-\nabla U_T(a), U_T(a)\rangle\\ &=I_{ST}(U_T, U_T)\geqq 0\,.\end{aligned}$$

So $\xi(t)$ must be a broken Jacobi field with $U_T\in$ Null space of $I_{ST|\mathfrak{N}}$. Since $w^i(b)=0$ for $i>t, \xi(t)$ may be expressed in the form $\xi(t)=X(t)$ modulo the null space of I_{ST} for some $X\in\mathscr{J}_T$ at least for $t_0\leqq t\leqq b$, where t_0 denotes the last conjugate point of $\mathscr{S}$, $\mathscr{T}$. Then it is easy to see that $\xi(t)$ belongs to ζ or the null space of I_{ST}.

REMARK. Ambrose defined the convexity as the index of $I_{ST(t)}$ on $\Xi(S, T(t))$ $(:=$ vector space of H'-vector fields along $c_{|[a,t]}$ such that $\xi\perp\dot{c}$, $\xi(a)\in S$, $\xi(t)\in T(t))$, where t is sufficiently near a. But this is equal to $\dim\zeta_0+\dim\zeta_1$ as the above proof shows.

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