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On the compact convex base of a dual cone

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Let X be a locally convex Hausdorff linear topological space over R, where R is the field of real numbers endowed with its usual topology, X^{*} be its topological dual and K be a closed proper cone with vertex θ, i. e., a closed subset of X with the following properties: i) $K+K\subset K$, ii) $\lambda K\subset K$ for all $\lambda\geqq 0$, and iii) $K\cap(-K)=\{\theta\}$, where θ denotes the zero element of the linear space X. Then K allows us to introduce, by virtue of "$x\leqq y$ if $y-x\in K$", a partial order $\leqq$, under which X is an ordered linear space with possitive cone K. Let Δ be a non-empty subset of the dual cone $K^{*}=\{x^{*}: x^{*}\in X^{*}.\ x^{*}(x)\geqq 0$ for all $x\in K.\}$ satisfying the following conditions:

(1) if $x^{*}(x)\geqq 0$ for all $x^{*}\in\Delta$, then $x\in K$;

(2) Δ is strongly compact and convex;

(3) $\theta^{*}\notin\Delta$.

Here θ^{*} is reserved for the zero element of X^{*}.

The use of Δ as a set of price systems is justified, when it is intended to treat the infinite-dimensional commodity space (see [2]). Although, in the finite-dimensional case, an example of Δ is easily found, it is not always easy and even impossible to find such an example in the infinite-dimensional case. The purpose of this paper is to discuss the existence of Δ in the infinite-dimensional case. In the infinite-dimensional Banach space, there exists no non-empty subset Δ of K^{*} with $X^{*}=K^{*}-K^{*}$ satisfying (1), (2) and (3). In fact, the following theorem holds:

THEOREM. *Let X be a Banach space, K be a closed proper cone in X and K^{*} be its dual cone. Assume $X^{*}=K^{*}-K^{*}$. If there exists a non-empty subset Δ of K^{*} which satisfies the above conditions* (1), (2) *and* (3), *then X is finite-dimensional.*

PROOF. Let $(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$ denote the weak*-closure of $\bigcup_{\lambda\geqq 0}\lambda\Delta$. Then obviously $K^{*}\supset(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$.

Suppose that $x_{0}^{*}\notin(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$ for some $x_{0}^{*}\in K^{*}$. Then, by making use of the separation theorem, there exists an $x_{0}\in X$ such that

$$\inf_{x^{*}\in\bigcup_{\lambda\geqq 0}\lambda\Delta} x^{*}(x_{0})\geqq 0>x_{0}^{*}(x_{0})\,.$$

Hence $x^*(x_0) \geqq 0$ for all $x^* \in \varDelta$. By (1), this implies $x_0 \in K$. Consequently $x_0^*(x_0) \geqq 0$, which is a contradiction. Therefore

$$K^* = (\bigcup_{\lambda \geqq 0} \lambda \varDelta)^{w-}.$$

Let $y^* \in K^*$. Then there exist nets $\{x_\alpha^*, \alpha \in A\}$ and $\{\lambda_\alpha, \alpha \in A\}$ such that $x_\alpha^* \in \varDelta$, $0 < \lambda_\alpha < +\infty$ and a net $\{\lambda_\alpha x_\alpha, \alpha \in A\}$ converges weakly to y^*. Since $\varDelta$ is strongly compact, it may be assumed that the net $\{x_\alpha^*, \alpha \in A\}$ converges strongly to x_0^* for some $x_0^* \in \varDelta$. On the other hand, by making use of the separation theorem, (2) and (3) imply that there exists an $x_0 \in X$ such that $x^*(x_0) \geqq \varepsilon > 0$ for all $x^* \in \varDelta$. Then the net $\{\lambda_\alpha x_\alpha^*(x_0), \alpha \in A\}$ converges to $y^*(x_0)$, and consequently $\{\lambda_\alpha, \alpha \in A\}$ converges to $\dfrac{y^*(x_0)}{x_0^*(x_0)}$ $(< \infty)$. Thus $\{\lambda_\alpha, \alpha \in A\}$ is a bounded set. Hence it may be assumed that $\{\lambda_\alpha, \alpha \in A\}$ converges to λ_0 for some λ_0. Then $y^* = \lambda_0 x_0^* \in \lambda_0 \varDelta$. Therefore, it has been proved that $K^* \subset \bigcup_{\lambda \geqq 0} \lambda \varDelta$. This shows $\bigcup_{\lambda \geqq 0} \lambda \varDelta = K^*$.

Put $\hat{\varDelta} = \bigcup_{0 \leqq \lambda \leqq 1} \lambda \varDelta$. Then $\hat{\varDelta}$ is strongly compact, too. By making use of $X^* = K^* - K^*$, $X^* = \bigcup_{n=1}^{\infty} n(\hat{\varDelta} - \hat{\varDelta})$. Here $\hat{\varDelta} - \hat{\varDelta}$ is strongly compact, and so it is strongly closed. It follows from the Baire's category theorem that $\hat{\varDelta}$ has a non-empty interior, while it is strongly compact. Hence X^* is finite-dimensional. Finally, X is also finite-dimensional.

Remark. Although the condition (1) is in general weaker than the condition (*) $K^* = \bigcup_{\lambda \geqq 0} \lambda \varDelta$, it is equivalent to the condition (*) under the condition (3), when X is a Banach space.

The following corollary is essentially another version of the above theorem.

Corollary 1. *Let X be an infinite-dimensional Banach space and $\varDelta_0 \subset X^*$ satisfy the following conditions:*

(i) *$\varDelta_0$ is strongly (norm-) compact and convex*

and

(ii) $\theta^* \notin \varDelta_0$.

Define K_0^* and K_0 by

(iii) $K_0^* = \bigcup_{\lambda \geqq 0} \lambda \varDelta_0$

and

$$K_0 = \left\{x : x^*(x) \geqq 0 \textit{ for all } x^* \in K_0^*\right\},$$

respectively. Assume

(iv) $K_0 \cap (-K_0) = \{\theta\}$.

$$K^* = \left\{x^* : x^* = (v_1, v_2, \cdots),\ v_n \geqq 0,\ \sum_{n=1}^{\infty} v_n < +\infty\right\}.$$

Put

$$x_0^* = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots\right),\ \Delta = \{x^* : \theta^* \leqq x^* \leqq x_0^*\} \text{ and}$$
$$\Delta_0 = \{x^* : x_0^* \leqq x^* \leqq 2x_0^*\}.$$

a) Δ satisfies (1) but does not satisfy (*). In fact, for example, $x'^* \in K^*$ but $x'^* \notin \bigcup_{\lambda \geqq 0} \lambda\Delta$, where

$$l^1 \ni x'^* = \left(1, \frac{1}{2^\alpha}, \frac{1}{3^\alpha}, \cdots\right) \qquad (1<\alpha<2).$$

b) Δ_0 is convex and strongly (i. e. norm-) compact. $\theta^* \notin \Delta_0$. Let $x_0 = (1, 0, 0, \cdots) \in (c_0)$. Then $x^*(x_0) \geqq 1$ for all $x^* \in \Delta_0$.

c) Let $K_0^* = \bigcup_{\lambda \geqq 0} \lambda\Delta_0$ and $K_0 = \{x : x^*(x) \geqq 0 \text{ for all } x^* \in K_0^*\}$. Then K_0 coincides with the set $\{x : x^*(x) \geqq 0 \text{ for all } x^* \in \Delta_0\}$. In the vector lattice (X, K), for each $x \in X$, x^+, x^- and $|x|$ denote sup (x, θ), sup $(-x, \theta)$ and sup $(x, -x)$, respectively. Since $x^* \in \Delta_0$ means $x^* = x_0^* + y^*$ for some y^* with $\theta^* \leqq y^* \leqq x_0^*$ and

$$\inf_{\theta^* \leqq y^* \leqq x_0^*} y^*(x) = -x_0^*(x^-),$$

the following chain of equivalences is valid:

$$x \in K_0 \Longleftrightarrow \inf_{x^* \in \Delta_0} x^*(x) \geqq 0 \Longleftrightarrow x_0^*(x) + \inf_{\theta^* \leqq y^* \leqq x_0^*} y^*(x) \geqq 0$$
$$\Longleftrightarrow x_0^*(x) - x_0^*(x^-) \geqq 0 \Longleftrightarrow x_0^*(x^+) \geqq 2x_0^*(x^-).$$

d) Let $x \in K_0 \cap (-K_0)$. Then, $x_0^*(x^+) \geqq 2x_0^*(x^-)$ and $x_0^*((-x)^+) \geqq 2x_0^*((-x)^-)$. Since $(-x)^+ = x^-$ and $(-x)^- = x^+$, $x_0^*(x^+) \geqq 2x_0^*(x^-)$ and $x_0^*(x^-) \geqq 2x_0^*(x^+)$.
Hence $x_0^*(x^+) = 0 = x_0^*(x^-)$. Remembering that $x_0^* = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdots\right)$, $x^+ = \theta = x^-$ and so $x = \theta$.
Thus it has been shown that K_0 is a proper cone. (n)

e) Let $x_0 = (1, 0, 0, \cdots)$ and $x_n = \left(\frac{1}{2}, 0, \cdots, 0, -\frac{n}{2}, 0, 0, \cdots\right) (n = 2, 3, \cdots)$.
Then $x_0 - x_n \geqq \theta$ and so $x_0^*((x_0 - x_n)^+) \geqq 2x_0^*((x_0 - x_n)^-)$.
Hence $x_0 - x_n \in K_0$, i. e., $x_n \underset{0}{\leqq} x_0$.

On the other hand, since $x_0^*(x_n^+) = \frac{1}{2}$ and $x_0^*(x_n^-) = \frac{1}{2n}$, $x_0^*(x_n^+) \geqq 2x_0^*(x_n^-)$ and so $\theta \underset{0}{\leqq} x_n$. Thus it has been shown that

Then

$$X^* \neq K_0^* - K_0^* .$$

REMARK. By utilizing (iv), $K_0^* - K_0^*$ is weak*-dense in X^*, i. e.,

$$(K_0^* - K_0^*)^{w-} = X^* .$$

COROLLARY 2. *Let (X, K) be an infinite-dimensional Banach lattice. Then there exists no strongly-compact and convex subset Δ of X^* such that*

$$K^* = \bigcup_{\lambda \geqq 0} \lambda\Delta \text{ and } \theta^* \notin \Delta .$$

In spite of corollary 2, even if (X, K) is an infinite-dimensional Banach lattice, it is possible to construct a subcone $K_0^* \subset K^*$ so that Δ_0 satisfies all the conditions (i), (ii) and (iii) of corollary 1. Here $K_0 \supset K$ may be called an augumented cone in comparison with the original cone K.

Let K be a closed proper cone in a locally convex Hausdoff linear topological space X. Take a non-empty subset Δ of K^* satisfying (2). For example, in the infinite-dimensional Banach spaces, a non-empty subset Δ of K^* satisfying the conditions (1) and (2) is easily found. Starting from this Δ, one method of the construction of Δ_0 satisfying the conditions (1′), (i) and (ii) (equivalently (i), (ii) and (iii) in the Banach space) is stated as follows, where the condition (1′) is a following modification of (1):

(1′) if $x^*(x) \geqq 0$ for all $x^* \in \Delta_0$, then $x \in K_0$.

Since K^* is not always proper, choose x_0^* so that $x_0^* \in K^*$ and $x_0^* \notin -K^*$. Put $\Delta_0 = x_0^* + \Delta$. Then Δ_0 satisfies (i) and (ii). Next put $K_0^* = \bigcup_{\lambda \geqq 0} \lambda\Delta_0$ and $K_0 = \{x : x^*(x) \geqq 0 \text{ for all } x^* \in K_0^*\}$. If $K_0^* - K_0^*$ is weak*-dense in X^* and not weak*-closed, then K_0 is proper and $K_0^* - K_0^* \neq X^*$. The partial order introduced by K_0 is denoted by $\underset{0}{\leqq}$. In a normed space X, it is sufficient for $K_0^* - K_0^* \neq X^*$ that there exists an order interval $\{z : z \in X,\ x \underset{0}{\leqq} z \underset{0}{\leqq} y\}$ which is not norm-bounded (see [1] p. 216 and p. 220). These Δ_0 and K_0 may satisfy the desired conditions (1′), (i) and (ii) in the infinite-dimensional spaces. The following examples show that this really occurs in the infinite-dimensional spaces.

EXAMPLE 1. Let $X = (c_0)$ and

$$K = \left\{x : x = (u_1, u_2, \cdots, u_n, \cdots),\ u_n \geqq 0,\ \lim_{n\to\infty} u_n = 0\right\}.$$

Then $X^* = l^1$, and

$$x_n \in \{x : \theta \underset{0}{\leqq} x \underset{0}{\leqq} x_0\} \qquad (n=2, 3, \cdots).$$

This order interval is not norm-bounded, because $\sup_n \|x_n\|_\infty = \infty$.
Hence $K_0^* - K_0^* \neq X^*$.

EXAMPLE 2. Let $X = C[0, 1]$. Then X^* is the space of signed measures on $[0, 1]$. Define $\phi_n^* \in X^*$ by

$$\phi_n^*(f) = \int_0^1 f(t)\, dt + \frac{1}{n} \int_0^1 \cos(n\pi t) f(t)\, dt \qquad \text{for all } f \in X.$$

Since

$$\|\phi_n^* - \phi_m^*\| = \int_0^1 \left| \frac{\cos(n\pi t)}{n} - \frac{\cos(m\pi t)}{m} \right| dt \leqq \frac{1}{n} + \frac{1}{m},$$

ϕ_n^* converges strongly to some $\phi_\infty^* \in X^*$. The set $\{\phi_1^*, \phi_2^*, \cdots, \phi_\infty^*\}$ is strongly (i. e. norm-) compact. Take $\varDelta_0$ as the (norm-) closed convex hull of this set. Then $\varDelta_0$ is (norm-) compact.

a) If $f_0(t) = 1$ for all $t \in [0, 1]$, then $\phi_n^*(f_0) = 1$ for all n. Hence $\phi^*(f_0) = 1$ for all $\phi^* \in \varDelta_0$. Therefore $\theta^* \notin \varDelta_0$.

b) Put $K_0^* = \bigcup_{\lambda \geqq 0} \lambda \varDelta_0$ and $K_0 = \{f : \phi^*(f) \geqq 0 \text{ for all } \phi^* \in K_0^*\}$. Then K_0 coincides with the set $\{f : \phi_n^*(f) \geqq 0\ (n=1, 2, \cdots)\}$.

c) Let $f \in K_0 \cap (-K_0)$. Then

$$\int_0^1 f(t)\, dt + \frac{1}{n} \int_0^1 \cos(n\pi t) f(t)\, dt = 0 \qquad \text{for all } n.$$

On the other hand,

$$\left| \frac{1}{n} \int_0^1 \cos(n\pi t) f(t)\, dt \right| \leqq \frac{1}{n} \sup_{0 \leqq t \leqq 1} |f(t)| \to 0, \qquad \text{as } n \to \infty.$$

Hence

$$\int_0^1 f(t)\, dt = 0 \text{ and } \int_0^1 \cos(n\pi t) f(t)\, dt = 0 \qquad \text{for all } n.$$

Since the set $\{\cos(\pi t), \cos(2\pi t), \cdots, \cos(n\pi t), \cdots\}$ is total, $f = \theta$. Thus it has been shown that K_0 is proper.

d) Define $f_m \in C[0, 1]$ $(m=0, 1, 2, \cdots)$ as follows:

$$f_0(t) = 1 \text{ for all } t \in [0, 1]$$

and

$$f_m(t) = m \cos(m\pi t) \text{ for all } t \in [0, 1] \qquad (m=1, 2, \cdots).$$

Then $\phi_n^*(f_0) = 1$ $(n=1, 2, \cdots)$ and

$$\phi_n^*(f_m) = m\int_0^1 \cos(m\pi t)\,dt + \frac{m}{n}\int_0^1 \cos(n\pi t)\cos(m\pi t)\,dt = \frac{1}{2}\delta_{nm}$$
$$(n=1, 2, \cdots;\ m=1, 2, \cdots),$$

where δ_{nm} denotes Kronecker's symbol. Hence, for each $m=1, 2, \cdots$, $\phi_n^*(f_0) \geqq \phi_n^*(f_m) \geqq 0$ for all $n=1, 2, \cdots$. Therefore, for each $m=1, 2, \cdots$, $\phi^*(f_0) \geqq \phi^*(f_m) \geqq \phi^*(\theta)$ for all $\phi^* \in K_0^*$. This means

$$f_m \in \left\{f : f \in C[0, 1],\ \theta \underset{0}{\leqq} f \underset{0}{\leqq} f_0\right\}.$$

On the other hand, $\sup_m \|f_m\|_\infty = \infty$. Thus the order interval which is not norm-bounded is obtained. Hence $K_0^* - K_0^* \neq X^*$, i. e., K_0^* is not generating.

References

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