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On a transfer theorem for Schur multipliers

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1. Introduction.

In this paper we shall give an alternative proof of the following theorem proved by D. F. Holt [3].

THEOREM* (Holt).

Let P be a Sylow p-subgroup of a finite group G, and suppose that P has nilpotency class at most $p/2$. Then the Sylow p-subgroups of the Schur multipliers of G and $N_G(P)$ are isomorphic.

We shall prove this theorem by using the method of cohomological G-functors.

Maps and functors will be written on the right in their arguments, with the corresponding convention for writing composites.

Let G be a finite group and k a commutative ring with identity element.

DEFINITION 1.

A *G-functor* over k is defined to be a quadruple

$$A=(a, \tau, \rho, \sigma),$$

where a, τ, ρ, σ are families of the following kind:

$a=(a(H))$ gives, for each subgroup H of G (notation $H\leq G$), a finitely generated k-module $a(H)$.

$\tau=(\tau_H{}^K)$ and $\rho=(\rho^K{}_H)$ give, for each pair (H, K) of subgroups of G such that $H\leq K$, the respective k-homomorphisms

$$\tau_H{}^K: a(H)\rightarrow a(K) \quad \text{and} \quad \rho^K{}_H: a(K)\rightarrow a(H).$$

$\sigma=(\sigma_H^g)$ gives, for each pair (H, g) where H is a subgroup of G and g an element in G, the k-homomorphism

$$\sigma_H^g: a(H)\rightarrow a(H^g).$$

These families of k-modules and k-homomorphisms must satisfy the following

Axioms for G-functors. (In these axioms, D, H, K, L are any subgroups of G; g, g' are any elements in G.)

(a) $\tau_H{}^H = 1_{a(H)}$, $\tau_H{}^K \tau_K{}^L = \tau_H{}^L$ if $H \leq K \leq L$;
(b) $\rho^H{}_H = 1_{a(H)}$, $\rho^K{}_H \rho^H{}_D = \rho^K{}_D$ if $K \geq H \geq D$;
(c) $\sigma_H^h = 1_{a(H)}$ if $h \in H$, $\sigma_H^g \sigma_{H^g}^{g'} = \sigma_H^{gg'}$;
(d) $\tau_H{}^K \sigma_K^g = \sigma_H^g \tau_{H^g}{}^{K^g}$, $\rho^K{}_H \sigma_H^g = \sigma_K^g \rho^{K^g}{}_{H^g}$;
(e) (*Mackey axiom*) If $H \leq L$, $K \leq L$ and Γ is a transversal of the (H, K)-double cosets in L, then

$$\tau_H{}^L \rho^L{}_K = \sum_{g \in \Gamma} \sigma_H^g \rho^{H^g}{}_{H^g \cap K} \tau_{H^g \cap K}{}^K .$$

The images by the k-homomorphisms $\tau_H{}^K$, $\rho^K{}_H$ and σ_H^g are simply written as follows;

$\alpha \tau_H{}^K = \alpha^K$ for α in $a(H)$, $\beta \rho^K{}_H = \beta_H$ for β in $a(K)$ and $\alpha \sigma_H^g = \alpha^g$ for α in $a(H)$, respectively.

A G-functor A is naturally considered to be an H-functor for any subgroup H of G. We denote such an H-functor by $A_{|H}$.

DEFINITION 2.

A G-functor $A = (a, \tau, \rho, \sigma)$ is called *cohomological* if it satisfies the following axiom (C):

(C) If $H \leq K \leq G$, then

$$\rho^K{}_H \tau_H{}^K = |K : H| 1_{a(K)} .$$

For examples of G-functors, see [2] and [8].

DEFINITION 3.

Let $A = (a, \tau, \rho, \sigma)$ be a cohomological G-functor and let S be a subgroup of G, α an element in $a(S)$, and X a subgroup of G. Then a triple (S, α, X) is called a *singularity* in G for A provided

(a) $\alpha^G{}_X \neq 0$,
(b) $\alpha_{S \cap Y^u} = 0$ for every proper subgroup Y of X (notation $Y < X$) and every element u in G.

The subgroup S is called the *singular subgroup* of the singularity. If the singular subgroup S is a proper subgroup of G, then the singularity is called *proper*.

Now we can state a transfer theorem for cohomological G-functors on which our proof of Theorem* depends.

THEOREM 1.

Let P be a Sylow p-subgroup of a finite group G and $A = (a, \tau, \rho, \sigma)$ a cohomological G-functor over a commutative ring k. Assume that the ring k is uniquely divisible by $|G : P|$ and P has no proper singularity in

P for $A_{|P}$. *Then*

$$\operatorname{Im} \rho^G{}_P = \operatorname{Im} \rho^N{}_P\,, \quad \textit{where} \quad N = N_G(P)\,.$$

And therefore

$$a(G) \simeq a(N)\,.$$

Let G be a finite group, M a right G-module, and let

$$\begin{tikzcd}
X_* : \cdots & X_{-n} \arrow[l] & \cdots \arrow[l] & X_{-1} \arrow[l] & & X_0 \arrow[ll] \arrow[dl] & X_1 \arrow[l] & \cdots \arrow[l] & X_n \arrow[l] & \cdots \arrow[l] \\
& & & & \boldsymbol{Z} \arrow[ul] \arrow[dl] & & & & & \\
& & & 0 & & 0 \arrow[ul] & & & &
\end{tikzcd}$$

be a complete resolution of G, where each G-free module X_n is a right G-module. For each subgroup H of G, the right G-module M and the complete resolution X_* of G are also a right H-module and a complete resolution of H, respectively.

Let (H, K) be a pair of subgroups of G such that $H \leq K$.

If Δ is a transversal of the H-left cosets in K, then for each element f in $\operatorname{Hom}_H(X_n, M)$, we can define an element f^K in $\operatorname{Hom}_K(X_n, M)$ by

$$(x)f^K = \sum_{g \in \Delta} (xg) f g^{-1} \qquad \text{for } x \text{ in } X_n\,.$$

The map $f \rightarrow f^K$ is a cochain morphism $\operatorname{Hom}_H(X_n, M) \rightarrow \operatorname{Hom}_K(X_n, M)$ and this morphism induces a homomorphism

$$\operatorname{cor}_{H,K} : \ H^n(H, M) \longrightarrow H^n(K, M)\,.$$

This homomorphism is called the *corestriction* from H to K.

Every element f in $\operatorname{Hom}_K(X_n, M)$ is also an element in $\operatorname{Hom}_H(X_n, M)$. If an element f in $\operatorname{Hom}_K(X_n, M)$ is viewed as an element in $\operatorname{Hom}_H(X_n, M)$, we write this element f_H. The map $f \rightarrow f_H$ is a cochain morphism $\operatorname{Hom}_K(X_n, M) \longrightarrow \operatorname{Hom}_H(X_n, M)$ and this morphism induces a homomorphism

$$\operatorname{res}_{K,H} : \ H^n(K, M) \longrightarrow H^n(H, M)\,.$$

This homomorphism is called the *restriction* from K to H.

For each pair (H, g) of a subgroup H of G and an element g in G and for each element f in $\operatorname{Hom}_H(X_n, M)$, we can define an element f^g in $\operatorname{Hom}_H(X_n, M)$ by

$$(x)f^g = (xg^{-1}) f g \qquad \text{for } x \text{ in } X_n\,.$$

The map $f \rightarrow f^g$ is a cochain morhism $\operatorname{Hom}_H(X_n, M) \longrightarrow \operatorname{Hom}_{H^g}(X_n, M)$ and this morphism induces a homomorphism

$$\mathrm{con}_H^g\colon H^n(H, M) \longrightarrow H^n(H^g, M).$$

This homomorphism is called the *conjugation* by g.

These three homomorphisms of cohomology groups have the following properties. (In what follows D, H, K, L are any subgroups of G and g, g' are any elements in G.)

(a) $\mathrm{cor}_{H,H} = 1_{H^n(H,M)}$, $\mathrm{cor}_{H,K}\,\mathrm{cor}_{K,L} = \mathrm{cor}_{H,L}$ if $H \le K \le L$;

(b) $\mathrm{res}_{H,H} = 1_{H^n(H,M)}$, $\mathrm{res}_{K,H}\,\mathrm{res}_{H,D} = \mathrm{res}_{K,D}$ if $K \ge H \ge D$;

(c) $\mathrm{con}_H^h = 1_{H^n(H,M)}$ if $h \in H$, $\mathrm{con}_H^g\,\mathrm{con}_{H^g}^{g'} = \mathrm{con}_H^{gg'}$;

(d) $\mathrm{cor}_{H,K}\,\mathrm{con}_K^g = \mathrm{con}_H^g\,\mathrm{cor}_{H^g,K^g}$, $\mathrm{res}_{K,H}\,\mathrm{con}_H^g = \mathrm{con}_K^g\,\mathrm{res}_{K^g,H^g}$;

(e) If $H \le L$, $K \le L$ and Γ is a transversal of the (H, K)-double cosets in L, then

$$\mathrm{cor}_{H,L}\,\mathrm{res}_{L,K} = \sum_{g \in \Gamma} \mathrm{con}_H^g\,\mathrm{res}_{H^g,H^g \cap K}\,\mathrm{cor}_{H^g \cap K,K};$$

(f) If $H \le K$, then

$$\mathrm{res}_{K,H}\,\mathrm{cor}_{H,K} = |K : H|\,1_{H^n(K,M)}.$$

Note that the axioms for the cohomological G-functors are abstracted from these properties.

Let $M(G)$ denote the Schur multiplier $H^2(G, \boldsymbol{C}^*)$ of a finite group G. For each subgroup H of G, put $a(H) = \Omega_1(M(H)_p)$, where $M(H)_p$ is the Sylow p-subgroup of $M(H)$ and $\Omega_1(M(H)_p)$ is the subgroup of $M(H)_p$ generated by the elements of order p. Then $a(H)$ is a finite dimensional $\boldsymbol{F}_p$-module. For each pair (H, K) of subgroups of G such that $H \le K$, let $\tau_H{}^K$ and $\rho^K{}_H$ be $\mathrm{cor}_{H,K|a(H)}$ and $\mathrm{res}_{K,H|a(K)}$, respectively. For each pair (H, g) of a subgroup H of G and an element g in G, we define $\sigma_H^g = \mathrm{con}_{H|a(H)}^g$. Then $A = (a, \tau, \rho, \sigma)$ is a cohomological G-functor over $\boldsymbol{F}_p$. We call this functor the *multiplier functor* (with respect to a prime p).

If a Sylow p-subgroup P of G has no proper singularity in P for the multiplier functor, then by Theorem 1 we have

$$\Omega_1\big(M(G)_p\big) \simeq \Omega_1\Big(M\big(N_G(P)\big)_p\Big).$$

Hence by Tate's theorem it follows that

$$M(G)_p \simeq M\big(N_G(P)\big)_p.$$

We shall establish Theorem* (Holt) by proving the following Theorem 2.

THEOREM 2.

Let P be a p-group of nilpotency class at most $p/2$. Then P has no *proper singularity in P for the multiplier functor.*

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2. A transfer theorem for cohomological G-functors.

In this section we shall prove Theorem 1.

Let G be a finite group and let $A=(a, \tau, \rho, \sigma)$ be a cohomological G-functor over a commutative ring k.

LEMMA 1.

Let H be a subgroup of G such that the ring k is uniquely divisible by $|G:H|$. Then the k-homomorphism $\rho^G{}_H$: $a(G)\rightarrow a(H)$ is a monomorphism and the k-homomorphism $\tau_H{}^G$: $a(H)\rightarrow a(G)$ is an epimorphism.

Moreover

$$a(H)=\mathrm{Im}\ \rho^G{}_H \oplus \mathrm{Ker}\ \tau_H{}^G\ .$$

PROOF. The composition homomorphism $\rho^G{}_H\tau_H{}^G$: $a(G)\rightarrow a(G)$ is equal to $|G:H|1_{a(G)}$ and this is an automorphism of $a(G)$ since the ring k is uniquely divisible by $|G:H|$. The lemma easily follows from this fact.

LEMMA 2.

Let (S, α, X) be a singularity in G for A. Then the following hold.

(1) *For every elements g, h in G, a triple (S^g, α^g, X^h) is also a singularity in G for A.*

(2) *There exists an element g in G such that $X^g\leq S$.*

(3) *If the ring k is uniquely divisible by $|S:H|$ for a subgroup H of S, then (H, α_H, X) is also a singularity in G for A.*

(4) *If a subgroup R of G contains S, then (R, α^R, X) is also a singularity in G for A.*

(5) *If a subgroup L of G contains S, then there exists an element g in G such that (S, α, X^g) is a singularity in L for $A|_L$.*

(6) *If a subgroup L of G contains X, then $(S^g\cap L, \alpha^g{}_{S^g\cap L}, X)$ is a singularity in L for $A|_L$ for some element g in G. If moreover $G=LS$, then $(S\cap L, \alpha_{S\cap L}, X)$ is a singularity in L for $A_{|L}$.*

PROOF. (1). This follows immediately from Definition 3.
(2). Let Γ be a transversal of the (S, X)-double cosets in G, then by Mackey axiom

$$\alpha^G{}_X=\sum_{g\in\Gamma}(\alpha^g{}_{S^g\cap X})^X .$$

Thus there exists an element g in G such that $\alpha_{S\cap X^g}\neq 0$ since $\alpha^G{}_X\neq 0$. If $S\cap X^g<X^g$, then by Definition 3 we have $\alpha_{S\cap X^g}=0$, a contradiction. So $X^g\leq S$.

(3). Since A is a cohomological G-functor, we have

$$\begin{aligned}(\alpha_H{}^G)_X&=(\alpha_H{}^S)^G{}_X\\&=|S:H|\alpha^G{}_X\\&\neq 0 .\end{aligned}$$

For every proper subgroup Y of X and every element u in G, we have

$$\begin{aligned}(\alpha_H)_{H\cap Y^u}&=(\alpha_{S\cap Y^u})_{H\cap Y^u}\\&=0 .\end{aligned}$$

Thus (H, α_H, X) is a singularity in G for A.

(4). It is clear that $(\alpha^R)^G{}_X\neq 0$.

Let Y be a proper subgroup of X and u an element in G. If Γ is a transversal of the $(S, R\cap Y^u)$-double cosets in R, then by Mackey axiom

$$\begin{aligned}\alpha^R{}_{R\cap Y^u}&=\sum_{g\in\Gamma}(\alpha^g{}_{S^g\cap R\cap Y^u})^{R\cap Y^u}\\&=\sum_{g\in\Gamma}\left((\alpha_{S\cap R^{g^{-1}}\cap Y^{ug^{-1}}})^g\right)^{R\cap Y^u}\\&=0 .\end{aligned}$$

Thus (R, α^R, X) is a singularity in G for A.

(5). Since $(\alpha^L)^G{}_X=\alpha^G{}_X\neq 0$, there exists an element g in G such that $\alpha^L{}_{L\cap X^g}\neq 0$ by Mackey axiom. Then again by Mackey axiom there exists an element s in S such that $\alpha_{S\cap L^s\cap X^{gs}}\neq 0$. Thus by Definition 3 we have $X^g\leq L$ and hence $\alpha^L{}_{X^g}\neq 0$. It is clear that $\alpha_{S\cap Z^u}=0$ for every proper subgroup Z of X^g and every element u in L. Thus (S, α, X^g) is a singularity in L for $A|_L$.

(6). Since $(\alpha^G{}_L)_X=\alpha^G{}_X\neq 0$, there exists an element g in G such that $(\alpha^g{}_{S^g\cap L})^L{}_X\neq 0$ by Mackey axiom. For every proper subgroup Y of X and every element u in L, we have

$$\begin{aligned}(\alpha^g{}_{S^g\cap L})_{S^g\cap L\cap Y^u}&=\alpha^g{}_{S^g\cap L\cap Y^u}\\&=(\alpha_{S\cap(L\cap Y^u)^{g^{-1}}})^g\\&=0 .\end{aligned}$$

Thus $(S^g\cap L, \alpha^g{}_{S^g\cap L}, X)$ is a singularity in L for $A_{|L}$. When $G=LS$, we can take $g=1$ so that $(S\cap L, \alpha_{S\cap L}, X)$ is a singularity in L for $A_{|L}$. The lemma is proved.

The following lemma gives us a technique for proving Theorem 1.

LEMMA 3.

Let H be a subgroup of G such that the ring k is uniquely divisible by $|G:H|$, R a subgroup of H, and let B be a k-submodule of $a(R)$. Assume that

$$\operatorname{Im} \rho^{G}{}_{R} < B \leq \operatorname{Im} \rho^{H}{}_{R} .$$

Then the following hold.

(1) *There exists an element α in $a(H)$ such that $\alpha \neq 0$, $\alpha^{G}=0$, and $0 \neq \alpha_{R} \in B$.*

(2) *Let X be a subgroup of R such that $\alpha_{X} \neq 0$ and $\alpha_{H \cap Y^{u}}=0$ for every proper subgroup Y of X and every element u in G. Then there exists an element g in $G-H$ such that*

(a) *$(H \cap H^{g}, \alpha^{g}{}_{H \cap H^{g}}-\alpha_{H \cap H^{g}}, X)$ is a singularity in H for $A_{|H}$; and*

(b) *$(R \cap H^{g}, \alpha^{g}{}_{R \cap H^{g}}-\alpha_{R \cap H^{g}}, X)$ is a singularity in R for $A_{|R}$.*

PROOF. (1). By Lemma 1 it follows that

$$a(H)=\operatorname{Im} \rho^{G}{}_{H} \oplus \operatorname{Ker} \tau_{H}{}^{G} .$$

Thus we have

$$\operatorname{Im} \rho^{H}{}_{R}=\operatorname{Im} \rho^{G}{}_{R} \oplus (\operatorname{Ker} \tau_{H}{}^{G})\, \rho^{H}{}_{R} .$$

Hence by our assumption on B it follows that

$$B \cap (\operatorname{Ker} \tau_{H}{}^{G})\, \rho^{H}{}_{R} \neq 0 .$$

Namely there exists an element α in $a(H)$ such that $\alpha \neq 0$, $\alpha^{G}=0$, and $0 \neq \alpha_{R} \in B$ as required.

(2). Let Γ be a transversal of the (H, H)-double cosets in G. Then

$$\begin{aligned}
&\sum_{g \in \Gamma}(\alpha^{g}{}_{H \cap H^{g}}-\alpha_{H \cap H^{g}})^{H}{}_{X} \\
&\quad =(\sum_{g \in \Gamma} \alpha^{g}{}_{H \cap H^{g}}{}^{H})_{X}-(\sum_{g \in \Gamma} \alpha_{H \cap H^{g}}{}^{H})_{X} \\
&\quad =\alpha^{G}{}_{X}-|G:H| \alpha_{X} \\
&\quad \neq 0 .
\end{aligned}$$

Thus there exists an element g in $G-H$ such that

$$(\alpha^{g}{}_{H \cap H^{g}}-\alpha_{H \cap H^{g}})^{H}{}_{X} \neq 0 .$$

By our assumption on the subgroup X we have

$$(\alpha^{g}{}_{H \cap H^{g}}-\alpha_{H \cap H^{g}})_{H \cap H^{g} \cap Y^{u}}=0$$

for every proper subgroup Y of X and every element u in H. Thus $(H\cap H^g,\ \alpha^g{}_{H\cap H^g}-\alpha_{H\cap H^g},\ X)$ is a singularity in H for $A_{|H}$. By Lemma 2 (6) there exists an element h in H such that $(R\cap (H\cap H^g)^h,\ (\alpha^g{}_{H\cap H^g}-\alpha_{H\cap H^g})^h{}_{R\cap(H\cap H^g)^h},\ X)$ is a singularity in R for $A_{|R}$. Since $R\cap(H\cap H^g)^h=R\cap H^{gh}$ and $(\alpha^g{}_{H\cap H^g}-\alpha_{H\cap H^g})^h{}_{R\cap(H\cap H^g)^h}=\alpha^{gh}{}_{R\cap H^{gh}}-\alpha_{R\cap H^{gh}}$, we have that $(R\cap H^g,\ \alpha^g{}_{R\cap H^g}-\alpha_{R\cap H^g},\ X)$ is a singularity in R for $A_{|R}$ by replacing g with $g^{-1}h$ if necessary. The lemma is proved.

REMARK. Let G, H, R, and α be as in Lemma 3. Assume that for every subgroup Q of R and every element g in G, there exist a subgroup T of R and an element h in H such that $H\cap Q^g\leq T^h$. Then a subgroup X of minimal order of R such that $\alpha_X\neq 0$ satisfies the assumption of Lemma 3 (2). Because for a proper subgroup Y of X and an element g in G, there exist a subgroup T of R and an element h in H such that $H\cap Y^g\leq T^h$. Hence $H\cap Y^{gh^{-1}}\leq T\leq R$. Since $h\in H$ and $\alpha\in a(H)$, we have $\alpha_{H\cap Y^g}=\alpha_{H\cap Y^{gh^{-1}}}$. Thus by the minimality of the order of X it follows that $\alpha_{H\cap Y^g}=0$.

THEOREM 1.

Let P be a Sylow p-subgroup of a finite group G and $A=(a, \tau, \rho, \sigma)$ a cohomological G-functor over a commutative ring k. Assume that the ring k is uniquely divisible by $|G:P|$ and P has no proper singularity in P for $A_{|P}$. Then

$$\mathrm{Im}\,\rho^G{}_P=\mathrm{Im}\,\rho^N{}_P\,,\quad \textit{where}\quad N=N_G(P)\,.$$

And therefore

$$a(G)\simeq a(N)\,.$$

PROOF. Suppose that $\mathrm{Im}\,\rho^G{}_P<\mathrm{Im}\,\rho^N{}_P$. Then by Lemma 3 there exists an element α in $a(N)$ such that $\alpha\neq 0$, $\alpha^G=0$, and $\alpha_P\neq 0$. Take a subgroup X of minimal order of P such that $\alpha_X\neq 0$. Then again by Lemma 3 there exists an element g in $G-N$ such that $(P\cap N^g,\ \alpha^g{}_{P\cap N^g}-\alpha_{P\cap N^g},\ X)$ is a singularity in P for $A_{|P}$. Then we have $P\cap N^g=P$ by our assumption on P. Hence it must hold that $P=P^g$, a contradiction. Thus we have

$$\mathrm{Im}\,\rho^G{}_P=\mathrm{Im}\,\rho^N{}_P\,.$$

The homomorphism $\rho^G{}_N$ gives an isomorphism of $a(G)$ to $a(N)$ since $\rho^G{}_N$ and $\rho^N{}_P$ are monomorphisms and $\rho^G{}_P=\rho^G{}_N\rho^N{}_P$. Theorem 1 is proved.

3. The proof of Theorem 2.

In this section we shall prove Theorem 2 and Theorem*.

THEOREM 2.

Let P be a p-group of nilpotency class at most $p/2$. Then P has no proper singularity in P for the multiplier functor.

PROOF. Suppose P has a proper singulaity (S, α, X) in P for the multiplier functior. By Lemma 2 we may assume that the singular subgroup S is a maximal subgroup of P and X is contained in S.

Let
$$1 \longrightarrow R \longrightarrow F \longrightarrow P \longrightarrow 1$$

be a free presentation of P and let F_S be the complete inverse image of S in F. The commutator subgroup $[F_S, R]$ of F_S and R is normal in F since S is normal in P. Thus we have two extensions

$$1 \longrightarrow \bar{R} \longrightarrow \bar{F} \longrightarrow P \longrightarrow 1 \quad \text{and} \quad 1 \longrightarrow \bar{R} \longrightarrow \bar{F}_S \longrightarrow S \longrightarrow 1,$$

where bars denote images modulo $[F_S, R]$. The latter extension is a central extension of S of free type. It is well known that $\bar{D}=\bar{R}\cap \bar{F}'_S$ is the torsion subgroup of $\bar{R}$ and $\bar{D}\simeq M(S)$. There exists a subgroup $\bar{J}$ of $\bar{R}$ such that $\bar{R}=\bar{J}\times\bar{D}$ as $\bar{R}$ is finitely generated. Thus we have a central extension of S

$$1 \longrightarrow Z \longrightarrow K \longrightarrow S \longrightarrow 1,$$

where $Z=\bar{R}/\bar{J}\simeq\bar{D}$ and $K=\bar{F}_S/\bar{J}$. In the Hochschild - Serre exact sequence

$$1 \longrightarrow \mathrm{Hom}\,(S, \boldsymbol{C}^*) \longrightarrow \mathrm{Hom}\,(K, \boldsymbol{C}^*) \longrightarrow \mathrm{Hom}\,(Z, \boldsymbol{C}^*) \longrightarrow M(S)$$

associated to this central extension of S, the transgression map

$$t : \mathrm{Hom}\,(Z, \boldsymbol{C}^*) \longrightarrow M(S)$$

is an isomorphism. For the proof of this fact, see [7] § 1, § 3 or [4] Kap. V § 23 or [5] Ch. 2 § 7, § 9. Hence there exists a unique element ϕ in $\Omega_1(\mathrm{Hom}\,(Z, \boldsymbol{C}^*))$ such that

$$\alpha=(\phi)\,t\,.$$

The factor group P/S acts on Z and therefore on $\mathrm{Hom}\,(Z, \boldsymbol{C}^*)$. On the other hand P/S acts on $M(S)$. These operations by P/S are commutative with the transgression t. Let u be an element in $P-S$. Then by Mackey axiom

$$\alpha^P{}_X=\sum_{i=0}^{p-1}\alpha^{u^i}{}_X\,.$$

Therefore

$$\left(\sum_{i=0}^{p-1}\phi^{u^i}\right)t=\sum_{i=0}^{p-1}\alpha^{u^i}\neq 0\,.$$

Let $\psi=\sum_{i=0}^{p-1}\phi^{u^i}$, then the order of ψ is p. Let I be the kernel of ϕ, then it follows that $\bigcap_{i=1}^{p-1}I^{u^i}\leq \mathrm{Ker}\,\psi$. On the other hand we have $[Z,u]\leq \mathrm{Ker}\psi$. Suppose $I=I^u$, then $I=\mathrm{Ker}\,\psi$ so that $[Z,u]\leq I$. Thus $\phi=\phi^u$ and hence $\psi=p\phi=0$, a contradiction. Therefore $I\neq I^u$. Hence if we put $L=\bigcap_{i=0}^{p-1}I^{u^i}$, then the factor group Z/L is an elementary abelian p-group of order p^p that has a basis on which u acts regularly. Let T be the semidirect product of Z by P. Since L is normalized by P, the semidirect product T involves the wreath product $\boldsymbol{Z}_p$ wr $\boldsymbol{Z}_p$ so that the nilpotency class of T is at least p.

On the other hand as in [3] Lemma 7 it follows that the nilpotency class of T is less than p by using the assumption that P has nilpotency class at most $p/2$. Thus we have a contradiction. Theorem 2 is proved.

PROOF of Theorem* (Holt). Theorem 1 and Theorem 2 imply that if a Sylow p-subgroup P of G is of nilpotency class at most $p/2$, then

$$\left(\Omega_1\left(M(G)_p\right)\right)\mathrm{res}_{G,P}=\left(\Omega_1\left(M(N)_p\right)\right)\mathrm{res}_{N,P}\,,\quad \text{where}\quad N=N_G(P)\,.$$

Since $M(P)=\mathrm{Im\ res}_{G,P}\oplus\mathrm{Ker\ cor}_{P,G}$, we have

$$\left(M(N)_p\right)\mathrm{res}_{N,P}=\left(M(G)_p\right)\mathrm{res}_{G,P}\oplus\left(\left(M(N)_p\right)\mathrm{res}_{N,P}\cap\mathrm{Ker\ cor}_{P,G}\right).$$

Hence by the first equation it follows that

$$\left(M(N)_p\right)\mathrm{res}_{N,P}\cap\mathrm{Ker\ cor}_{P,G}=0$$

so that

$$\left(M(G)_p\right)\mathrm{res}_{G,P}=\left(M(N)_p\right)\mathrm{res}_{N,P}\,.$$

As in the proof of Theorem 1 $\mathrm{res}_{G,N|M(G)_p}$ gives an isomorphism of $M(G)_p$ to $M(N)_p$.

REMARK. As we have seen in the proof of Theorem 2, if a p-group P has a proper singularity in P for the multiplier functor, then P has a maximal subgroup S whose Schur multiplier $M(S)$ has a factor group isomorphic to an elementary abelian p-group of order p^p. Therefore a p-group which has no such maximal subgroup has no proper singularity. For example a 2-group of miximal class has no proper singularity. However it is still open to determine a necessary and sufficient condition for a p-group to have no proper singularity for the multiplier functor.

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