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Standard subgroups of type $2\Omega^{+}(8,2)$

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§ 1. Introduction

As introduced by M. Aschbacher in [2], a quasi-simple subgroup L of a finite group G is said to be a standard subgroup of G if $C_G(L)$ has even order, $C_G(L)\cap C_G(L)^g$ has odd order for $g\notin N_G(L)$, and $[L, L^g]\neq 1$ for $g\in G$.

In this paper, we prove

MAIN THEOREM. *If G is a finite group with $O(G)=1$ having a standard subgroup L isomorphic to a double cover of $\Omega^{+}(8,2)$ such that $C_G(L)$ has cyclic Sylow 2-subgroups, then $L\lhd G$.*

REMARK. The Schur multiplier of $\Omega^{+}(8,2)$ is E_4. But since what is called the triality automorphism permutes the involutions of the Schur multiplier, a double cover $2\Omega^{+}(8,2)$ is uniquely determined up to isomorphism.

Our proof consists of showing $Z(L)\subseteq Z(G)$ by Glauberman's Z^*-theorem.

The notation is standard except possibly the following:

X^{∞} the final term of the derived series of X.

$X=YZ$ means that $Y\lhd X$ and $X=\langle Y, Z\rangle$. If $Y\cap Z=1$ and if an emphasis is to be placed on that fact, we write $X=Y\cdot Z$.

If X is a 2-group, then by $J(X)$ we denote the usual Thompson subgroup generated by the abelian subgroups of maximal order.

In Section 3, we let G denote a group which satisfies the hypotheses of the Main Theorem, and use symbols such as $N(X)$ and $C(X)$ to denote $N_G(X)$ and $C_G(X)$, respectively.

§ 2. Properties of $2\Omega^{+}(8,2)$

We fix notation for $2\Omega^{+}(8,2)$ in the following lemma. For more detailed information, the reader is referred to J. S. Frame [4].

LEMMA 2.1. (i) *Let $X=W(E_8)$, the Weyl group of type (E_8). Then $L=X'$ is a double cover of $\Omega^{+}(8,2)$, and* $\mathrm{Aut}(L)\cong\bar{X}=X/Z(X)\cong O^{+}(8,2)$. *Let z be the involution of $Z(X)$. $\bar{L}$ contains five classes of involutions. Let $\bar{a}$ be a central involution. Then*

$$C_{\bar{L}}(\bar{a}) \cong (D_8 * D_8 * D_8 * D_8) \cdot (S_3 \times S_3 \times S_3),$$

$$C_{\bar{X}}(\bar{a}) \cong (D_8 * D_8 * D_8 * D_8) \cdot \bigl(S_3 \times (S_3 \operatorname{wr} Z_2)\bigr)$$

Let $\bar{b}_1, \bar{b}_2, \bar{b}_3$ *be representatives of the three classes of involutions such that*

$$C_{\bar{L}}(\bar{b}_i) \cong E_{64} \cdot S_6, \quad 1 \leq i \leq 3.$$

We choose our notation so that $\bar{b}_2$ *and* $\bar{b}_3$ *are conjugate in* $\bar{X}$. *Then*

$$C_{\bar{L}}(\bar{b}_1) \cong E_{64} \cdot (Z_2 \times S_6),$$

$$C_{\bar{X}}(\bar{b}_i) \cong E_{64} \cdot S_6, \quad i = 2, 3.$$

Let $\bar{d}$ *be an involution from the remaining class. Then*

$$C_{\bar{L}}(\bar{d}) \cong E_{64} \cdot (Z_2 \times S_4), \quad C_{\bar{X}}(\bar{d}) \cong E_{128} \cdot (Z_2 \times S_4).$$

$\bar{X}$ *contains six classes of involutions. We denote by* $\bar{v}$ *and* $\bar{w}$ *involutions of* $\bar{X} - \bar{L}$ *such that*

$$C(\bar{v}) \cong Z_2 \times Sp(6, 2),$$

$$C_{\bar{X}}(\bar{w}) \cong \bigl(E_8 \times (D_8 * D_8)\bigr) \cdot (S_3 \times S_3),$$

respectively. Note that for every involution $\bar{x}$ *of* $\bar{X}$, $O(C_{\bar{L}}(\bar{x})) = O(C_{\bar{X}}(\bar{x})) = 1$ *and neither* $C_{\bar{L}}(\bar{x})$ *nor* $C_{\bar{X}}(\bar{x})$ *has any* Z_4 *or* $Z_2 \times Z_4$ *normal subgroup.*

(ii) *By* [4], a, b_1, d, u, v *are involutions, and* b_2, b_3 *are of order 4. Again by* [4],

$$a \sim az, \quad b_1 \not\sim b_1 z, \quad d \sim dz \text{ in } L,$$

$$v \not\sim vz, \quad w \not\sim wz \text{ in } X.$$

(iii) *Let* U *be a Sylow 2-subgroup of* L. $\bar{U}$ *contains exactly six elementary abelian subgroups,* $\bar{B}_i$ $(1 \leq i \leq 6)$, *of order* 64, *and all of them are normal in* $\bar{U}$. *At the cost of relabeling, we may assume that* $\bar{B}_i \cap \{\bar{b}_j^{\bar{L}}\} \neq \emptyset$ *if and only if* $i = j$ *or* $i + j = 7$, *and that* $N_{\bar{L}}(\bar{B}_i)/\bar{B}_i \cong A_8$ *for* $i \leq 3$ *and* $N_{\bar{L}}(\bar{B}_i)/\bar{B}_i \cong E_8 \cdot GL(3, 2)$ *for* $i \geq 4$. *Let* B_i *be the full inverse image of* $\bar{B}_i$. *Then, by* (ii) *and by our choice of labeling,* B_1 *and* B_6 *are the only elementary abelian subgroups of order 128 of* U.

(iv) *Let* $\bar{K}$ *be a complement to* $\bar{B}_1$ *in* $N_{\bar{L}}(\bar{B}_1)$.

(v) *Choose* $\bar{v}$ *so that* $\bar{v}$ *normalizes* $\bar{U}$ *and* $\bar{K}$ *and so that* $\bar{v}$ *centralizes* $\bar{B}_6$. *Then* $U\langle v \rangle \in \mathrm{Syl}_2(X)$, $Z(U\langle v \rangle) = Z(U) = \langle z \rangle$, $J(U\langle v \rangle) = B_6 \times \langle v \rangle \cong E_{256}$.

(vi) $\left|\{a^L\} \cap B_6\right| = 14,$

$$\left|\{b_1^L\} \cap B_6\right| = \left|\{(b_1 z)^L\} \cap B_6\right| = 28,$$

$$\left|\{d^L\}\cap B_6\right| = 56\,,$$

$$\left|\{v^X\}\cap B_6\langle v\rangle\right| = \left|\{(vz)^X\}\cap B_6\langle v\rangle\right| = 8\,,$$

$$\left|\{w^X\}\cap B_6\langle v\rangle\right| = \left|\{(wz)^X\}\cap B_6\langle v\rangle\right| = 56\,.$$

LEMMA 2.2. (i) *$N_X(B_1)$ acts indecomposably on B_1.*

(ii) *$N_X(B_1)$ splits over B_1.*

PROOF: We may assume $a\in B_1$. By way of contradiction, suppose the action of $N_X(B_1)$ on B_1 is decomposable. Then $|C_{N_X(B_1)}(a)|_2=2^{14}$. On the other hand, since $a\sim az$ in L, $[C_{\bar X}(\bar a):\overline{C_X(a)}]=2$ and so $|C_X(a)|_2=2^{13}$. This is a contradiction. Thus (i) holds. There are exactly two classes of complements to $\bar B_1$ in $N_{\bar L}(\bar B_1)$. (See, for example, Lemma 11.3 of M. Aschbacher [1]). The involutions of one of them are from $\{\bar b_1^{\bar L}\}$ and $\{\bar a^{\bar L}\}$, and those of the others are from $\{\bar b_1^{\bar L}\}$ and $\{\bar d^{\bar L}\}$. Hence the full inverse image of $\bar K$ is isomorphic to $Z_2\times A_8$ by Lemma 2.1 (ii). Thus $N_L(B_1)$ splits over B_1. By Lemma 1.1 (v), v is an involution of $N_X(B_1)$, and $\bar v$ normalizes the full inverse image of $\bar K$. Therefore v normalizes the commutator subgroup of the full inverse image of $\bar K$, which is a complement to B_1 in $N_L(B_1)$. Thus $N_X(B_1)$ splits over B_1.

Let K be the commutator subgroup of the full inverse image of $\bar K$. Thus $K\cong A_8$ and $K\langle v\rangle\cong S_8$. Lemma 2.2 (i) shows that the action of $K\langle v\rangle$ on B_1 comes from the standard permutation module. Thus, in proving the following three lemmas, we regard $K\langle v\rangle$ as the symmetric group on $\Omega=\{i:\ 1\le i\le 8\}$, and write an element t of B_1 in the form

$$t=\prod_{i\in\Omega}e_i^{t_i}\,;\ t_i=0 \text{ or } 1,\ \left|\{i:\ t_i=1\}\right|=\text{even}\,.$$

LEMMA 2.3. $O_2(C_L(a))\cong(D_8*D_8)\times(D_8*D_8)$. *Here a is in the commutator subgroup of an indecomposable component of $O_2(C_L(a))$, and so is az.*

PROOF: We may assume $a=e_1e_2e_3e_4\in B_1$. Then we have $O_2(C_{B_1K}(a)')=F_1\times F_2$ with $F_i=\langle V_i, W_i\rangle$, where

$$V_1=\langle e_1e_2,\, e_2e_3,\, e_3e_4\rangle\subseteq B_1\,,\qquad V_2=\langle e_5e_6,\, e_6e_7,\, e_7e_8\rangle\subseteq B_1\,,$$

$$W_1=\langle(12)(34),\,(13)(24)\rangle\subseteq K\,,\qquad W_2=\langle(56)(78),\,(57)(68)\rangle\subseteq K\,.$$

Clearly $F_i\cong D_8*D_8$, $a\in F_1$ and $az=e_5e_6e_7e_8\in F_2$. Since $O_2(C_L(a))\in\mathrm{Syl}_2(C_L(a'))$ and $|C_L(a)'|_2=1024$, $O_2(C_L(a))=O_2(C_{B_1K}(a)')$. This proves the lemma.

LEMMA 2.4. *$\langle z\rangle$, $\langle b_1\rangle$ and $\langle b_1z\rangle$ are all characteristic both in $C_L(b_1)$ in $C_X(b_1)$.*

PROOF: We may assume $b_1=e_1e_2\in B_1$. Then $C_L(b_1)=B_1\cdot C_K(b_1)$, $C_K(b_1)\cong S_6$, $B_1=O_2(C_L(b_1))$. $C_K(b_1)'$ is the alternating group on $\{i\in\Omega: 3\leq i\leq 8\}$, and $C_K(b_1)=\langle C_K(b_1)', (12)(34)\rangle$. Therefore $\langle z, b_1\rangle=Z(C_L(b_1))$. Since $B_1\cap C_L(b_1)^{\infty}=\langle e_ie_{i+1}: 3\leq i\leq 7\rangle$, $\langle b_1z\rangle=\langle z, b_1\rangle\cap C_L(b_1)^{\infty}$. Hence $\langle b_1z\rangle$ is characteristic in $C_L(b_1)$. Every element x of order 4 of $C_K(b_1)-C_K(b_1)'$ is of the form $x=(12)(i_1i_2i_3i_4)$, $3\leq i_k\leq 8$. Therefore $\langle b_1\rangle=\langle z, b_1\rangle\cap[B_1, x]$ for every such x. Hence $\langle b_1\rangle$ is characteristic in $C_L(b_1)$, and so $\langle z\rangle$ is also characteristic. Next note that $C_X(b_1)=B_1\cdot C_{K\langle v\rangle}(b_1)$ and $\langle z, b_1\rangle=Z(C_X(b_1))$. As before, we have $\langle b_1z\rangle=\langle z, b_1\rangle\cap C_X(b_1)^{\infty}$. Therefore $\langle b_1z\rangle$ is characteristic in $C_X(b_1)$. Since $\langle B_1, (12)\rangle=O_2(C_X(b_1))$, $\langle b_1\rangle=\langle B_1, (12)\rangle'$ is characteristic. Hence $\langle z\rangle$ is also characteristic in $C_X(b_1)$.

LEMMA 2.5. $[B_6\langle v\rangle, x]\geq 4$ *for every 2-element* x *of* $N_X(B_6\langle v\rangle)-B_6\langle v\rangle$.

PROOF. Let Λ denote $\{1, 3, 5, 7\}$, and set $V=\{e_ie_{i+1}: i\in\Lambda\}\subseteq B_1$ and $W=\{(i, i+1): i\in\Lambda\}\subseteq K$. We may assume $B_6\langle v\rangle=V\times W$. Let x be an arbitrary 2-element of $N_X(B_6\langle v\rangle)-B_6\langle v\rangle$. Since $N_{B_1K\langle v\rangle}(B_6\langle v\rangle))$ contains a Sylow 2-subgroup of X, we may assume $x\in B_1K\langle v\rangle$. First suppose $x\in B_6\langle v\rangle B_1$. By replacing x by a suitable element of the coset $xB_6\langle v\rangle$, we may assume x is in the form

$$x=\prod_{i\in\Lambda}e_i^{t_i};\ t_i=0 \text{ or } 1,\ \left|\{i\in\Lambda: t_i=1\}\right|=\text{even}.$$

Then $[B_6\langle v\rangle, x]=\langle e_ie_{i+}1: i\in\Lambda, t_i=1\rangle$. Therefore $|[B_6\langle v\rangle, x]|=2^{|\{i\in\Lambda: t_i=1\}|}\geq 2^2$. Next suppose $x\notin B_6\langle v\rangle B_1$. Then the action of x on V is nontrivial and is the same as that on $(B_6\langle v\rangle)/V$. Hence $|[B_6\langle v\rangle, x]|\geq|[V, x]|^2\geq 2^2$.

LEMMA 2.6. *There exists an involution* $\bar{x}$ *of* $C_{\bar{L}}(\bar{v})$ *such that* $\bar{x}\in\{\bar{b}_1^{\bar{X}}\}$ *and* $\bar{v}\bar{x}\in\{\bar{v}^{\bar{X}}\}$.

PROOF. We let $\bar{X}$ act on a vector space V of dimension 8 over GF(2) with a quadratic form of plus type so that $\bar{X}$ leaves the quadratic form invariant. We then choose $\bar{x}$ to be an element of $\{\bar{b}_1^{\bar{X}}\}$ such that $[V, \bar{x}]\supseteq[V, \bar{v}]$. Then $\bar{x}$ satisfies all the requirements of the lemma.

§ 3. Proof of Main Theorem.

In the remainder of this paper, we let G denote a group which satisfies the hypotheses of the Main Theorem, and use the description of L given in Section 2. Let S be a Sylow 2-subgroup of $N(L)$ containing U.

LEMMA 3.1. *If* $|C(L)|_2\geq 4$, *then* $L\lhd G$.

PROOF: This follows immediately from Theorem 2 of L. Finkelstein [3] and Lemma 2.1 (i).

Throughout the rest of this paper, we assume $|C(L)|_2=2$.

LEMMA 3.2. *If $[N(L):LC(L)]=2$ and if there is no involution in $N(L)-LC(L)$, then $z\in Z(G)$.*

PROOF: By Lemma 2.1(v), $J(S)\cong Z_4\times E_{64}$ and $\mho^1(J(S))=\langle z\rangle$. Therefore $S\in \mathrm{Syl}_2(G)$, and $\{z^{N(J(S))}\}=\{z\}$. Note that our assumption implies that every involution of S is conjugate to some involution of $B_6(\subseteq J(S))$ by Lemma 2.1(vi). Since $N(J(S))$ controls the fusion of $J(S)$, this means $\{z^G\}\cap S=\{z\}$. Hence Glauberman's Z^*-theorem yields the desired conclusion.

From now on, we assume that either $[N(L):LC(L)]=2$ and $N(L)-LC(L)$ contains involutions or $[N(L):LC(L)]=1$.

LEMMA 3.3. $S\in \mathrm{Syl}_2(G)$.

PROOF: This is because $Z(S)=\langle z\rangle$ by Lemma 2.1(v).

LEMMA 3.4. *$z\not\sim a$ in G.*

PROOF: By way of contradiction, suppose $a^g=z$, $g\in G$. Then $C_{C(z)}(a)^g \subseteq C(z)$. From the structures of the centralizers of the involutions of $C(z)$ (Lemma 2.1(i)), we observe that every involution x of $C(z)$ such that $|C_{C(z)}(x)|$ is divisible by $|S|/2$ is conjugate to a in $C(z)$. Therefore there exists an element h of $C(z)$ such that $(z^g)^h=a$. Hence, regarding gh as g, we may assume $z^g=a$. Then g normalizes $\langle z,a\rangle$, and so g normalizes also $C_{C(z)}(a)$. Hence we may regard g as an automorphism of $O_2(C_{C(z)}(a)/O(C_{C(z)}(a)))$ $(\cong O_2(C_L(a))$ which sends $aO(C_{C(z)}(a))$ to $zO(C_{C(z)}(a))$. But by Lemma 2.3 and Krull-Remak-Schmidt's theorem, we have that there is no such automorphism. This is a contradiction.

LEMMA 3.5. *$z\not\sim b_1$ and $z\not\sim b_1z$ in G.*

PROOF: Suppose $b_1^g=z$ or $(b_1z)^g=z$, $g\in G$. As in Lemma 3.4, we may assume g normalizes $C_{C(z)}(b_1)$, for every involution x of $C(z)$ such that $x\in C_{C(z)}(x)'$ and such that $C_{C(z)}(x)$ contains a subgroup isomorphic to $E_{128}\cdot S_6$ is conjugate to either b_1 or b_1z in $C(z)$. Again arguing as in Lemma 3.4, we get a contradiction to Lemma 2.4.

Now we finish the proof of the Main Theorem, distinguishing two cases.

LEMMA 3.6. *If $N(L)=LC(L)$, then $z\in Z(G)$.*

PROOF: We first prove B_6 is weakly closed in U. By way of contradiction, suppose $B_6^g=B_1$, $g\in G$. Since B_6 and B_1 are both normal in U, we may assume $U^g=U$. But by Lemma 2.1(v), this implies $g\in C(z)$, which is absurd. Therefore B_6 is weakly closed by Lemma 2.1(iii). On the other hand, since $|\{z^{N(B_6)}\}|$ must divide $|GL(7,2)|$, we have that $\{z^{N(B_6)}\}=\{z\}$ by Lemmas 2.1 (vi), 3.4 and 3.5. Since every involution of U is conjugate to some involution of B_6 in L by Lemma 2.1 (vi) and since B_6 is weakly closed in U, Glauberman's Z^*-theorem yields the desired conclusion.

LEMMA 3.7. *If* $[N(L):LC(L)]=2$ *and* $N(L)-LC(L)$ *contains involutions, then* $z\in Z(G)$.

PROOF: $N(L)$ contains a subgroup X isomorphic to $W(E_8)$. We use the description of X given in Lemma 2.1. We may assume $S=U\langle v\rangle$. Thus $J(S)=B_6\langle v\rangle$. We first prove $z\not\sim v$ in G. Suppose $v^g=z$, $g\in G$. As in Lemma 3.4, we may assume that either $z^g=v$ or $z^g=vz$. Thus g normalizes $C_{C(z)}(v)$, and so g normalizes also $C_{C(z)}(v)^\infty\cong Sp(6,2)$. Since the outer automorphism group of $Sp(6,2)$ is trivial, we may assume g centralizes $C_{C(z)}(v)^\infty$. First assume $z^g=vz$. By Lemma 2.6, there is an involution x of $C_{C(z)}(v)^\infty$ such that $\bar{x}\in\{\bar{b}_1^{\bar{X}}\}$ and $\bar{v}\bar{x}\in\{\bar{v}^{\bar{X}}\}$. Since $x^g=x$, $(vx)^g=xz$. Since vx and xz are conjugate to either v or vz and either b_1 or b_1z, respectively, in X by our choice of x, and since z is conjugate to both v and vz in G by our assumption, this means that z is conjugate to either b_1 or b_1z. This contradicts Lemma 3.5. Therefore $z^g=v$. Then by taking a suitable odd power of g, we may assume g is a 2-element. Since g centralizes $C_{C(z)}(v)^\infty$, $|[B_6\langle v\rangle, g]|=|\langle vz\rangle|=2$. But since $S\in\mathrm{Syl}_2(G)$, this contradicts Lemma 2.5. Thus $z\not\sim v$. Similarly $z\not\sim vz$. Since $|\{z^{N(b_6\langle v\rangle)}\}|$ must divide $|GL(8,2)|$, those antifusions together with Lemmas 3.4 and 3.5 show that $\{z^{N(B_6\langle v\rangle)}\}=\{z\}$. Now the desired conclusion follows again from Glauberman's Z^*-theorem.

Thus the proof of our Main Theorem is complete.

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