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Characterization of Poisson Integrals of Vector-Valued Functions and Measures on the Unit Circle

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Introduction.

An answer to the question whether, for a given complex-valued harmonic function f in the open unit disk D, there exists a finite measure on $[-\pi, \pi]$ (i. e. on the unit circle Π) such that f is the Poisson integral of this measure can be given in terms of the family of functions $\{f_r;\ 0\leq r<1\}$ defined on the unit circle by

(1) $$f_r : e^{i\theta}\mapsto f(r e^{i\theta}),\ \theta\in[-\pi, \pi].$$

Namely, such a measure exists if and only if there exists a constant α, independent of r, such that

$$\int_{-\pi}^{\pi}|f_r(e^{i\theta})|\,d\theta\leq\alpha,$$

for each $0\leq r<1$. This condition means that the linear maps Φ_r, $0\leq r<1$, from the space $C(\Pi)$ of continuous functions on the unit circle (equipped with the uniform norm) into the complex numbers defined by

(2) $$\Phi_r(\psi)=\int_{-\pi}^{\pi}\psi(\theta)f_r(e^{i\theta})\,d\theta,\ \psi\in C(\Pi),$$

map the unit ball of this space into a bounded set independent of r.

Just as well known is the criterion that f is the Poisson integral of an integrable function on Π if and only if the net of functions $\{f_r;\ 0\leq r<1\}$ is Cauchy in the sace $L^1(\Pi)$.

If f is a harmonic function in D, but now with values in a Banach space X, in which case the family of functions $\{f_r;\ 0\leq r<1\}$ also assumes its values in the space X, then it is natural to ask whether the classical results for numerical-valued functions have vector analogues which characterize f as the Poisson integral of an X-valued measure or integrable function on the unit circle. The aim of this note is to show that this is indeed the case.

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It turns out that for a harmonic function $f : D \to X$ the maps (2) can again be formed, but now with values in X. Then there exists an X-valued vector measure on the unit circle such that f is its Poisson integral if and only if the associated operators $\{\Phi_r;\ 0 \le r < 1\}$ map the unit ball of $C(\Pi)$ into a weakly compact set not depending on r (cf. Theorem 2. 1).

It is well known that f is the Poisson integral of an X-valued Bochner integrable function on Π if and only if the net $\{f_r;\ 0 \le r < 1\}$ is Cauchy in the space $L^1(\Pi, X)$ of X-valued Bochner integrable functions on the unit circle Π; see [4; Théorème 3] or [1; Theorem 2. 1]. The essential point in this case is that the space $L^1(\Pi, X)$ is complete, [2; p. 50]. In practice the requirement of Bochner integrability is often unduely restrictive and it is, therefore, desirable to have available criteria which ensure that f is the Poisson integral of a Pettis integrable function. Unfortunately, unlike $L^1(\Pi, X)$ the space of X-valued Pettis integrable functions is not in general complete for the topology of uniform convergence of indefinite integrals. This difficulty is overcome in a somewhat novel way in § 3 and is possible due to some recent work of S. Okada which characterizes the completion of the space of strongly measurable, Pettis integrable functions for the topology of uniform convergence of indefinite integrals, as a space of Pettis integrable functions, not with values in X itself, but with values in an auxiliary space containing a copy of the original space X.

1. Preliminaries.

For a vector-valued function there are many possible ways of defining measurability and integrability. Some of these definitions may be considered as natural extensions of the numerical-valued case. This is in particular true of the notion of strong measurability which will suffice for the purposes of this note. In this section we give the basic definitions and results concerning vector measures and integrability of vector-valued functions which are needed in the sequel.

Let $D = \{z \in \boldsymbol{C};\ |z| < 1\}$ denote the open unit disc in the complex plane $\boldsymbol{C}$ and $\Pi = \{z \in \boldsymbol{C};\ |z| = 1\}$ denote its boundary which we will often identify with the interval $[-\pi, \pi]$ in the obvious way. Accordingly, it is tacitly assumed that functions defined on $[-\pi, \pi]$ have equal values at the endpoints. The σ-algebra of Borel subsets of the unit circle is denoted by B. The space of continuous linear functionals and the space of all linear functionals on a Banach space X are denoted by X' and X^*, respectively.

Let X be a Banach space. Then an X-valued function f defined in an interval (a, b) is said to be differentiable at a point $\xi \in (a, b)$ if there exists an element $f'(\xi)$ of X, necessarily unique, such that

$$f'(\xi)=\lim_{\omega\to\xi}(f(\omega)-f(\xi))/(\omega-\xi),$$

where the limit exists in the norm topology of X. The function f is said to be differentiable in (a, b) if it is differentiable at each point of the interval. This definition extends to higher derivatives and also to partial derivatives of X-valued functions of two variables with domain an open subset of $\boldsymbol{R}^2$ in an obvious way.

A function $f : D\to X$ is called harmonic if it has continuous partial derivatives up to order (at least) two and satisfies Laplace's equation

$$((\partial^2 f/\partial x^2)+(\partial^2 f/\partial y^2))(z)=0,$$

at each point $z\in D$ where, as usual, x and y denote the (real) variables corresponding to the real and imaginary parts of points in D, respectively, and elements of D are considered as points in $\boldsymbol{R}^2$.

Let X be a Banach space. A map $m : B\to X$ is a vector measure if it is σ-additive. For each $x'\in X'$, the $\boldsymbol{C}$-valued measure $E\to\langle m(E), x'\rangle$, $E\in B$, is denoted by $\langle m, x'\rangle$. Its variation is denoted by $|\langle m, x'\rangle|$. The semivariation of m is the set function $\|m\|$ defined by

$$\|m\|(E)=\sup\{|\langle m, x'\rangle|(E) ; \|x'\|\leq 1\},\ E\in B.$$

The number $\|m\|(\Pi)$ is called the total semivariation of m. The function $m\to\|m\|(\Pi)$ is a norm for the space of X-valued vector measures on B.

A vector measure $m : B\to X$ is of finite variation if there exists a non-negative finite measure ν on B such that $\|m(E)\|\leq\nu(E)$ for each set $E\in B$. The smallest such measure ν (in the sense of [2 ; p. 3]) is called the variation measure of m and is denoted by $|m|$.

Let $m : B\to X$ be a vector measure. A complex-valued, B-measurable function f on Π is said to be m-integrable if it is integrable with respect to every measure $\langle m, x'\rangle$, $x'\in X'$, and if, for every set $E\in B$, there exitst an element $\int_E fdm$ of X such that

$$\langle \int_E fdm, x'\rangle=\int_E fd\langle m, x'\rangle,\ x'\in X'.$$

The X-valued mapping

$$fm : E\to\int_E fdm,\ E\in B,$$

is called the indefinite integral of f with respect to the measure m. The Orlicz-Pettis lemma implies that it is a vector measure.

For each $0 \le r < 1$ define a non-negative, 2π-periodic, continuous function P_r by

(3) $$P_r(\theta) = (1-r^2)/(1+r^2-2r\cos\theta), \quad \theta \in [-\pi, \pi].$$

It is clear that each function P_r, $0 \le r < 1$, is symmetric about $\theta = 0$. The family of functions P_r, $0 \le r < 1$, usually called the Poisson kernel, is an approximate identity for L^1 of the circle.

Let $m : B \to X$ be a vector measure. If $0 \le r < 1$, then for each $\theta \in [-\pi, \pi]$ the function $t \to P_r(\theta - t)$, $t \in [-\pi, \pi]$, is certainly bounded and B-measurable and, hence, is m-integrable, [8; II Lemma 3.1]. Accordingly, it is possible to define the Poisson integral of m to be the function $m*P : D \to X$ given by

(4) $$(m*P)(r\,e^{i\theta}) = (2\pi)^{-1}\int_{-\pi}^{\pi} P_r(\theta - t)\,dm(t),$$

for each point $r\,e^{i\theta} \in D$. It follows, from the inequality $\| \int f dm \| \le \|f\|_\infty \|m\|(\Pi)$, for example, valid for bounded Borel functions f on Π, that $m*P$ is an X-valued, harmonic function in D.

Let X be a Banach space. A function $f : \Pi \to X$ is said to be Pettis integrable with respect to Lebesgue measure λ on Π if the function

$$\langle f, x' \rangle : t \to \langle f(t), x' \rangle, \quad t \in \Pi,$$

is λ-integrable for each $x' \in X'$, and if, for every set $E \in B$, there exists an element $\int_E f d\lambda$ of X such that

$$\left\langle \int_E f d\lambda, x' \right\rangle = \int_E \langle f, x' \rangle d\lambda, \quad x' \in X'.$$

The Orlicz-Pettis lemma implies that the indefinite integral of f with respect to λ, that is, the set function $f\lambda : E \to \int_E f d\lambda$, $E \in B$, is an X-valued vector measure; its total semivariation is given by

(5) $$\|f\lambda\| = \sup\left\{ \int_\Pi |\langle f, x' \rangle| \, d\lambda \; ; \; \|x'\| \le 1 \right\}.$$

The Poisson integral, $f*P$, of f, is defined to be the Poisson integral of its indefinite injegral $f\lambda$. Since for each point $r\,e^{i\theta} \in D$ the X-valued function $t \to P_r(\theta - t) f(t)$, $t \in [-\pi, \pi]$, is again Pettis λ-integrable, it follows that

$$(f*P)(r\,e^{i\theta}) = (2\pi)^{-1}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\, d\lambda(t).$$

An X-valued function f on the unit circle is said to be strongly measurable (with respect to λ) if there exists a sequence of B-simple functions $f_n : \Pi \to X$, $n=1, 2, \dots$, such that $f_n(t) \to f(t)$, in the norm topology of X, for λ-a.e. point $t \in \Pi$. If in addition $\int_\Pi \|f(t)\| dt$ is finite, then f is said to be Bochner integrable. The Banach space of all (equivalence classes of) X-valued Bochner integrable functions on the unit circle, equipped with the norm

$$\|f\|_B = \int_\Pi \|f(t)\| dt,$$

is denoted by $L^1(\Pi, X)$. If $f \in L^1(\Pi, X)$, then its indefinite integral $f\lambda$ is a vector measure of finite variation and $\|f\|_B = |f\lambda|$; see [2; II Theorem 2.4 (iv)]. Furthermore, it follows from (5) that

$$\|f\lambda\| \leq \|f\|_B, \quad f \in L^1(\Pi, X).$$

Since Bochner integrable functions are Pettis λ-integrable, their Poisson integral in already defined. We remark that any continuous function $f : \Pi \to X$ is necessarily Bochner integrable.

2. Poisson integral of vector measures.

Throughout this section X denotes a Banach space. Let $f : D \to X$ be a harmonic function and f_r denote the continuous X-valued function on the unit circle defined by (1), for each $0 \leq r < 1$. Then for each $\psi \in C(\Pi)$ and each $0 \leq r < 1$ the function ψf_r is Bochner integrable and, hence, the family of linear operators Φ_r, $0 \leq r < 1$, given by (2) is certainly defined. The family of maps $\{\Phi_r : 0 \leq r < 1\}$ is said to be equicompact (weakly equicompact) whenever

$$\{\Phi_r(\psi) ;\ 0 \leq r < 1,\ \psi \in C(\Pi),\ \|\psi\|_\infty \leq 1\} \tag{6}$$

is a relatively compact (weakly compact) subset of X. It is worth noting that for each $0 \leq r < 1$ the operator norm $\|\Phi_r\|$ of Φ_r is precisely the total semivariation $\|f_r\lambda\|$ of the indefinite integral of f_r; this follows easily from (5) and the formula (10) below.

THEOREM 2.1. *Let $f : D \to X$ be a harmonic function. Then f is the Poisson integral of an (unique) X-valued measure with domain B if and only if the associated family of maps $\{\Phi_r ;\ 0 \leq r < 1\}$ defined by (2), is weakly equicompact.*

PROOF. Let $m : B \to X$ be a vector measure and $f = m * P$ be its Poisson integral. If $\psi \in C(\Pi)$, then

$$(7) \qquad \Phi_r(\psi)=\int_{-\pi}^{\pi}\psi(\theta)(2\pi)^{-1}\int_{-\pi}^{\pi}P_r(\theta-t)dm(t)d\theta, \ 0\leq r<1.$$

Since for a fixed $0\leq r<1$, both ψ and P_r are continuous functions on the circle and the function

$$h_{r,\psi}:\ t\to(2\pi)^{-1}\int_{-\pi}^{\pi}P_r(\theta-t)\psi(\theta)d\theta, \ t\in[-\pi, \pi],$$

satisfies $\|h_{r,\psi}\|_\infty\leq\|\psi\|_\infty$, it is permissible to interchange the order of iterated integrals in (7) giving

$$(8) \qquad \Phi_r(\psi)=\int_{-\pi}^{\pi}h_{r,\psi}(t)dm(t), \ \psi\in C(\Pi),$$

for each $0\leq r<1$. If $\overline{bco}(m(B))$ denotes the closed convex balanced hull in X of the range $m(B)=\{m(E)\ ;\ E\in B\}$, of m, then it follows from (8) that $\Phi_r(\psi)\in\|\psi\|_\infty\ \overline{bco}(m(B))$ for each $\psi\in C(\Pi)$ and each $0\leq r<1$. Accordingly, the set (6) is contained in $\overline{bco}(m(B))$ and, hence, is relatively weakly compact (by [2; IX Lemma 1.3] and Krein's theorem). This shows that the family of maps (2) associated with $f=m*P$ is weakly equicompact.

Conversely, let $f: D\to X$ be a harmonic function for which the associated family of maps $\{\Phi_r;\ 0\leq r<1\}$ is weakly equicompact. Fix $x'\in X'$. Then

$$(9) \qquad \langle\Phi_r(\psi), x'\rangle=\int_\Pi\psi\langle f_r, x'\rangle\ d\lambda, \ \psi\in C(\Pi),$$

for each $0\leq r<1$. Since $\langle f_r, x'\rangle\in L^1(\Pi)$ and

$$(10) \qquad \|\langle f_r, x'\rangle\|_1=\sup\{\,|\int_\Pi\psi\langle f_r, x'\rangle d\lambda\,|\ ;\ \psi\in C(\Pi),\ \|\psi\|_\infty\leq 1\}$$

for each $0\leq r<1$, it follows from (9) and the boundedness of the set (6) that there is a constant $B(x')$ such that

$$\sup\{\|\langle f_r, x'\rangle\|_1\ ;\ 0\leq r<1\}\leq B(x').$$

Accordingly, there exists a unique measure $\mu_{x'}: B\to \boldsymbol{C}$ such that $\langle f, x'\rangle$ is its Poisson integral, that is,

$$(11) \qquad \langle f_r, x'\rangle=P_r*\mu_{x'}, \ 0\leq r<1\ ;$$

see [7; p. 38]. Since the net of measures $\{(2\pi)^{-1}(P_r*\mu_{x'})(t)dt\ ;\ 0\leq r<1\}$ converges weak-star to $\mu_{x'}$, [7; p. 33], it follows that

$$(12) \qquad \lim_{r\to 1-} (2\pi)^{-1}\int_{-\pi}^{\pi}\psi(t)(P_r * \mu_{x'})(t)\,dt = \int_{\Pi}\psi d\mu_{x'},$$

for each $\psi\in C(\Pi)$. Then (9), (11) and (12) imply that, given $x'\in X'$,

$$\lim_{r\to 1-}\langle\Phi_r(\psi), x'\rangle = 2\pi\int_{\Pi}\psi d\mu_{x'}$$

exists for each $\psi\in C(\Pi)$.

Let V be the closed convex balanced hull of (6). Since, for a given element $\psi\in C(\Pi)$, the values $\Phi_r(\psi)$, $0\le r<1$, belong to $\|\psi\|_\infty V$ and $\lim_{r\to 1-}\langle\Phi_r(\psi), x'\rangle$ exists for each $x'\in X'$, it follows from the weak compactness of $\|\psi\|_\infty V$ that there exists an element $\Phi(\psi)$ in $\|\psi\|_\infty V$ such that

$$\lim_{r\to 1-}\langle\Phi_r(\psi), x'\rangle = \langle\Phi(\psi), x'\rangle,\quad x'\in X'.$$

So, the linear map $\Phi : C(\Pi)\to X$ defined by $\psi\to\Phi(\psi)$ for each $\psi\in C(\Pi)$ is weakly compact as it maps the unit ball of $C(\Pi)$ into V. Hence, there exists a vector measure $m : B\to X$ such that

$$(13) \qquad \Phi(\psi) = \int_{\Pi}\psi dm,\quad \psi\in C(\Pi);$$

see [2; p. 153, Theorem 5], for example. In particular, if $x'\in X'$, then the identities

$$2\pi\int_{\Pi}\psi d\mu_{x'} = \langle\Phi(\psi), x'\rangle\int_{\Pi}\psi d\langle m, x'\rangle,$$

valid for each $\psi\in C(\Pi)$, show that $\langle m, x'\rangle = \mu_{x'}$. Since the family of numerical measures $\{\mu_{x'}; x'\in X'\}$ is unique it follows that m is the unique vector measure on B such that

$$\langle P_r * m, x'\rangle = P_r * \langle m, x'\rangle = \langle f_r, x'\rangle,\quad x'\in X'.$$

In particular, f is the Poisson integral of $(2\pi)^{-1}m$.

REMARK. It is worth noting that if the Banach space X is reflexive then bounded sets are relatively weakly compact and, hence, for such spaces it follows from Theorem 2.1 that a harmonic function $f : D\to X$ is the Poisson integral of an X-valued measure on the unit circle if and only if the associated operators $\{\Phi_r;\ 0\le r<1\}$ are equibounded, that is,

$$\sup\{\|f_r\lambda\|;\ 0\le r<1\}<\infty.$$

This statement is an exact vector analogue of the criterion stated in the introduction characterizing those complex-valued harmonic functions on D which are the Poisson integral of a complex Borel measure on the circle.

The criterion of Theorem 2.1 does not allow us to decide whether the vector measure whose Poisson integral is f has special properties, such as finite variation or relatively compact range, for example. The latter case is relatively easy to formulate.

THEOREM 2.2. *A harmonic function $f: D \to X$ is the Poisson integral of an X-valued measure on B with relatively compact range if and only if the associated family of linear maps $\{\Phi_r;\ 0 \leq r < 1\}$ defined by (2) is equicompact.*

PROOF. If $m: B \to X$ is a measure with relatively compact range then the same calculation as in the proof of Theorem 2.1 shows that the set (6) is contained in $\overline{bco}(m(B))$. Since $\overline{bco}(m(B))$ is compact this shows that the family $\{\Phi_r;\ 0 \leq r < 1\}$ is equicompact.

Conversely, suppose that $\{\Phi_r;\ 0 \leq r < 1\}$ is equicompact. If V denotes the closed convex balanced hull of (6) then V is compact and so in particular, weakly compact. An examination of the proof of Theorem 2.1 shows that the weakly compact operator $\Phi: C(\Pi) \to X$ constructed there maps the unit ball of $C(\Pi)$ into V. Accordingly, Φ is actually compact and so the representing measure $m: B \to X$ satisfying (13) and whose Poisson integral is f (cf. proof of Theorem 2.1) has relatively compact range, [2; VI Theorem 2.18].

The case for measures of finite variation is due to M. Heins, namely

THEOREM 2.3. *A harmonic function $f: D \to X$ is the Poisson integral of an X-valued measure of finite variation on B if and only if*

$$\sup\{\|f_r\|_B;\ 0 \leq r < 1\} < \infty. \tag{14}$$

PROOF. Since $w \mapsto \|f(w)\|$ is subharmonic in D, [5; p. 89], it follows that (14) holds if and only if $w \mapsto \|f(w)\|$ has a harmonic majorant, [3; p. 38, Theorem 6.7]. The desired conclusion follows from [6; Theorem 3.1].

REMARK. If the Banach space X has the Radon-Nikodym property and $f: D \to X$ is a harmonic function satisfying the hypothesis of Theorem 2.3, then the (unique) X-valued measure of finite variation on B whose Poisson integral is f necessarily has relatively compact range (cf. proof of IX Theorem 1.10 in [2]). This assertion is false if the assumption that X has the Radon-Nikodym property is removed, [2; IX Example 1.1].

3. Poisson integrals of Pettis integrable functions.

The first result concerning the Poisson integral of Pettis integrable functions is the following

THEOREM 3.1. *Let $f : D \to X$ be a harmonic function and $F : \Pi \to X$ be a Pettis integrable function. Let $\{\Phi_r;\ 0 \leq r < 1\}$ be the family of maps associated with f via the formulae* (2) *and $\Phi_F : C(\Pi) \to X$ be the linear map defined by*

$$\Phi_F : \psi \to \int_\Pi \psi F d\lambda, \quad \psi \in C(\Pi).$$

Then f is the Poisson integral of F if and only if in the weak topology,

$$\lim_{r \to 1-} \Phi_r(\psi) = \Phi_F(\psi) \tag{15}$$

uniformly with respect to $\psi \in C(\Pi)$, $\|\psi\|_\infty \leq 1$.

PROOF. Suppose that (15) holds uniformly with respect to ψ in the unit ball of $C(\Pi)$. Fix $x' \in X'$. Then

$$\lim_{r \to 1-} \langle \Phi_r(\psi), x' \rangle = \langle \Phi_F(\psi), x' \rangle$$

uniformly for $\|\psi\|_\infty \leq 1$ or, equivalently,

$$\lim_{r \to 1-} \sup\{ \left| \int_\Pi \psi \langle f_r - F, x' \rangle d\lambda \right| ;\ \psi \in C(\Pi),\ \|\psi\|_\infty \leq 1\} = 0.$$

Since $L^1(\Pi)$ is part of the topological dual space to $C(\Pi)$, this means that $\lim_{r \to 1-} \|\langle f_r, x' \rangle - \langle F, x' \rangle\|_1 = 0$ and, hence, $\langle f, x' \rangle$ is the Poisson integral of $\langle F, x' \rangle$; see [7 ; p. 33]. Since this is the case for every $x' \in X'$ it follows that $f = F * P$.

Conversely, suppose that f is the Poisson integral of F. Then it follows from the numerical case that

$$\text{(weak)} \quad \lim_{r \to 1-} \int_\Pi \psi((P_r * F) - F) d\lambda = 0$$

uniformly with respect to $\psi \in C(\Pi)$, $\|\psi\|_\infty \leq 1$. But $P_r * F = f_r$ for each $0 \leq r < 1$ and, hence, the limit (15) holds uniformly for ψ in the unit ball of $C(\Pi)$.

It is clear that Theorem 3.1 is of little use in determining whether or not a given harmonic function $f : D \to X$ is the Poisson integral of some X-valued Pettis integrable function on Π as it can only confirm or refute whether f is the Poisson integral of a *particular* Pettis integrable function, assumed known in advance. A natural starting point for finding a criterion allowing such a determination for f would be to examine the corresponding family of functions f_r, $0 \leq r < 1$, considered as being Pettis integrable rather than Bochner integrable, with respect to some natural topology. Since the indefinite integral of a Pettis integrable function is a vector measure, the space $P^1(\Pi, X)$ of all (equivalence classes of) X-valued Pettis integrable functions on the unit circle comes equipped with a ready made norm topology, namely the total semivariation topology induced from the space of X-valued measures on B (cf. § 1). However, as noted previously, the space $P^1(\Pi, X)$ is not usually complete. In fact, if X is an infinite dimensional, separable Banach space then $P^1(\Pi, X)$, in contrast to $L^1(\Pi, X)$, is never complete, [10; p. 131]. Accordingly, if the net $\{f_r;\ 0 \leq r < 1\}$ is Cauchy in $P^1(\Pi, X)$, then it is not in general possible to deduce the existence of a limit of this net in the space $P^1(\Pi, X)$. Of course, if $P^1(\Pi, X)$ is considered as a subspace of the X-valued measures on B, then this net does have a limit, a measure of a special type; see Theorem 3. 3. However, it is desirable, if possible, to remain within the realm of *functions*. If we relax the requirement that the function whose Poisson integral is to be f must assume its values in X, then due to some recent results of S. Okada, [9], this is indeed possible; see Corollaries 3. 3. 1 and 3. 3. 2 below. For the sake of self containment we summarise those aspects of [9] which are relevant to this paper.

Let Y be a locally convex Hausdorff space such that there exists a continuous linear injection of X into Y. Then the space Y' can be identified with a subspace of X' which separates the points of X.

A Y-valued function f defined on Π is said to be (X, Y)-Archimedes integrable with respect to Lebesgue measure λ, [9], if there exist vectors $c_i \in X$ and sets $E(i) \in B$, $i=1, 2, \ldots$ such that

(i) the sequence of sets $\{c_i\lambda(F);\ F \in B,\ F \subseteq E(i)\}_{i=1}^{\infty}$ is summable in X, in the sense of [9], and

(ii) if $y' \in Y'$, then the equality

$$\langle f(t), y' \rangle = \sum_{i=1}^{\infty} \langle c_i, y' \rangle \chi_{E(i)}(t)$$

holds for every $t \in \Pi$ for which $\sum_{i=1}^{\infty} |\langle c_i, y' \rangle| \chi_{E(i)}(t)$ is finite.

The indefinite integral of f with respect to λ is the X-valued vector measure $f\lambda$ given by

$$f\lambda : E \to \sum_{i=1}^{\infty} \lambda(E \cap E(i)) c_i, \quad E \in B.$$

The Poisson integral $P*f$, of f, is defined as the Poisson integral of its indefinite integral. Hence, even though f itself is Y-valued its Poisson integral assumes its values in X. Equivalently, for each point $r\, e^{i\theta} \in D$ the Y-valued function $t \to P_r(\theta - t) f(t)$, $t \in [-\pi, \pi]$, is Pettis λ-integrable (with the values of its indefinite integral belonging to X) and

$$((f\lambda) * P_r)(\theta) = (2\pi)^{-1} \int_{-\pi}^{\pi} P_r(\theta - t) f(t)\, dt.$$

The vector space of all (X, Y)-Archimedes integrable functions on Π is denoted by $L_0(\lambda\,; X, Y)$. The total semivariation induces a seminorm

$$f \to \|f\lambda\|\ (\Pi),\ f \in L_0(\lambda\,; X, Y)\,;$$

see § 1 for the notation. The so defined seminormed space $L_0(\lambda\,; X, Y)$ may not be Hausdorff. This can be overcome in the usual way by declaring two elements f and g of $L_0(\lambda\,; X, Y)$ to be equal if $\|(f-g)\lambda\|\ (\Pi) = 0$. This is equivalent to the requirement that $\langle f, y' \rangle = \langle g, y' \rangle$, λ-a. e. for each $y' \in Y'$, [9; Proposition 9]. The resulting normed space (of equivalence classes) is denoted by $L(\lambda\,; X, Y)$. In particular, if $L_0(\lambda\,; X, Y)$ is complete then $L(\lambda\,; X, Y)$ is a Banach space.

The existence of spaces Y for which $L_0(\lambda\,; X, Y)$ is complete is guaranteed by the following result, [9].

Proposition 3. 2. *The seminormed space $L_0(\lambda\,; X, X'^*)$ is complete and contains as dense subspaces the space of X-valued, B-simple functions and the space of strongly measurable, X-valued Pettis integrable functions on the unit circle.*

Remark. It is not claimed in Proposition 3. 2 that $L(\lambda\,; X, X'^*)$ is the completion of $P^1(\Pi, X)$ but only of its subspace $P_0^1(\Pi, X)$, consisting of strongly measurable functions. This will suffice for our purposes.

The space of all λ-continuous vector measures $m : B \to X$ with relatively compact range in X, equipped with the semivariation norm, is denoted by $K(\Pi, X)$.

Theorem 3. 3. *A harmonic function $f : D \to X$ is the Poisson integral of an element from $K(\Pi, X)$ if and only if the associated net of functions $\{f_r\,;\ 0 \le r < 1\}$, considered as belonging to the space $P^1(\Pi, X)$, is Cauchy.*

Proof. Suppose that $\{f_r\,;\ 0 \le r < 1\}$ is Cauchy in $P^1(\Pi, X)$. Noting that each element f_r, $0 \le r < 1$, actually belongs to $P_0^1(\Pi, X)$, by continuity for example, it follows from [2; VIII Theorem 5] that there exists $m \in K(\Pi, X)$ such that $f_r\lambda \to m$ with respect to the semivariation norm. In particular, for $x' \in X'$, we have that $\langle f_r\lambda, x' \rangle \to \langle m, x' \rangle$ in the space of complex measures. But, $\{\langle f_r, x'\rangle\,;\ 0\leq r<1\}$ is Cauchy in $L^1(\Pi)$, as $\{f_r\,;\ 0\leq r<1\}$ is Cauchy in $P^1(\Pi, X)$, and hence there exists $g_{x'}\in L^1(\Pi)$ such that $\langle f_r, x'\rangle\to g_{x'}$ in $L^1(\Pi)$ and $\langle f, x'\rangle=P*g_{x'}$, [7 ; p. 33]. Then $\langle f_r\lambda, x'\rangle=\langle f_r, x'\rangle\lambda \to g_{x'}\lambda$ in the space of complex measures and hence, $\langle m, x'\rangle=g_{x'}\lambda$. Accordingly,

$$\langle P*m, x'\rangle=P*\langle m, x'\rangle=P*g_{x'}\lambda=P*g_{x'}=\langle f, x'\rangle.$$

Since this is the case for every $x'\in X'$ it follows that f is the Poisson integral of m.

Conversely, suppose that $f=P*m$ for some $m\in K(\Pi, X)$. We have already noted (cf. § 1) that $\{f_r\,;\ 0\leq r<1\}$ is contained in $P_0^1(\Pi, X)$ in this case and so it remains to show that it is a Cauchy net.

Let $\varepsilon>0$. Since simple functions are dense in $K(\Pi, X)$, [2 ; p. 224], there exist elements $c_1, \ldots, c_n$ in X and sets $E(1), \ldots, E(n)$ in B such that $h=\sum_{j=1}^{n} c_j\chi_{E(j)}$ satisfies $\|h\lambda-m\|<\varepsilon/3$. Using (5), (8), (9) and (10) we have

$$\|f_r\lambda\|=\sup\ \{\,|\langle\Phi_r(\psi), x'\rangle|\ ;\ \psi\in C(\Pi),\ \|\psi\|_\infty\leq 1,\ x'\in X',\ \|x'\|\leq 1\}\leq\|m\|.$$

Replacing m by $h\lambda-m$ in this inequality, it follows that

$$\|f_r\lambda-h_r\lambda\|\leq\|h\lambda-m\|<\varepsilon/3,$$

for all $0\leq r<1$. Since $h\in L^1(\Pi, X)$, there exists $\delta>0$ such that $(1-\delta)<r<1$ implies $\|h_r\lambda-h\lambda\|<\varepsilon/3$. Hence, $\|f_r\lambda-m\|<\varepsilon$ whenever $(1-\delta)<r<1$. This proves the theorem.

COROLLARY 3. 3. 1. *Let $f : D\to X$ be harmonic and suppose that the associated net of functions $\{f_r\,;\ 0\leq r<1\}$ as defined by (1) and considered as a part of the space $P^1(\Pi, X)$, is Cauchy. Then there exists an (unique) Archimedes (X, X'^*)-integrable function, in particular, X'^*-valued Pettis integrable function, whose Poisson integral is f.*

PROOF. It follows from Theorem 3. 3 that $f=P*m$ for some measure $m\in K(\Pi, X)$ and hence, $f=P*F$ for some $F\in L(\lambda\ ;\ X, X'^*)$, [9 ; Proposition 15(ii)]. The uniqueness of F follows from the uniqueness of m and [9 ; Proposition 9].

REMARK. It is worth noting that for particular spaces X it may be possible to replace the space X'^* in Corollary 3. 3. 1 by a substantially smaller space. For example, if X is a separable Hilbert space and Γ is a complete orthonormal basis for X', then the vector space $\boldsymbol{C}^\Gamma$, consisting of all $\boldsymbol{C}$-

valued functions on Γ equipped with the natural linear operations is a Fréchet space with respect to the topology of pointwise convergance. Furthermore, X is continuously imbedded in $\boldsymbol{C}^{\Gamma}$ and the space $L(\lambda\,;\, X, \boldsymbol{C}^{\Gamma})$ is complete, [9 ; § 2]. Accordingly, if a harmonic function f : $\mathrm{D}\rightarrow X$ satisfies the hypothesis of Corollary 3. 3. 1, then there exists an unique $(X, \boldsymbol{C}^{\Gamma})$-Archimedes integrable function on the unit circle whose Poisson integral is f.

COROLLARY 3. 3. 2. *Let Y be a locally conuex Hausdorff space into which X is continuously imbedded. If F : $\Pi\rightarrow Y$ is an (X, Y)-Archimedes integrable function, then the net $\{P_r*F\,;\, 0\leq r<1\}$ is contained in $P^1(\Pi, X)$ and is Cauchy in that space.*

PROOF. Let $m=F\lambda$. Since Y' separates points of X and $\langle m, y'\rangle$ is absolutely continuous with respect to λ, for each $y'\in Y'$, it follows that m is absolutely continuous with respect to λ. Furthermore, m has relatively compact range in X, [9 ; Proposition 15(i)], and so $m\in K(\Pi, X)$. Noting that $P*F$ is precisely $P*m$ (by definition), it follows from Theorem 3. 3 that $\{P_r*F\,;\, 0\leq r<1\}$ is a Cauchy net in $P^1(\Pi, X)$.

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