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Complex powers of a class of pseudodifferential operators in R^n and the asymptotic behavior of eigenvalues

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§ 0. Introduction

In the previous paper [2], we constructed complex powers for some hypoelliptic pseudodifferential operators P in $OPL^{m,M}(\Omega;\Sigma)$ (for the notation, see Sjöstrand [18]) on a compact manifold Ω of dimension n without boundary and examined the asymptotic behavior of the eigenvalues of P. Here the principal symbol vanished exactly to M-th order on the characteristic set Σ of codimension d in $T^*\Omega\backslash 0$. The hypoellipticity of these operators is well known by Boutet de Monvel [3] for $M=2$ and Helffer [6] for general M. Moreover Menikoff-Sjöstrand [11], [12], [13], Sjöstrand [19] and Iwasaki [9] studied the asymptotic behavior of eigenvalues of P under various assumptions on Σ in the case $M=2$. Their methods are based on the constructions of heat kernel and an application of Karamata's Tauberian theorem. For general M, Mohamed [14], [15] and [16] gave the asymptotic formula for the eigenvalues of P by using Carleman's method in which the Hardy-Littlewood Tauberian theorem was used.

However the method in [2] was essentially due to Minakshinsundaram's method (c. f. Seeley [17] and Smagin [20]). The essentials of the theory in [2] were as follows: At first we construct complex powers $\{P^z\}_{z\in C}$ of P. When the real part of z is negative and $|z|$ is sufficiently large, P^z is of trace class and the trace is extended to a meromorphic function in C which is written by Trace(P^z). Secondly we examine the first singularity of Trace(P^z). Finally we apply the extended Ikehara Tauberian theorem. (See [2: Lemma 5.2] and Wiener [21]). Here since Trace(P^z) is a meromorphic function in C, we call the pole with the smallest real part the first singularity throughout this paper. More precisely, denoting the counting function of eigenvalues by $N(\lambda)$, the first term of the asymptotic behavior of $N(\lambda)$ as λ tends to infinity is closely related to the position and the order of the pole at the first singularity. In the case where $n/m=d/M$, the first singularity situates at $z=-n/m$ and is a double pole and then we have for a constant c

$$N(\lambda)=c\,\lambda^{n/m}\log\lambda+o(\lambda^{n/m}\log\lambda)\ \text{ as } \lambda\to\infty.$$

In the other cases they are only simple poles and $\log\lambda$ does not appear in the first term of $N(\lambda)$.

However in the framework of [2], for example, we can not treat the following operator on $\boldsymbol{R}^3$:

$$P=(D_{x_1}^2+x_1^2)^2(D_{x_2}^2+x_2^2)^2(|D_x|^2+|x|^2)^2+$$
$$\mu(D_{x_1}^2+D_{x_2}^2+x_1^2+x_2^2)^2(|D_x|^2+|x|^2)^3+\nu(|D_x|^2+|x|^2)^4(\mu,\ \nu>0).$$

Our purpose in the present paper is to study the asymptotic behavior of $N(\lambda)$ for such operators. In order to do so we consider a class $OPL^{m,M_1,M_2}(\Sigma_1,\Sigma_2)$ where the characteristic set Σ is a union of two closed submanifolds Σ_1 and Σ_2 of codimension d_1 and d_2 in $\boldsymbol{R}^{2n}\backslash 0$ and the principal symbol vanishes exactly to M_i-th order on Σ_i $(i=1,2)$ respectively. Under some appropriate conditions, we construct complex powers $\{P^z\}$ and examine the first singularity of $\mathrm{Trace}(P^z)$ in the same way as [2]. But it is necessary to construct different symbols of P^z according to the order relations among real numbers $2n/m$, d_1/M_1 and d_2/M_2. In particular, we have a new result that for the case $2n/m=d_1/M_1=d_2/M_2$ with a constant c

$$N(\lambda)=c\,\lambda^{2n/m}(\log\lambda)^2+o(\lambda^{2n/m}(\log\lambda)^2)\ \text{ as } \lambda\to\infty.$$

The plan of this paper is as follows. In § 1 we give the precise definition of the operators mentioned above and give some hypotheses. In § 2 we introduce two classes of operators in which we construct the parametrices of $P-\zeta$ for some $\zeta\in\boldsymbol{C}$. By taking an application in § 5 and § 6 into consideration, we construct in § 3 various parametrices of $P-\zeta$ for some $\zeta\in\boldsymbol{C}$. In § 4 we construct symbols of complex powers corresponding to parametrices in § 3 respectively. In § 5 we examine the first singularity of the trace of complex powers. Finally in § 6 we study asymptotic behavior of the eigenvalues using the results in § 5 and give some examples.

For brevity of the notations, we use the followings which are held from § 1 to § 5:

$$M_0=M_1+M_2,\ \ d_0=d_1+d_2$$
$$\Sigma_0=\Sigma_1\cap\Sigma_2,\ \ \Sigma=\Sigma_1\cup\Sigma_2$$

$N(a,b)=a-b/2$ for any real numbers a and b.

§ 1. Definitions of operators and some hypotheses

In this section we introduce a class of pseudodifferential operators on $\boldsymbol{R}^n$ and give our hypotheses.

Let Σ_1 and Σ_2 be closed conic submanifolds of codimension d_1 and d_2 in $\boldsymbol{R}^n \times \boldsymbol{R}^n$ respectively such that $d_0 = d_1 + d_2 < 2n$. Here the conicity of Σ_i means that $(x, \xi) \in \Sigma_i$ implies $(\lambda x, \lambda \xi) \in \Sigma_i$ for any $\lambda > 0$.

DEFINITION 1.1. *(c. f. [1] and [18]) Let m be a real number and M_i $(i=1, 2)$ be non-negative integers. Then the space $OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$ is the set of all pseudodifferential operators $P(x, D) \in L^m(\boldsymbol{R}^n)$ (for the notation $L^m(\boldsymbol{R}^n)$ see Hörmander [7] and [8]) such that $P(x, D)$ has a symbol $p(x, \xi) \in C^\infty(\boldsymbol{R}^{2n})$ satisfying the following (1. 1) and (1. 2):*

(1.1) *There exists a sequence of functions $\{p_{m-j/2}(x, \xi)\}_{j=0,1,\ldots}$ such that $p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j/2}(x, \xi)$ where $p_{m-j/2}(x, \xi)$ are elements of $C^\infty(\boldsymbol{R}^{2n} \backslash 0)$ and positively homogeneous of degree $m - j/2$ in $(x, \xi) \in \boldsymbol{R}^{2n} \backslash 0$. Here the asymptotic sum in (1. 1) means that for every positive integer N and every multi-indices α, β, there exists a constant $C_{\alpha, \beta, N} > 0$ such that*

$$|D_x^\alpha D_\xi^\beta (p(x, \xi) - \sum_{j=0}^{N-1} p_{m-j/2}(x, \xi))| \leq C_{\alpha, \beta, N}\ r(x, \xi)^{m-N/2-|\alpha|-|\beta|}$$

for $r(x, \xi) \geq 1$ where $r = r(x, \xi) = (|x|^2 + |\xi|^2)^{1/2}$.

(1.2) *There exists a positive constant C such that*

$$\frac{|p_{m-j/2}(x, \xi)|}{r(x, \xi)^{m-j/2}} \leq C \sum_{\substack{k_1+k_2=j \\ k_i \leq M_i}} d_{\Sigma_1}(x, \xi)^{M_1-k_1}\, d_{\Sigma_2}(x, \xi)^{M_2-k_2}, \quad j=0, 1, \ldots, M_0,$$

where $d_{\Sigma_i}(x, \xi) = \inf_{(x', \xi') \in \Sigma_i} (|x' - \frac{x}{r}| + |\xi' - \frac{\xi}{r}|),\ i=1, 2.$

The class of symbols satisfying (1. 1) and (1. 2) in an open conic set U in $\boldsymbol{R}^{2n} \backslash 0$ is denoted by $L^{m, M_1, M_2}(U\,;\ \Sigma_1, \Sigma_2)$. Finally we say that $P(x, D)$ is regularly degenerate if moreover $p(x, \xi)$ satisfies:

$$(1.3) \qquad \frac{|p_m(x, \xi)|}{r(x, \xi)^m} \geq C\ d_{\Sigma_1}(x, \xi)^{M_1}\, d_{\Sigma_2}(x, \xi)^{M_2}.$$

For brevity of the notations, we denote:

$$OPL^{m, M_1, 0}(\Sigma_1, \Sigma_2) = OPL^{m, M_1}(\Sigma_1)$$
$$OPL^{m, 0, M_2}(\Sigma_1, \Sigma_2) = OPL^{m, M_2}(\Sigma_2).$$

If necessary, by relabelling of Σ_i, we may assume:

$$(1.4) \qquad \frac{d_2}{M_2} \leq \frac{d_1}{M_1}.$$

For the construction of parametrices of $P(x, D) - \zeta$ as in introduction, we have to keep the following hypotheses (H. 1)$\sim$(H. 4).

(H.1) $P_m(x, \xi)\geq 0$ for all $(x, \xi)\in \boldsymbol{R}^{2n}\backslash 0$.

(H.2) Σ_1 and Σ_2 intersect transversally. That is, $\Sigma_0=\Sigma_1\cap\Sigma_2$ is a closed conic submanifold such that for every point $\rho\in\Sigma_0$,

$$T_\rho\Sigma_0=T_\rho\Sigma_1\cap T_\rho\Sigma_2.$$

Now for every $\rho\in\Sigma_0$ and $j=0, 1, \ldots, M_0$, we can define a multi-linear form $\tilde{\tilde{p}}_{m-j/2}(\rho)$ on $N_\rho\Sigma_0=\boldsymbol{R}^{2n}/T_\rho\Sigma_0$ which may be identified with $\boldsymbol{R}^{d_1}\times\boldsymbol{R}^{d_2}$: For $X_1, X_2, \ldots, X_{M_0-j}\in N_\rho\Sigma_0$,

$$\tilde{\tilde{p}}_{m-j/2}(\rho)(X_1, \ldots, X_{M_0-j})=\frac{1}{(M_0-j)\,!}(\tilde{X}_1\ldots\tilde{X}_{M_0-j}\, p_{m-j/2})(\rho)$$

where $\tilde{X}$ means a vector field extending X to a neighborhood of ρ. For every $\rho\in\Sigma_i\backslash\Sigma_0$ and $j=0, \ldots, M_i$, we also define $\tilde{\tilde{p}}_{m-j/2}(\rho)$ similarly. Thus we define the followings: If $\rho\in\Sigma_0$,

$$\tilde{p}(\rho, X)=\sum_{j=0}^{M_0}\tilde{p}_{m-j/2}(\rho)(X),\quad X\in N_\rho\Sigma_0$$

where $\tilde{p}_{m-j/2}(\rho)(X)=\tilde{\tilde{p}}_{m-j/2}(\rho)(X, \ldots, X)$ and similarly if $\rho\in\Sigma_i\backslash\Sigma_0$,

$$\tilde{p}(\rho, X_i)=\sum_{j=0}^{M_i}\tilde{p}_{m-j/2}(\rho)(X_i),\quad X_i\in N_\rho\Sigma_i.$$

REMARK 1.2. For example, if $\rho\in\Sigma_0$ and W is a conic neighborhood of ρ, the class $[\sum_{j=0}^{M_0} p_{m-j/2}]\in L^{m, M_1, M_2}(W;\Sigma_1, \Sigma_2)/L^{m, M_1+M_2+1}(W;\Sigma_1\cap\Sigma_2)$ is invariant under a transformation of local coordinates. (c. f. [1] and Proposition 2. 2). Therefore $\tilde{p}(\rho, X)$ is defined invariantly.

(H.3) There exists a positive constant δ such that for any $\rho\in\Sigma_0\cap S^*\boldsymbol{R}^{2n}$ (where $S^*\boldsymbol{R}^{2n}=\{(x, \xi)\in\boldsymbol{R}^{2n};\ r(x, \xi)=1\}$)

$$\tilde{p}(\rho, X)\geq 2\delta(|X_1|^2+1)^{M_1/2}(|X_2|^2+1)^{M_2/2}\text{ for all }X=(X_1, X_2)\in\boldsymbol{R}^{d_1}\times\boldsymbol{R}^{d_2},$$

and for any $\rho\in(\Sigma_i\backslash\Sigma_0)\cap S^*\boldsymbol{R}^{2n}\quad(i=1, 2)$,

$$\tilde{p}(\rho, X_i)\geq 2\delta(|X_i|^2+1)^{M_i/2}\text{ for all }X_i\in\boldsymbol{R}^{d_i}.$$

(H.4) M_1 and M_2 are positive integers and $m>M_0/2$.

REMARK 1.3. If $P(x, D)\in OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$ satisfies (H. 1)$\sim$(H. 4), it is well known that $P(x, D)$ is hypoelliptic with loss of $M_0/2$-derivatives. (c.f. [1]).

§ 2. The preparations for constructions of parametrices

In this section we introduce two classes of symbols in which we construct parametrices of $P(x, D)-\zeta I$ for some $\zeta\in C$ and complex powers of $P(x, D) \in OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$. In order to do, let $\rho\in\Sigma_0$. By (H. 2) we can choose a local coordinate system in a conic neighborhood W of ρ : $w=(u_1, u_2, v, r)$ where $u_1=(u_{11}, u_{12}, \ldots, u_{1d_1})$, $u_2=(u_{21}, u_{22}, \ldots, u_{2d_2})$, $v=(v_1, v_2, \ldots, v_{2n-d_0-1})$ such that u_{ij}, v_k are positively homogeneous functions of degree 0 with du_{ij} $(j=1, \ldots, d_i, i=1, 2)$, dv_k $(k=1, \ldots, 2n-d_0-1)$ being linearly independent and $\Sigma_i\cap W=\{u_i=0\}$, $i=1, 2$. When $\rho\in\Sigma_i\backslash\Sigma_0$, we can choose a local coordinate system (u_i, v, r) in a conic neighborhood W of $\rho\in\Sigma_i\backslash\Sigma_0$ such that $W\cap\Sigma_0=\phi$ and $\Sigma_i\cap W=\{u_i=0\}$, $i=1, 2$.

DEFINITION 2.1. *(c.f. [2] and [3]) Let m, k_1 and k_2 be real numbers and W a conic neighborhood of $\rho\in\Sigma_0$. We denote by $S^{m, k_1, k_2}(W; \Sigma_1, \Sigma_2)$ the set of all C^∞ functions $a(w)$ defined in W such that for any non-negative integer p and any multi-indices $(\alpha_1, \alpha_2, \beta)$, there exists a constant $C>0$ such that for all $r\geq 1$,*

$$(2.1)\qquad \left|\left(\frac{\partial}{\partial u_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial u_2}\right)^{\alpha_2}\left(\frac{\partial}{\partial v}\right)^{\beta}\left(\frac{\partial}{\partial r}\right)^{p} a(w)\right|\leq C\ r^{m-p}\rho_{\Sigma_1}^{k_1-|\alpha_1|}\rho_{\Sigma_2}^{k_2-|\alpha_2|}\quad \textit{where}$$

$\rho_{\Sigma_i}=(d_{\Sigma_i}^2+r^{-1})^{1/2}$. Similarly if W is a conic neighborhood of $\rho\in\Sigma_i\backslash\Sigma_0$ such that $W\cap\Sigma_0=\phi$, we also define $S^{m, k_i}(W; \Sigma_i)$.

Note that $S^{m, k_1, k_2}(W; \Sigma_1, \Sigma_2)$ and $S^{m, k_i}(W; \Sigma_i)$ are Fréchet spaces when equipped with the semi-norms defined by the best possible constants in (2. 1). Then we have:

PROPOSITION 2. 2. *If W is a conic neighborhood of $\rho\in\Sigma_0$ or $\rho\in\Sigma_i\backslash\Sigma_0$ such that $W\cap\Sigma_0=\phi$, then $\dfrac{\partial}{\partial x_i}$ and $\dfrac{\partial}{\partial \xi_i}$ are continuous from $S^{m, k_1, k_2}(W; \Sigma_1, \Sigma_2)$ to $S^{m-1/2, k_1, k_2}(W; \Sigma_1, \Sigma_2)$ or from $S^{m, k_i}(W; \Sigma_4)$ to $S^{m-1/2, k_i}(W; \Sigma_i)$ respectively.*

In fact we can write $\dfrac{\partial}{\partial x_i}=\dfrac{\partial u_1}{\partial x_i}\dfrac{\partial}{\partial u_1}+\dfrac{\partial u_2}{\partial x_i}\dfrac{\partial}{\partial x_2}+\dfrac{\partial v}{\partial x_i}\dfrac{\partial}{\partial v}+\dfrac{\partial r}{\partial x_i}\dfrac{\partial}{\partial r}$. Thus it suffices to note that $\dfrac{\partial u_j}{\partial x_i}$, $\dfrac{\partial v}{\partial x_i}$ and $\dfrac{\partial r}{\partial x_i}$ are homogeneous of degree -1, -1 and 0 respectively and

$$S^{m, k_1, k_2}\subset S^{m+1/2, k_1+1, k_2}\cap S^{m+1/2, k_1, k_2+1}.$$

Let W be a conic neighborhood of $\rho\in\Sigma_0$. Then we need the following three propositions which follow from a routine consideration (c.f. [2], [3]).

PROPOSITION 2.3. *For non-negative integers M_1 and M_2, we have*

$$L^{m,M_1,M_2}(W\,;\,\Sigma_1,\Sigma_2)\subset S^{m,M_1,M_2}(W\,;\,\Sigma_1,\Sigma_2).$$

PROPOSITION 2.4. *If* $p_1\in S^{m,M_1,M_2}(W\,;\,\Sigma_1,\Sigma_2)$ *and* $p_2\in S^{m',M_1',M_2'}(W\,;\,\Sigma_1,\Sigma_2)$, *then we have* $p_1\# p_2\in S^{m+m',M_1+M_1',M_2+M_2'}(W\,;\,\Sigma_1,\Sigma_2)$ *where* $\#$ *means the composition of the symbols*:

$$p_1\# p_2\sim\sum_\alpha\frac{1}{\alpha\,!}\partial_\xi^\alpha p_1 D_x^\alpha p_2.$$

PROPOSITION 2.5. *If* $p\in S^{m,M_1,M_2}(W\,;\,\Sigma_1,\Sigma_2)$ *satisfies*

$$|p|\geq C\ r^m\rho_{\Sigma_1}^{M_1}\rho_{\Sigma_2}^{M_2}$$

for a positive constant C, then we have

$$p^{-1}\in S^{-m,-M_1,-M_2}(W\,;\,\Sigma_1,\Sigma_2).$$

Finally we define a symbol class with a parameter ζ in order to consider parametrices of $P(x,D)-\zeta$ for some $\zeta\in\boldsymbol{C}$.

DEFINITION 2.6. *Let m, M_1 and M_2 be fixed numbers as in* (H.4) *and let l, k_1 and k_2 be real numbers, W a conic neighborhood of $\rho\in\Sigma_0$ and Λ an open set in the complex plane $\boldsymbol{C}$. Then we denote by* $S_\Lambda^{l,k_1,k_2}(W\,;\,\Sigma_1,\Sigma_2)$ *the set of all* $a(w,\zeta)\in C^\infty(W\times\Lambda)$ *satisfying the following* (i) *and* (ii),

(i) *for every* $\zeta\in\Lambda$, $a(w,\zeta)\in S^{l,k_1,k_2}(W\,;\,\Sigma_1,\Sigma_2)$

(ii) *for every* $\zeta\in\Lambda$, $|\zeta|a(w,\zeta)\in S^{m+l,M_1+k_1,M_2+k_2}(W\,;\,\Sigma_1,\Sigma_2)$ *and for every non-negative integer p and multi-indices* $(\alpha_1,\alpha_2,\beta)$, *there exists a positive constant C independing in $\zeta\in\Lambda$ such that*

$$\left|\left(\frac{\partial}{\partial u_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial u_2}\right)^{\alpha_2}\left(\frac{\partial}{\partial v}\right)^{\beta}\left(\frac{\partial}{\partial r}\right)^{p}[\,|\zeta|a(w,\zeta)]\right|\leq$$
$$Cr^{m+l-p}\rho_{\Sigma_1}^{M_1+k_1-|\alpha_1|}\rho_{\Sigma_2}^{M_2+k_2-|\alpha_2|}\ \text{for all}\ (w,\zeta)\in W\times\Lambda.$$

§ 3. Constructions of parametrices

In this section we construct the parametrices of $P(x,D)-\zeta I$ for some $\zeta\in\Lambda$ with various top symbols where Λ is the union of a small open convex cone containing the negative real line and $\{\zeta\in\boldsymbol{C}\,;\,|\zeta|<\delta\}$ where δ is as in (H.3). Let $\rho\in\Sigma_0$ and $w=(u_1,u_2,v,r)$ be a local coordinate system in a small conic neighborhood W of ρ as in § 2. By (1.2) and Taylor's theorem, we can write

$$(3.1)\qquad p_{m-j/2}=\sum_{\substack{|\alpha_1|+|\alpha_2|=M_0-j\\ |\alpha_1|\le M_1,\ |\alpha_2|\le M_2}} a_{\alpha_1,\alpha_2}(u_1, u_2, v, r)u_1^{\alpha_1}u_2^{\alpha_2} \quad \text{in } W.$$

Thus we have for $X=(X_1, X_2)\in N_\rho\Sigma_0=\boldsymbol{R}^{d_1}\times\boldsymbol{R}^{d_2}$,

$$\tilde{p}(\rho, X)=\sum_{j=0}^{M_0}\sum_{|\alpha_1|+|\alpha_2|=M_0-j,\ |\alpha_i|\le M_i} a_{\alpha_1,\alpha_2}(\rho)X_1^{\alpha_1}X_2^{\alpha_2}.$$

Then we need the following three symbols which are needed in order to examine the first singularity in various cases.

PROPOSITION 3.1. *Let $\rho\in\Sigma_0$. Then there exists a small conic neighborhood W of ρ and $a^{(j)}(x, \xi)\in S_\Lambda^{-m,\,-M_1,\,-M_2}(W\,;\,\Sigma_1, \Sigma_2)(j=1, 2, 3)$ such that*

$$(p-\zeta)\ \#\ a_\zeta^{(j)}=1+\sum_{i=1}^{3} c_\zeta^{(ji)}$$

where $c_\zeta^{(11)}\in S_\Lambda^{0,\,1,\,0}$, $c_\zeta^{(12)}$, $c_\zeta^{(22)}\in S_\Lambda^{0,\,0,\,1}$, $c_\zeta^{(21)}\in S_\Lambda^{-1/2,\,-1,\,0}$, $c_\zeta^{(13)}$, $c_\zeta^{(31)}\in S_\Lambda^{-1/2,\,0,\,0}$ and $c_\zeta^{(23)}=c_\zeta^{(32)}=c_\zeta^{(33)}=0$.

PROOF. We choose a function $\chi\in C^\infty(\boldsymbol{R}^{2n})$:

$$\chi(x, \xi)=1 \text{ if } |x|+|\xi|\geqslant 1 \text{ and } \quad =0 \text{ if } |x|+|\xi|\le 1/2.$$

Existence of $a_\zeta^{(1)}$: Let (u_1, u_2, v, r) be a local coordinate system in W as above. We identify (X_1, X_2) with (u_1, u_2) and ρ with $(0, 0, v, r)$ and write $\tilde{p}(\rho, X)=\tilde{p}(u_1, u_2, v, r)$. Define for $\zeta\in\Lambda$,

$$(3.2)\qquad a_\zeta^{(1)}(u_1, u_2, v, r)=\chi(u_1, u_2, v, r)(\tilde{p}(u_1, u_2, v, r)-\zeta)^{-1}.$$

Then we have

$$(p-\zeta)\#a_\zeta^{(1)}=\chi\{(\tilde{p}-\zeta)\#(\tilde{p}-\zeta)^{-1}+(p-\sum_{j=0}^{M_0}p_{m-j/2})\#(\tilde{p}-\zeta)^{-1}+ \\ +\sum_{j=0}^{M_0}(p_{m-j/2}-\tilde{p}_{m-j/2})\#(\tilde{p}-\zeta)^{-1}\}+[p-\zeta, \chi](\tilde{p}-\zeta)^{-1}.$$

Here we note that by Proposition 2.5 and (H.3) we have

$$(\tilde{p}-\zeta)^{-1}\in S_\Lambda^{-m,\,-M_1,\,-M_2}(W\,;\,\Sigma_1, \Sigma_2).$$

Thus it suffices to apply Proposition 2.2 and 2.4.

Existence of $a_\zeta^{(2)}$: By (1.4) we have $\tilde{p}(u_1, u_2, v, r)-\tilde{p}_{\Sigma_2}(u_1, u_2, v, r)=r_1+r_2$ where

$$\tilde{p}_{\Sigma_2}=\sum_{|\alpha_1|=M_1}\{\sum_{j=0}^{M_2}\sum_{|\alpha_2|=M_2-j}a_{\alpha_1,\alpha_2}(u_1, 0, v, r)u_2^{\alpha_2}\}u_1^{\alpha_1},$$

$r_1\in S^{m-1/2, M_1-1, M_2}$ and $r_2\in S^{m, M_1, M_2+1}$. On the other hand, by (H. 3), we have for $\lambda>0$,

$$\begin{aligned}\lambda^{-M_1}\tilde{p}(\lambda u_1, u_2, v, r)&=\sum_{|\alpha_1|=M_1}\{\sum_{j=0}^{M_2}\sum_{|\alpha_2|=M_2-j}a_{\alpha_1,\alpha_2}(0, 0, v, r)u_2^{\alpha_2}\}u_1^{\alpha_1}+O(\lambda^{-1})\\ &\geq 2\delta\lambda^{-M_1}r^m(|\lambda u_1|^2+r^{-1})^{M_1/2}(|u_2|^2+r^{-1})^{M_2/2}.\end{aligned}$$

Letting $\lambda\to\infty$, we see

$$\begin{aligned}&\sum_{|\alpha_1|=M_1}\{\sum_{j=0}^{M_2}\sum_{|\alpha_2|=M_2-j}a_{\alpha_1,\alpha_2}(0, 0, v, r)u_2^{\alpha_2}\}u_1^{\alpha_1}\\ &\geq 2\delta r^m|u_1|^{M_1}(|u_2|^2+r^{-1})^{M_2/2}.\end{aligned}$$

Since W is small enough, for any $\varepsilon>0$,

$$|a_{\alpha_1,\alpha_2}(u_1, 0, v, r)-a_{\alpha_1,\alpha_2}(0, 0, v, r)|\leq\varepsilon r^{m-(M_2-|\alpha_2|)/2}$$

if $|\alpha_1|=M_1$. Therefore we have

$$\tilde{p}_{\Sigma_2}(u_1, u_2, v, r)\geq(3\delta/2)r^m|u_1|^{M_1}(|u_2|^2+r^{-1})^{M_2/2}.$$

Thus it suffices to define for $\zeta\in\Lambda$,

$$\begin{aligned}a_\zeta^{(2)}(u_1, u_2, v, r)=\chi(u_1, u_2, v, r)[\tilde{p}_{\Sigma_2}(u_1, u_2, v, r)+&\\ +r^{m-M_1/2}(|u_2|^2+r^{-1})^{M_2/2}-\zeta]^{-1}.&\end{aligned}\tag{3.3}$$

Existence of $a_\zeta^{(3)}$: Since W is small enough, it suffices to define

$$a_\zeta^{(3)}(x, \xi)=\chi(x, \xi)(\sum_{j=0}^{M_0}p_{m-j/2}(x, \xi)-\zeta)^{-1}.\tag{3.4}$$

This completes the proof.

Now we can construct microlocal parametrices of $P(x, D)-\zeta I$, $\zeta\in\Lambda$. Let $\psi(x, \xi)$ be a C^∞ function of positively homogeneous of degree 0 and supp $\psi\in W$. We define

$$P_{\zeta,0}^{(1)}(x, D)=\psi(x, D)a_\zeta^{(3)}(x, D)\tag{3.5}$$

$$P_{\zeta,0}^{(2)}(x, D)=\psi(x, D)\{a_\zeta^{(1)}(x, D)-a_\zeta^{(3)}(x, D)(\sum_{i=1}^{3}c_\zeta^{(1i)}(x, D))\}\tag{3.6}$$

$$P_{\zeta,0}^{(3)}(x, D)=\psi(x, D)\{a_\zeta^{(2)}(x, D)-a_\zeta^{(3)}(x, D)(\sum_{i=1}^{2}c_\zeta^{(2i)}(x, D))\}.\tag{3.7}$$

Then we have $(P(x, D)-\zeta I)P_{\zeta,0}^{(j)}(x, D)=\psi(x, D)+d_\zeta^{(j)}(x, D)$ where $d_\zeta^{(j)}(x, \xi)\in S^{-1/2,0,0}$ for $j=1, 2, 3$. If we put

$$P^{(j)}_{\xi, l}(x, D)=P^{(j)}_{\xi, 0}(x, D)(-d^{(j)}_{\xi}(x, D))^l, \quad l=0, 1, 2, \dots,$$

we see that $P^{(j)}_{\xi, l}(x, D)\in OPS_{\Lambda}^{-m-l/2, -M_1, -M_2}$ and there exist $q^{(j)}_{\xi}(x, D)\in OPS_{\Lambda}^{-m, -M_1, -M_2}$ such that for every $N>0$,

$$q^{(j)}_{\xi}(x, D)-\sum_{l=0}^{N-1}P^{(j)}_{\xi, l}(x, D)\in OPS_{\Lambda}^{-m-N/2, -M_1, -M_2}, \quad j=1, 2, 3.$$

Then we have $(P(x, D)-\xi I)q^{(j)}_{\xi}(x, D)\equiv\psi(x, D)$ mod $OPS_{\Lambda}^{-\infty}=\bigcap_{m>0} OPS_{\Lambda}^{-m, -M_1, -M_2}$.

Next we consider the case where W is a small conic neighborhood of $\rho\in\Sigma_i\backslash\Sigma_0$ such that $W\cap\Sigma_0=\phi$, $i=1, 2$. In this case, we can write as in (3.1):

$$\tilde{p}(\rho, X_i)=\sum_{j=0}^{M_i}\sum_{|\alpha_i|=M_i-j}a_{\alpha_i}(\rho)X_i^{\alpha_i} \text{ for } X_i\in\boldsymbol{R}^{d_i}.$$

PROPOSITION 3.2. *Let $\rho\in\Sigma_i\backslash\Sigma_0$. Then there exist a conic neighborhood W of ρ and $a^{(ij)}_{\xi}(x, \xi)\in S_{\Lambda}^{-m, -M_i}(W\,;\,\Sigma_i)(j=1, 2)$ such that*

$$(p-\xi)\# a^{(ij)}_{\xi}=1+c^{(ij)}_{\xi}$$

where $c^{(i1)}_{\xi}\in S_{\Lambda}^{0, 1}(W\,;\,\Sigma_i)$ and $c^{(i2)}_{\xi}\in S_{\Lambda}^{-1/2, -1}(W\,;\,\Sigma_i)$.

PROOF. If we consider as in the proof of Proposition 3.1, it suffices to define as follows:
Existence of $a^{(i1)}_{\xi}$: $a^{(i1)}_{\xi}(u_i, v, r)=\chi(u_i, v, r)(\tilde{p}(u_i, v, r)-\xi)^{-1}$
Existence of $a^{(i2)}_{\xi}$: $a^{(i2)}_{\xi}(x, \xi)=\chi(x, \xi)(p_m(x, \xi)+r^{m-M_i/2}-\xi)^{-1}$.
This completes the proof.

Let $\psi(x, \xi)$ be a C^∞ function of positively homogeneous of degree 0 and supp $\psi\subset W$. Define

$$P^{(i1)}_{\xi, 0}(x, D)=\psi(x, D)(a^{(i1)}_{\xi}(x, D)-a^{(i2)}_{\xi}(x, D)c^{(i1)}_{\xi}(x, D)),$$
$$P^{(i2)}_{\xi, 0}(x, D)=\psi(x, D)(a^{(i2)}_{\xi}(x, D)-a^{(i1)}_{\xi}(x, D)c^{(i2)}_{\xi}(x, D)).$$

As the same way as the preceding arguments, we can construct $q^{(ij)}_{\xi}(x, D)\in OPS_{\Lambda}^{-m, -M_i}(i=1, 2$ and $j=1, 2)$ such that for every $N>0$, we have

$$q^{(ij)}_{\xi}(x, D)-\sum_{l=0}^{M-1}p^{(ij)}_{\xi, l}(x, D)\in OPS_{\Lambda}^{-m-N/2, -M_i}$$

and $(P(x, D)-\xi I)q^{(ij)}_{\xi}(x, D)\equiv\psi(x, D)$ mod $OPS_{\Lambda}^{-\infty}$.

Finally we have

PROPOSITION 3.3. *Let W be an open cone such that $W\cap\Sigma=\phi$. Then there exists $a_{\zeta}^{(3)}(x, \xi)\in S^{-m}(W)$ such that*

$$(p-\zeta)\# a_{\zeta}^{(3)}=1+c_{\zeta}^{(3)} \text{ where } c_{\zeta}^{(3)}\in S_{\Lambda}^{-1/2}.$$

PROOF. If necessary, we replace δ as in (H.3) with smaller one. So we may assume $p_m(x, \xi)\geq\delta$ in W. Thus if we put

$$a_{\zeta}^{(3)}(x, \xi)=\chi(x, \xi)(p_m(x, \xi)-\zeta)^{-1},$$

the proof is complete.

§ 4. Construction of complex powers

In this section we consider complex powers of an operator P associated to $P(x, D)$. Assume that $P(x, D)\in OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$ satisfies (1.3), (1.4) and (H.1)$\sim$(H.4). Moreover we assume:
(H.5) $P(x, D)$ is formally self-adjoint, i.e., for every u, $v\in\mathscr{S}(\boldsymbol{R}^n)$.

$$\int_{\boldsymbol{R}^n} P(x, D)u\,\bar{v}\,dx=\int_{\boldsymbol{R}^n} u\,\overline{P(x, D)v}\,dx.$$

Let P_0 be an operator on $L^2(\boldsymbol{R}^n)$ with the definition domain $D(P_0)=\mathscr{S}(\boldsymbol{R}^n)$ such that $P_0 u=P(x, D)u$ for $u\in D(P_0)$. By Remark 1.3 and (H.4), $P(x, D)$ is hypoelliptic with loss of $M_0/2$-derivatives and $m-M_0/2>0$. Therefore P_0 is essentially self-adjoint and the closure P of P_0 is an unbounded self-adjoint operator with the definition domain
$D(P)=\{u\in L^2(\boldsymbol{R}^n)\,;\, P(x, D)u\in L^2(\boldsymbol{R}^n)\}$,

$$P\,u=P(x, D)u \text{ for } u\in D(P).$$

Since $P(x, D)$ has a parametrix $Q(x, D)\in OPS^{-m, -M_1, -M_2}(\Sigma_1, \Sigma_2)$, P has a compact regularizer on $L^2(\boldsymbol{R}^n)$. (c.f. Kumano-go [10] and also Grushin [5]). Thus P has the spectrum consist only of eigenvalues of finite multiplicity. Finally we assume:
(H.6) P is positive definite, i.e., there exists a positive real number γ such that $(P\,u, u)\geq\gamma\|u\|_{L^2(\boldsymbol{R}^n)}^2$ for all $u\in D(P)$.

Then we can define complex powers P^z by the spectral resolution of P. Let Γ be a curve beginning at infinity, passing along the negative real line to a circle $\{\zeta\,;\,|\zeta|=\delta\}$ (where δ is in (H.3) and we may assume $\delta\leq\gamma$), then clockwise about the circle and back to infinity along the negative real line. For $\mathscr{R}_e\, z<0$, we see

$$(4.1)\qquad P^z=\frac{i}{2\pi}\int_\Gamma \zeta^z(P-\zeta)^{-1}d\zeta$$

where ζ^z takes the principal value in $\boldsymbol{C}\backslash\boldsymbol{R}^-$. Here we note that $\|(P-\zeta)^{-1}\|_{\mathscr{L}(L^2,L^2)}\leq[\mathrm{dist}(\zeta,[\gamma,\infty)]^{-1}=O(|\zeta|^{-1})$ as $|\zeta|\to\infty$ and $\zeta\in\Lambda$. Therefore the integral in the right hand side in (4.1) is convergent.

On the other hand we define operators $P_z(x, D)$ with the symbol $\sigma(P_z)$ by the formula:

$$\sigma(P_z)(x,\xi)=\frac{i}{2\pi}\int_\Gamma \zeta^z q_\zeta(x,\xi)\,d\zeta. \tag{4.2}$$

Here for brevity of the notations we have dropped the upper indices of $q_\zeta^{(j)}(x, D)(j=1,2,3)$ in § 3. Since $q_\zeta\in S_\Lambda^{-m,-M_1,M_2}(\Sigma_1,\Sigma_2)$, we see easily that the integral in (4.2) is absolutely convergent when $\mathscr{R}_e\, z<0$. For $\mathscr{R}_e\, z\geq 0$, choose an integer k such that $-1\leq\mathscr{R}_e\, z-k<0$ and define

$$P_z(x,D)=P(x,D)^k P_{z-k}(x,D). \tag{4.3}$$

Then we have:

THEOREM 4.1. *Assume that* $P(x,D)\in OPL^{m,M_1,M_2}(\Sigma_1,\Sigma_2)$ *satisfies* (1.3), (1.4) *and* (H.1)$\sim$(H.6). *Then we have the followings*:

(i) $P^z\in OPS^{m\mathscr{R}_e z,\,M_1\mathscr{R}_e z,\,M_2\mathscr{R}_e z}(\Sigma_1,\Sigma_2)$.

(ii) *For any negative real number a and real numbers m', k_1 and k_2 satisfying $ma<m'$, $N(m,M_i)a<N(m',k_i)(i=1,2)$ and $N(m,M_0)a<N(m',k_1+k_2)$, $\sigma(P^z)$ is holomorphic on any compact set in $\{z\,;\,\mathscr{R}_e z<a\}$ with value in $S^{m',k_1,k_2}(\Sigma_1,\Sigma_2)$.*

Later from now we write such class of symbols satisfying (i) and (ii) by $S_0^{m\mathscr{R}_e z,\,M_1\mathscr{R}_e z,\,M_2\mathscr{R}_e z}$.

PROOF. Let $\mathscr{R}_e z<0$. Near Σ_0, we see that by (H.3), $q_\zeta(x,\xi)$ is holomorphic in $\{\zeta\,;\,\mathscr{I}_m\zeta=0,\ \mathscr{R}_e\zeta\leq 0\}\cup\{\zeta\,;\,|\zeta|\leq\delta\, R(r,u_1,u_2)\}$ where

$$R(r,u_1,u_2)=r^m\rho_{\Sigma_1}^{M_1}\rho_{\Sigma_2}^{M_2}. \tag{4.4}$$

So we may replace the contour Γ in (4.2) with $\Gamma'=\Gamma_1'+\Gamma_2'+\Gamma_3'$
where Γ_1': $\zeta=-s$ $\quad\delta R(r,u_1,u_2)\leq s\leq+\infty$,
Γ_2': $\zeta=\delta R(r,u_1,u_2)e^{-i\theta}$ $\quad -\pi\leq\theta\leq\pi$,
Γ_3': $\zeta=s$ $\quad\delta R(r,u_1,u_2)\leq s\leq+\infty$.

On the other hand since $q_\zeta(x,\xi)\in S_\Lambda^{-m,-M_1,-M_2}(\Sigma_1,\Sigma_2)$, for any multi-index $(\alpha_1,\alpha_2,\beta)$ and non-negative integer p there exists a constant $C=C_{\alpha_1,\alpha_2,\beta,p}$ such that

$$\left|\left(\frac{\partial}{\partial u_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial u_2}\right)^{\alpha_2}\left(\frac{\partial}{\partial v}\right)^{\beta}\left(\frac{\partial}{\partial r}\right)^{p} q_\zeta(u_1,u_2,v,r)\right|\leq C|\zeta|^{-1}r^{-p}\rho_{\Sigma_1}^{-|\alpha_1|}\rho_{\Sigma_2}^{-|\alpha_2|}.$$

In order to estimate $\sigma(P^z)$, put for each $j=1, 2, 3$,

$$I_j=\frac{i}{2\pi}\int_{\Gamma_j}, \zeta^z\Big(\frac{\partial}{\partial u_1}\Big)^{\alpha_1}\Big(\frac{\partial}{\partial u_2}\Big)^{\alpha_2}\Big(\frac{\partial}{\partial v}\Big)^{\beta}\Big(\frac{\partial}{\partial r}\Big)^{p}q_\zeta(u_1, u_2, v, r)\ d\zeta.$$

Then we have for $j=1$ or 3,

$$\begin{aligned}|I_j| &\le C\ r^{-p}\rho_{\Sigma_1}^{-|\alpha_1|}\rho_{\Sigma_2}^{-|\alpha_2|}\int_{\delta R(r,u_1,u_2)}^{\infty} s^{\mathscr{R}_e z-1}\ ds\\ &\le C_z\ R(r, u_1, u_2)^{\mathscr{R}_e z}r^{-p}\rho_{\Sigma_1}^{-|\alpha_1|}\ \rho_{\Sigma_2}^{-|\alpha_2|}\end{aligned}$$

where C_z is a constant depending on z. For $j=2$, we have easily

$$|I_j|\le C'_z\ R(r, u_1, u_2)^{\mathscr{R}_e z}\ r^{-p}\rho_{\Sigma_1}^{-|\alpha_1|}\ \rho_{\Sigma_2}^{-|\alpha_2|}$$

where C'_z is a constant depending on z. Similarly we can estimate (4.2) also in the other cases of Σ_1 and Σ_2. Thus we have

$$\sigma(P^z)(x, \xi)\in S_0^{m\mathscr{R}_e z,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z}(\Sigma_1, \Sigma_2).$$

Moreover since $(P-\zeta)^{-1}-q_\zeta(x, D)\in OPS_\Lambda^{-\infty}$, then we see that

$$\sigma(P^z)-\frac{i}{2\pi}\int_\Gamma \zeta^z q_\zeta(x, \xi)\ d\zeta\in S_0^{-\infty}.$$

Thus we have (i) for $\mathscr{R}_e\, z<0$ and (ii). For $\mathscr{R}_e\, z\ge 0$, by Proposition 2.4 and (4.3), (i) is clear. This completes the proof.

For the symbols of P^z we have the following Propositions corresponding to Proposition 3.1, 3.2 and 3.3 respectively whose proofs are omitted. (c.f. [2]).

PROPOSITION 4.2. *Let W be a small conic neighborhood of $\rho\in\Sigma_0$ and χ a function of positively homogeneous of degree* 0 *such that* supp$\chi\subset W$. *Then we have in W*

(i) $\sigma(P^z)=\chi\tilde{p}(u_1, u_2, v, r)^z+d_z^{(11)}+d_z^{(12)}+d_z^{(13)}$
where $d_z^{(11)}\in S_0^{m\mathscr{R}_e z,\ M_1\mathscr{R}_e z+1,\ M_2\mathscr{R}_e z}$, $d_z^{(12)}\in S_0^{m\mathscr{R}_e z,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z+1}$ *and* $d_z^{(13)}\in S_0^{m\mathscr{R}_e z-1/2,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z}$.

(ii) $\sigma(P^z)=\chi[\tilde{p}_{\Sigma_2}(u_1, u_2, v, r)+r^{m-M_2/2}(|u_2|^2+r^{-1})^{M_2/2}]^z+d_z^{(21)}+d_z^{(22)}$
where $d_z^{(21)}\in S_0^{m\mathscr{R}_e z-1/2,\ M_1\mathscr{R}_e z-1,\ M_2\mathscr{R}_e z}$ *amd* $d_z^{(22)}\in S_0^{m\mathscr{R}_e z,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z+1}$.

(iii) $\sigma(P^z)=(\sum_{j=0}^{M_0} p_{m-j/2})^z+d_z^{(3)}$ *where* $d_z^{(3)}\in S_0^{m\mathscr{R}_e z-1/2,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z}$.

Next for every $i=1, 2$, we have:

PROPOSITION 4.3$_{(i)}$. *Let W be a small conic neighborhood of $\rho\in\Sigma_i\backslash\Sigma_0$*

such that $W\cap\Sigma_0=\phi$. *And also let* χ *be a function of positively homogeneous of degree* 0 *such that* $\operatorname{supp}\chi\subset W$. *Then we have in* W :

(i) $\sigma(P^z)=\chi\tilde{p}(u_i, v, r)^z+d_z^{(i1)}+d_z^{(i2)}$

where $d_z^{(i1)}\in S_0^{m\mathscr{R}_e z,\ M_i\mathscr{R}_e z+1}(W\ ;\ \Sigma_i)$ *and* $d_z^{(i2)}\in S_0^{m\mathscr{R}_e z-1/2,\ M_i\mathscr{R}_e z}$.

(ii) $\sigma(P^z)=\chi(p_m+r^{m-M_i/2})^z+d_z^{(i2)}$

where $d_z^{(i2)}\in S_0^{m\mathscr{R}_e z-1/2,\ M_i\mathscr{R}_e z-1}(W\ ;\ \Sigma_i)$.

Proposition 4.4. *Let* W *be an open cone such that* $W\cap\Sigma=\phi$ *and* χ *be a function of positively homogeneous of degree* 0 *such that* $\operatorname{supp}\chi\subset W$. *Then we have in* W,

$$\sigma(P^z)=\chi p_m^z+d_z$$

where $d_z\in S_0^{m\mathscr{R}_e z-1/2}(W)$.

§ 5. The first singularity of Trace$(\mathbf{P}^z)$

In this section we consider the first singularity of Trace(P^z) and determine the order of the pole and the coefficient at the point. Let $p_z(x, \xi)$ be the symbol of P^z. It is well known that if

$$\int_{R^n\times R^n}|p_z(x, \xi)|\,dx\ d\xi\le C_z$$

for some constant C_z, then P^z is an operator of trace class and the trace is given by:

$$\operatorname{Tr}(P^z)=(2\pi)^{-n}\int_{R^n\times R^n}p_z(x, \xi)\,dx\ d\xi.$$

Since

$$\int_{r\le 1}p_z(x, \xi)\,dx\ d\xi$$

is entire, we may consider:

$$I(z)=(2\pi)^{-n}\int_{r\ge 1}p_z(x, \xi)\,dx\ d\xi.$$

Proposition 5.1. *Let* $p_z\in S_0^{m\mathscr{R}_e z-j,\ M_1\mathscr{R}_e z-k_1,\ M_2\mathscr{R}_e z-k_2}(\Sigma_1, \Sigma_2)$ *and* W *be an open cone and* χ *a* C^∞ *function of positively homogeneous of degree* 0 *such that* $\operatorname{supp}\chi\subset W$. *Put*

$$I_\chi(z)=\int_{r\ge 1}\chi(x, \xi)p_z(x, \xi)\,dx\ d\xi.$$

(I) *The case*: W *is a small conic neighborhood of* $\rho\in\Sigma_0$. *Then* $I_\chi(z)$ *is* *holomorphic in* $\{z\,;\,\mathscr{R}_e\, z<a\}$ *if* a *satisfies any one of the followings.*

(I.1) $a<-\dfrac{d_i-k_i}{M_i}(i=1,2)$ *and* $a<-\dfrac{N(2n-j,\ d_0-k_1-k_2)}{N(m,\ M_0)}$,

(I.2) $-\dfrac{d_1-k_1}{M_1}\leq a<-\dfrac{d_2-k_2}{M_2}$ *and* $a<-\dfrac{N(2n-j,\ d_2-k_2)}{N(m,\ M_2)}$,

(I.3) $-\dfrac{d_2-k_2}{M_2}\leq a<-\dfrac{d_1-k_1}{M_1}$ *and* $a<-\dfrac{N(2n-j,\ d_1-k_1)}{N(m,\ M_1)}$,

(I.4) $-\dfrac{d_i-k_i}{M_i}\leq a\ (i=1,2)$ *and* $a<-\dfrac{2n-j}{m}$.

$(\mathrm{II})_{(i)}$ *The case*: W is a small conic neighborhood of $\rho\in\Sigma_i\setminus\Sigma_0$ $(\mathrm{i}=1,2)$ *such that* $W\cap\Sigma_0=\phi$. *Then* $I_\chi(z)$ *is holomorphic in* $\{z\,;\,\mathscr{R}_e\, z<a\}$ *if* a *satisfies any one of the followings.*

(II.1.i) $a<-\dfrac{d_i-k_i}{M_i}$ *and* $a<-\dfrac{N(2n-j,\ d_i-k_i)}{N(m,\ M_i)}$,

(II.2.i) $-\dfrac{d_i-k_i}{M_i}\leq a$ *and* $a<-\dfrac{2n-j}{m}$.

(III) *The case*: W *is outside of* Σ. *Then* $I_\chi(z)$ *is holomorphic in* $\{z\,;\,\mathscr{R}_e\, z<a\}$ *if* $a<-\dfrac{2n-j}{m}$.

Proof. (I) We choose a local coordinate system $w=(u_1,\ u_2,\ v,\ r)$ as in §2. We may assume that $W\subset\{w=(u_1,\ u_2,\ v,\ r)\,;\ |u_i|\leq 1,\ i=1,\ 2\}$. Let K be an arbitrary compact set in $\{z\,;\,\mathscr{R}_e\, z<a\}$. Then by Theorem 4. 1, there exists a constant C which is independent of $z\in K$ such that

$$|p_z(x,\ \xi)|\leq C\ R(r,\ u_1,\ u_2)^a r^{-j}(|u_1|^2+r^{-1})^{-k_1/2}(|u_2|^2+r^{-1})^{-k_2/2}.$$

Note that $dx\, d\xi=J(u_1,\ u_2,\ v,\ r)\ du_1\ du_2\ dv\ dr$ where $J(u_1,\ u_2,\ v,\ r)=|\det\dfrac{D(u_1,\ u_2,\ v,\ r)}{D(x,\ \xi)}|^{-1}$ is positively homogeneous of degree $2n-1$. Thus if $\mathscr{R}_e\, z<a$, we have for some constants $C,\ C'$ and T,

$$\begin{aligned}(5.1)\qquad &\int_{r\geq 1}|\chi(x,\ \xi)\ p_z(x,\ \xi)|\ dx\ d\xi\\ &\leq C\int_1^\infty\int_{|v|\leq T,|u_i|\leq 1}R(r,\ u_1,\ u_2)^a\ r^{-j+2n-1}(|u_1|^2+r^{-1})^{-k_1/2}\times\\ &\quad(|u_2|^2+r^{-1})^{-k_2/2}\ du_1du_2dvdr\\ &\leq C'\int_1^\infty r^{N(m,\ M_0)a+N(2n,\ d_0)-1-j+(k_1+k_2)/2}\ dr\ \prod_{i=1}^2\int_0^{r^{1/2}}(t_i^2+1)^{(M_ia-k_i)/2}\ t_i^{d_i-1}\ dt_i.\end{aligned}$$

Here we have that if $M_ia-k_i+d_i<0$,

$$\int_0^{r^{1/2}} (t_i^2+1)^{(M_ia-k_i)/2}\, t_i^{d_i-1}\, dt_i \leq \int_0^{\infty} (t_i^2+1)^{(M_ia-k_i)/2}\, t_i^{d_i-1}\, dt_i < \infty$$

and if $M_ia-k_i+d_i \geq 0$,

$$\int_0^{r^{1/2}} (t_i^2+1)^{(M_ia-k_i)/2}\, t_i^{d_i-1}\, dt_i = O(r^{(M_ia+d_i-k_i)/2} \log r) \text{ as } r \to \infty.$$

Thus (I) holds. Also (II) and (III) follows from the same arguments, so we omit them.

Now we have results on the first singularity of $\mathrm{Tr}(P^z)$ for each case.

PROPOSITION 5. 2. *When* $\frac{d_1}{M_1} \geq \frac{d_2}{M_2} > \frac{2n}{m}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\, \mathscr{R}_e\, z < -\frac{2n}{m}\}$ *and has a simple pole at* $z=-\frac{2n}{m}$ *as the first singularity with the residue* $\mathrm{Res}(-\frac{2n}{m}) = \frac{2\mathrm{n}\ \mathrm{A}_1}{m}$ *where*

$$(5.2) \qquad A_1 = (2\pi)^{-n} \int_{p_m(x,\xi) \leq 1} dx\, d\xi.$$

PROOF. That $\mathrm{Tr}(P^z)$ is holomorphic in $\{z;\, \mathscr{R}_e\, z < -\frac{2n}{m}\}$ follows from Proposition 5. 1 with $j=k_1=k_2=0$. In this case we use Proposition 4. 2(iii), 4. 3(ii), 4. 4 and slso 5. 1. Then we can write $\mathrm{Tr}(P^z) = I_0(z) + I_1(z)$ where

$$I_0(z) = (2\pi)^{-n} \int_{r \geq 1} (p_m + r^{m - \mathrm{Min}(M_1,\, M_2)/2})^z\, dx\, d\xi$$

and $I_1(z)$ is holomorhic in $\{z\,;\, \mathscr{R}_e\, z \leq -\frac{2n}{m}\}$. Here by using the mean value theorem, for any $a<0$ and any ε, $0<\varepsilon<1$, there exists a constant C such that

$$\begin{aligned}
&\Big| \int_{r \geq 1} \{(p_m + r^{m-\mathrm{Min}(M_1,\, M_2)/2})^a - (p_m+1)^a\}\, dx\, d\xi \Big| \\
&= \Big| \int_{r \geq 1} [a(r^{m-\mathrm{Min}(M_1,\, M_2)/2} - 1) \times \\
&\int_0^1 \{p_m + 1 + \theta(r^{m-\mathrm{Min}(M_1,\, M_2)/2} - 1)\}^{a-1}\, d\theta]\, dx\, d\xi \Big| \\
&\leq C \int_1^{\infty} r^{ma+2n-1-\varepsilon \mathrm{Min}(M_1,\, M_2)/2}\, dr \prod_{i=1}^{2} \int_0^1 t_i^{M_ia - M_i\varepsilon + d_i - 1}\, dt_i.
\end{aligned}$$

Thus if we choose a such that $a > -\frac{2n}{m}$, we see that the integral is convergent. So we are reduced to (c. f. [2]) :

$$\int (p_m+1)^z\, dx\, d\xi = \frac{2n}{m}\sigma(1)\frac{\Gamma(2n/m)\Gamma(-(z+2n/m))}{\Gamma(-z)}$$

where $\sigma(\lambda)=(2\pi)^{-n}\int_{p_m(x,\xi)\leq\lambda} dx\, d\xi$.

Therefore by the properties of Γ-function, we reach the conclusion.

PROPOSITION 5.3. *When* $\frac{N(2n, d_2)}{N(m, M_2)} > \frac{2n}{m}, \frac{d_1}{M_1}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\,\mathscr{R}_e\, z < -\frac{N(2n, d_0)}{N(m, M_0)}\}$ *and has a simple pole at* $z=-\frac{N(2n, d_0)}{N(m, M_0)}$ *as the first singularity with the residue* $\mathrm{Res}(-\frac{N(2n, d_0)}{N(m, M_0)})=\frac{A_2}{N(m, M_0)}$ *where*

$$A_2=(2\pi)^{-n}\int_{(\Sigma_0\cap S^*\boldsymbol{R}^{2n})\times \boldsymbol{R}^{d_1}\times \boldsymbol{R}^{d_2}} J(0, 0, v, 1)\times \tilde{p}(u_1, u_2, v, 1)^{-N(2n, d_0)/N(m, M_0)}\, du_1\, du_2\, dv. \tag{5.3}$$

PROOF. We have $\frac{N(2n, d_0)}{N(m, M_0)} > \frac{N(2n, d_i)}{N(m, M_i)}$ $(i=1, 2)$ in this case. By Proposition 4. 2(i), 4. 3(i), 4. 4 and 5. 1, we may consider with W and χ as in Proposition 5. 1(I),

$$\int_{r\geq 1} h(u_1, u_2, v, r)\tilde{p}(u_1, u_2, v, r)^z\, du_1\, du_2\, dv\, dr$$

where $h(u_1, u_2, v, r)=\chi(u_1, u_2, v, r)J(u_1, u_2, v, r)$. Since we have $\{h(u_1, u_2, v, r)-h(0, 0, v, r)\}\tilde{p}(u_1, u_2, v, r)^z=r_z'+r_z''$ where $r_z'\in S_0^{m\mathscr{R}_e z,\, M_1\mathscr{R}_e z+1,\, M_2\mathscr{R}_e z}$ and $r_z''\in S_0^{m\mathscr{R}_e z,\, M_1\mathscr{R}_e z,\, M_2\mathscr{R}_e z+1}$, again by Proposition 5. 1 we are reduced to the integral $I(z)=$

$$(2\pi)^{-n}\int_{(\Sigma_0\cap\{r\geq 1\})\times \boldsymbol{R}^{d_1}\times \boldsymbol{R}^{d_2}} h(0, 0, v, r)\tilde{p}(u_1, u_2, v, r)^z\, du_1\, du_2\, dv\, dr.$$

By quasi-homogeneity of $\tilde{p}$ and the change of variable: $u_i\to r^{-1/2}u_i$ $(i=1, 2)$, we see that

$$I(z)=(2\pi)^{-n}\int_1^\infty r^{N(m, M_0)z+N(2n, d_0)-1}\, dr\; I_1(z)$$

where

$$I_1(z)=\int_{(\Sigma_0\cap S^*R^{2n})\times \boldsymbol{R}^{d_1}\times \boldsymbol{R}^{d_2}} h(0, 0, v, 1)\tilde{p}(u_1, u_2, v, 1)^z du_1 du_2\, dv.$$

Since it is clear that $I_1(z)$ is holomorphic in $\{z\,;\,\mathscr{R}_e\, z\leq -\frac{N(2n, d_0)}{N(m, M_0)}\}$, we reach the conclusion.

PROPOSITION 5.4. *When* $\frac{d_1}{M_1}>\frac{N(2n, d_2)}{N(m, M_2)}>\frac{2n}{m}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\ \mathscr{R}_e\, z<-\frac{N(2n, d_2)}{N(m, M_2)}\}$ *and has a simple pole at* $z=-\frac{N(2n, d_2)}{N(m, M_2)}$ *as the first singularity with the residue* $\mathrm{Res}(-\frac{N(2n, d_2)}{N(m, M_2)})=-\frac{A_3}{N(m, N_2)}$ *where*

$$(5.4)\quad A_3=(2\pi)^{-n}\int_{(\Sigma_2\cap S^*\boldsymbol{R}^{2n})\times \boldsymbol{R}^{d_2}}(\tilde{p}(u_2, v, 1)+1)^{-N(2n, d_2)/N(m, M_2)}J(0, v, 1)\,du_2dv$$

PROOF That $\mathrm{Tr}(P^z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\, z<-\frac{N(2n, d_2)}{N(m, M_2)}\}$ follows from Proposition 5. 1. By Proposition 4. 2(ii), 4. $3_{(2)}$(i), 4. $3_{(1)}$(ii), 4. 4 and 5. 1, we may consider the integral of $p_z(x, \xi)$ near Σ_2. First let W and χ be as in Proposition 5. 1(II)$_{(2)}$. Then by the same way as the proof of Proposition 5. 3, we have modulo holomorphic functions for $\mathscr{R}_e\, z\le -\frac{N(2n, d_2)}{N(m, M_2)}$,

$$I_\chi(z)\equiv$$
$$(2\pi)^{-n}\int_{r\ge 1}h(0, v, r)\{\tilde{p}(u_2, v, r)+r^{m-M_1/2}(|u_2|^2+r^{-1})^{M_2/2}\}^z\,du_2\,dv\,dr.$$

Secondly let W and χ be as in Proposition 5. 1(I). Then we have

$$\tilde{p}(u_1, u_2, v, r)^z-\{\tilde{p}_{\Sigma_2}(u_2, u_1, v, r)+r^{m-M_1/2}(|u_2|^2+r^{-1})^{M_2/2})\}^z$$
$$=r_z^1+r_z^2$$

where $r_z^1\in S_0^{m\mathscr{R}_e z-1/2,\ M_1\mathscr{R}_e z-1,\ M_2\mathscr{R}_e z}$ and $r_z^2\in S_0^{m\mathscr{R}_e z,\ M_1\mathscr{R}_e z,\ M_2\mathscr{R}_e z+1}$. So we have

$$I_\chi(z)\equiv(2\pi)^{-n}\int_{r\ge 1}h(u_1, 0, v, r)\times$$
$$\{\tilde{p}_{\Sigma_2}(u_2, u_1, v, r)+r^{m-M_1/2}(|u_2|^2+r^{-1})^{M_2/2}\}^z\,du_1du_2dvdr.$$

By the quasi-homogeneity of $\tilde{p}(u_2, v, r)$ and $\tilde{p}_{\Sigma_2}(u_2, u_1, v, r)$ and the change of variable $u_2\to r^{-1/2}u_2$, we reach the conclusion.

PROPOSITION 5.5. *When* $\frac{d_1}{M_1}=\frac{d_2}{M_2}=\frac{2n}{m}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\ \mathscr{R}_e\, z<-\frac{2n}{m}\}$ *and has a triple pole at* $z=-\frac{2n}{m}$ *as the first singularity with the coefficient of* $(z+\frac{2n}{m})^{-3}$ *equal to* $-\frac{N(2m, M_0)A_4}{4mN(m, M_1)N(m, M_2)N(m, M_0)}$

where

$$(5.5)\qquad A_4=(2\pi)^{-n}\int_{(\Sigma_0\cap S^*\boldsymbol{R}^{2n})\times S^*\boldsymbol{R}^{d_1}\times S^*\boldsymbol{R}^{d_2}}\tilde{p}_m(\omega_1,\omega_2,v,1)^{-2n/m}\times J(0,0,v,1)\ d\omega_1 d\omega_2 dv.$$

PROOF. In this proposition if a function $f(z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\ z<-\frac{2n}{m}\}$ and has at most a double pole at $z=-\frac{2n}{m}$ as the first singularity, we say that the function is negligible and write $f(z)\equiv 0$.

That $\mathrm{Tr}(P^z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\ z<-\frac{2n}{m}\}$ follows from Proposition 5. 1. Let W and χ be as in Proposition 5. 1(I). By Proposition 4. 2 (i), 4. $3_{(j)}$(i) and 4. 4, we may consider

$$J(z)=(2\pi)^{-n}\int_{r\geq 1}h(u_1,u_2,v,r)\{\tilde{p}(u_1,u_2,v,r)^z+d_z^{(1)}+d_z^{(2)}\}\ du_1\,du_2\,dv\,dr$$

where $d_z^{(1)}=$

$$=\{\sum_{\substack{|\alpha_1|\leq M_1\\|\alpha_2|\leq M_2}}a_{\alpha_1,\alpha_2}(u_1,u_2,v,r)u_1^{\alpha_1}u_2^{\alpha_2}\}^z-\{\sum_{\substack{|\alpha_1|\leq M_1\\|\alpha_2|\leq M_2}}a_{\alpha_1,\alpha_2}(0,u_2,v,r)\ u_1^{\alpha_1}u_2^{\alpha_2}\}^z$$

and $d_z^{(2)}=\{\sum_{\substack{|\alpha_1|\leq M_1\\|\alpha_2|\leq M_2}}a_{\alpha_1,\alpha_2}(0,u_2,v,r)u_1^{\alpha_1}u_2^{\alpha_2}\}^z-\{\sum_{\substack{|\alpha_1|\leq M_1\\|\alpha_2|\leq M_2}}a_{\alpha_1,\alpha_2}(0,0,v,r)u_1^{\alpha_1}u_2^{\alpha_2}\}^z.$

Here we may assume that $\mathrm{supp}h\subset\{(u_1,u_2,v,r)\,;\ |u_i|\leq 1,\ i=1,2\}$. Moreover we shall prove:

$$(5.6)\qquad J(z)\equiv J_0(z)\ \text{where}\ J_0(z)=$$
$$=(2\pi)^{-n}\int_{r\geq 1,\ r^{-1/2}\leq|u_i|\leq 1}h(0,0,v,r)\tilde{p}(u_1,u_2,v,r)^z\ du_1\,du_2\,dv\,dr.$$

In order to prove (5. 6) we need the following lemmas.

LEMMA 5. 6. *If we put* $J_1(z)=$

$$=\int_{|u_i|\leq r^{-1/2}}h(u_1,u_2,v,r)\tilde{p}(u_1,u_2,v,r)^z\ du_1\,du_2\,dv\,dr,$$

then $J_1(z)\equiv 0$.

PROOF. By the preceding arguments, we have for $\mathscr{R}_e\ z<-\dfrac{N(2n,d_0)}{N(m,M_0)}$,

$$J_1(z)=\int_1^\infty r^{N(m,M_0)z+N(2n,d_0)-1}\,dr\times$$
$$\times\int_{|u_i|\leq 1}h(r^{-1/2}u_1,r^{-1/2}u_2,v,1)\tilde{p}(u_1,u_2,v,1)^z\ du_1\,du_2\,dv$$

$$= -\frac{1}{N(m, M_0)z + N(2n, d_0)} \int_{|u_i| \leq 1} h(u_1, u_2, v, 1) \times$$

$$\tilde{p}(u_1, u_2, v, 1)^z \, du_1 \, du_2 \, dv - \int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 3/2} \, dr \times$$

$$\int_{|u_i| \leq 1} \sum_{i=1}^{2} u_i \, \tilde{h}_i(r^{-1/2}u_1, r^{-1/2}u_2, v, 1) \tilde{p}(u_1, u_2, v, 1)^z \, du_1 \, du_2 \, dv.$$

Thus we see that $J_1(z) \equiv 0$ and this completes the proof.

LEMMA 5.7. *If we put* $J_2(z) =$

$$\int_{\substack{|u_1| \leq r^{-1/2} \\ r^{-1/2} \leq |u_2| \leq 1}} h(u_1, u_2, v, r) \tilde{p}(u_1 u_2, v, r)^z \, du_1 \, dv \, dr,$$

then $J_2(z) \equiv 0$.

PROOF. 1^{st}-*step*: If we put $J_3(z) =$

$$\int_{\substack{|u_1| \leq r^{-1/2} \\ r^{-1/2} \leq |u_2| \leq 1}} \{h(u_1, u_2, v, r) - h(u_1, 0, v, r)\} \tilde{p}(u_1, u_2, v, r)^z \, du_1 \, du_2 \, dv \, dr,$$

we can prove $J_3(z) \equiv 0$. In fact, if we put $h(u_1, u_2, v, r) - h(u_1, 0, v, r) = u_2 \cdot \tilde{h}(u_1, u_2, v, r)$, we have

$$J_3(z) = \int_1^\infty r^{N(m, M_1)z + N(2n, d_1) - 1} J_4(r, z) \, dr.$$

Here $J_4(r, z) =$

$$\int_{\substack{|u_1| \leq 1 \\ r^{-1/2} \leq |u_2| \leq 1}} u_2 \cdot \tilde{h}(r^{-1/2}u_1, u_2, v, r) \{\sum_{i=1}^{2} \hat{p}_i(u_1, u_2, v, r)\}^z \, du_1 \, du_2 \, dv \, dr \cdot$$

where $\hat{p}_1(u_1, u_2, v, r) = \sum_{\substack{|\alpha_2| = M_2 \\ |\alpha_1| \leq M_1}} a_{\alpha_1, \alpha_2}(0, 0, v, r) \; u_1^{\alpha_1} \; u_2^{\alpha_2}$ and

$$\hat{p}_2(u_1, u_2, v, r) = \sum_{\substack{|\alpha_2| < M_2 \\ |\alpha_1| \leq M_1}} r^{(|\alpha_2| - M_2)/2} \, a_{\alpha_1, \alpha_2}(0, 0, v, 1) \; u_1^{\alpha_1} \; u_2^{\alpha_2}.$$

Moreover we can write

$$J_4(r, z) = \int_{r^{-1/2}}^{1} t^{M_2 z + d_2} J_5(t, r, z) \, dt$$

where $J_5(t, r, z) =$

$$\int \omega_2 \cdot h(r^{-1/2}u_1, t\omega_2, v, 1)[\hat{p}_1(u_1, \omega_2, v, 1) + \hat{p}_2(u_1, \omega_2, v, t^2 r)]^z \, du_1 \, d\omega_2 \, dv.$$

Thus by the integration by parts, we have $J_4(r, z) = \frac{1}{M_2 z + d_2 + 1} \times$ $[J_5(1, r, z) - r^{-(M_2 z + d_2 + 1)/2} J_5(r^{-1/2}, r, z) - \int_{r^{-1/2}}^{1} t^{M_2 z + d_2 + 1} \frac{\partial}{\partial t} J_5(t, r, z) dt]$.

Here we have

$$\frac{\partial}{\partial t} J_5(t, r, z) = \int \{\tilde{h}_1(r^{-1/2}u_1, t\omega_2, v, 1)[\hat{p}_1(u_1, \omega_2, v, 1) + \hat{p}_2(u_1, \omega_2, v, t^2 r)]^z + z\omega_2 \cdot \tilde{h}_2(r^{-1/2}u_1, t\omega_2, v, 1)[\hat{p}_1(u_1, \omega_2, v, 1) + \hat{p}_2(u_1, \omega_2, v, t^2 r)]^{z-1} \times r^{(|\alpha_2| - M_2)/2} t^{|\alpha_2| - M_2 - 1}\} du_1 \, d\omega_2 \, dv$$

where $\tilde{h}_1$ and $\tilde{h}_2$ are bounded functions. Thus we have

$$J_3(z) = \frac{-1}{N(m, M_1)z + N(2n, d_1)} \int_1^\infty r^{N(m, M_1)z + N(2n, d_1)} \frac{\partial}{\partial r} J_4(r, z) \, dr.$$

Here we note

$$\frac{\partial}{\partial r} J_5(1, r, z) = O(r^{-3/2}), \ \frac{\partial}{\partial r}[r^{-(M_2 z + d_2 + 1)/2} J_5(r^{-1/2}, r, z)] = O(r^{-(M_2 z + d_2 + 3)/2}) \text{ and}$$
$$\frac{\partial}{\partial r}[\int_{r^{-1/2}}^{1} t^{M_2 z + d_2 + 1} \frac{\partial}{\partial t} J_5(t, r, z) \, dt] = O(r^{-3/2})$$

as $r \to \infty$ uniformly on $\{z ; \mathcal{R}_e z \leq -\frac{2n}{m} + \varepsilon\}$ for any $\varepsilon > 0$. Therefore we see that $J_3(z)$ is negligible.

2^{nd}*-step* : If we put $J_6(z) =$

$$\int_{\substack{|u_1| \leq r^{-1/2} \\ r^{-1/2} \leq |u_2| \leq 1}} h(u_1, 0, v, r) \ \tilde{p}(u_1, u_2, v, r)^z \, du_1 \, du_2 \, dv \, dr,$$

we can prove $J_6(z) \equiv 0$. In fact, we have $J_6(z) =$

$$\int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1} dr \int_{\substack{|u_1| \leq 1 \\ 1 \leq |u_2| \leq r^{1/2}}} h(r^{-1/2}u_1, 0, v, 1) \tilde{p}(u_1, u_2, v, 1)^z \, du_1 du_2 dv.$$

Here if we write $\tilde{p}(u_1, u_2, v, 1)^z = \hat{p}_1(u_1, u_2, v, 1)^z + r_z(u_1, u_2, v)$, we have $|r_z(u_1, u_2, v)| \leq C \ |u_2|^{M_2 \mathcal{R}_e z - 1}$. Therefore we have $J_6(z)$

$$\equiv \int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1} dr \times \int_{\substack{|u_1| \leq 1 \\ 1 \leq |u_2| \leq r^{1/2}}} h(r^{-1/2}u_1, 0, v, 1) \hat{p}_1(u_1, u_2, v, r)^z du_1 du_2 dv dr$$

$$= \int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1} dr \int_1^{r^{1/2}} t^{M_2 z + d_2 - 1} dt$$

$$\times \int_{|u_1|\le 1,\, |\omega_2|=1} h(r^{-1/2}u_1, 0, v, 1)\ \hat{p}_1(u_1, \omega_2, v, r)^z\, du_1\, d\omega_2\, dv$$
$$=\frac{1}{M_2 z+d_2}\int_1^\infty r^{N(m, M_0)z+N(2n, d_0)-1}\ (r^{(M_2z+d_2)/2}-1)$$
$$\times \int_{|u_1|\le 1,\, |\omega_2|=1} h(r^{-1/2}u_1, 0, v, 1)\ \hat{p}_1(u_1, \omega_2, v, 1)^z\, du_1\, d\omega_2\, dv.$$

By the integration by parts with respect to r, we see that $J_6(z)\equiv 0$. This completes the proof.

Similarly we see that

$$\int_{\substack{r^{-1/2}\le|u_1|\le 1\\ |u_2|\le r^{-1/2}}} h(u_1, u_2, v, r)\ \tilde{p}(u_1, u_2, v, r)^z\, du_1\, du_2\, dv\, dr \equiv 0.$$

Thus we are reduced to study $J_7(z)$ where

$$J_7(z)=\int_{r^{-1/2}\le|u_i|\le 1} h(u_1, u_2, v, r)\ \tilde{p}(u_1, u_2, v, r)^z\, du_1\, du_2\, dv\, dr.$$

However we have

LEMMA 5.8. *If we put $J_7(z)$ as above, we have $J_7(z)\equiv J_0(z)$.*

PROOF. We put $h(u_1, u_2, v, r)-h(0, 0, v, r)=u_1\cdot h_1(u_1, u_2, v, r)+u_2\cdot h_2(u_1, u_2, v, r)$. Then by the same way as the proof of Lemma 5.7 (2^{nd}-step), the proof is clear.

Finally we must prove

LEMMA 5.9. *If we put*

$$K_i(z)=\int_{r\ge 1} d_z^{(i)}(u_1, u_2, v, r)\ h(u_1, u_2, v, r)\ du_1\, du_2\, dv\, dr,$$

then we have $K_1(z)+K_2(z)\equiv 0$.

PROOF. By Proposition 3. 1 and the construction of parametrices (c. f. [2; § 4]), we have $K_1(z)+K_2(z)=$

$$\int_{r\ge 1} h(u_1, u_2, v, r)[\{\sum_{j=0}^{M_0}\tilde{p}_{m-j/2}\}^z-\{\sum_{j=0}^{M_0}p_{m-j/2}\}^z]\ du_1\, du_2\, dv\, dr.$$

Here by the mean value theorem, we have $K_1(z)+K_2(z)=$

$$\int_{r\ge 1} h(u_1, u_2, v, r)\ z\{\sum_{j=0}^{M_0}(p_{m-j/2}-\tilde{p}_{m-j/2})\}$$
$$\times \int_0^1 [\sum_{j=0}^{M_0}\tilde{p}_{m-j/2}+\theta\{\sum_{j=0}^{M_0}(p_{m-j/2}-\tilde{p}_{m-j/2})\}^{z-1}]\ d\theta\, du_1\, du_2\, dv\, dr.$$

As the same way as the proof of Lemma 5.7 (2^{nd}-step), we see that $K_1(z)+$

$K_2(z)$ is negligible. This completes the proof.

End of the proof of Proposition 5. 5.

By (5. 6), we may consider $J_0(z)$. If we write

$\tilde{p}(u_1, u_2, v, 1)^z = \tilde{p}_m(u_1, u_2, v, 1)^z + r_z(u_1, u_2, v)$ for $1 \leq |u_i| \leq r^{1/2}$,

we have

$$|r_z(u_1, u_2, v)| \leq C|u_1|^{M_1 \mathscr{R}_e z - 1}|u_2|^{M_2 \mathscr{R}_e z - 1}(|u_1| + |u_2|).$$

So we can see that the integral corresponding to r_z is negligible. Therefore we have $J_0(z) \equiv (2\pi)^{-n} \times$

$$\int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1}\, dr \int_{1 \leq |u_i| \leq r^{1/2}} h(0, 0, v, 1) \tilde{p}_m(u_1, u_2, v, 1)^z\, du_1\, du_2\, dv$$

$$= A_4'(z) \int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1}\, dr \prod_{i=1}^2 \int_1^{r^{1/2}} t_i^{M_i z + d_i - 1}\, dt_i$$

$$= A_4'(z) \int_1^\infty r^{N(m, M_0)z + N(2n, d_0) - 1}\, dr \prod_{i=1}^2 \frac{(r^{(M_i z + d_i)/2} - 1)}{M_i z + d_i}$$

where $A_4'(z)$ is defined by

$$(2\pi)^{-n} \int_{(\Sigma_0 \cap S^* \boldsymbol{R}^{2n}) \times S^* \boldsymbol{R}^{d_1} \times S^* \boldsymbol{R}^{d_2}} h(0, 0, v, 1)\ \tilde{p}_m(\omega_1, \omega_2, v, 1)^z\, d\omega_1\, d\omega_2\, dv.$$

and $A_4(z)$ is an entire function. By using an appropriate partition of unity, we reach the conclusion of Proposition 5. 5.

PROPOSITION 5. 10. *When* $\dfrac{d_1}{M_1} = \dfrac{N(2n, d_2)}{N(m, M_2)} > \dfrac{2n}{m}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\ \mathscr{R}_e\, z < -\dfrac{N(2n, d_2)}{N(m, M_2)}\}$ *and has a double pole at* $z = -\dfrac{N(2n, d_2)}{N(m, M_2)}$ *as the first singularity with the coefficient of* $(z + \dfrac{N(2n, d_2)}{N(m, M_2)})^{-2}$ *equal to* $\dfrac{A_5}{2(M_1 d_2 - M_2 d_1)\ N(m, M_2)\ N(m, M_0)}$ *where*

(5.7) $\quad A_5 = (2\pi)^{-n} \times$

$$\int_{(\Sigma_0 \cap S^* \boldsymbol{R}^{2n}) \times S^* \boldsymbol{R}^{d_1} \times S^* \boldsymbol{R}^{d_2}} \tilde{p}_m(\omega_1, \omega_2, v, 1)^{-N(2n, d_2)/N(m, M_2)} J(0, 0, v, 1)\, d\omega_1\, d\omega_2\, dv.$$

PROOF. In this proposition if a function $f(z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\, z < -\dfrac{N(2n, d_2)}{N(m, M_2)}\}$ and has at most a simple pole at $z = -\dfrac{N(2n, d_2)}{N(m, M_2)}$ as the first singularity, we say that $f(z)$ is negligible and write $f(z) \equiv 0$.

That $\mathrm{Tr}(P^z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\ z<-\frac{N(2n,d_2)}{N(m,M_2)}\}$ follows from Proposition 5.1. In $\Sigma_2\backslash\Sigma_1$, by using $\tilde{p}^z_{\Sigma_2}$, we see that the corresponding integral is negligible. Also outside Σ_2, by using p^z_m, we see that the corresponding integral is negligible. Near Σ_0 by the same way as the proof of Proposition 5.5, we see that if we define an entire function

$$A_5'(z)=\int_{(\Sigma_0\cap S^*\boldsymbol{R}^{2n})\times S^*\boldsymbol{R}^{d_1}\times S^*\boldsymbol{R}^{d_2}} h(0,0,v,1)\ \tilde{p}_m(\omega_1,\omega_2,v,1)^z\, d\omega_1\, d\omega_2\, dv,$$

then we have $I(z)=$

$$A_5'(z)\int_1^\infty r^{N(m,M_0)z+N(2n,d_0)-1}\,dr\ \prod_{i=1}^{2}\int_0^{r^{1/2}} t_i^{M_iz+d_i-1}\,dt_i$$
$$\equiv\frac{-A_5'(z)}{(M_1z+d_1)(M_2z+d_2)}\int_1^\infty r^{N(m,M_0)z+N(2n,d_0)-1}\ (r^{(M_1z+d_1)/2}-1)\ dr$$

modulo negligible terms. This completes the proof.

PROPOSITION 5.11. *When* $\frac{d_1}{M_1}>\frac{d_2}{M_2}=\frac{2n}{m}$, $\mathrm{Tr}(P^z)$ *is holomorphic in* $\{z\,;\ \mathscr{R}_e\ z<-\frac{2n}{m}\}$ *and has a double pole at* $z=-\frac{2n}{m}$ *as the first singularity with the coefficient of* $(z+\frac{2n}{m})^{-2}$ *equal to* $\frac{A_6}{2m\ N(m,M_2)}$ *where*

$$(5.8)\qquad A_6=(2\pi)^{-n}\int_{(\Sigma_2\cap S^*\boldsymbol{R}^{2n})\times S^*\boldsymbol{R}^{d_2}}(\tilde{p}_{\Sigma_2,m}(\omega_2,v,1)+1)^{-2n/m}J(0,v,1)\ d\omega_2\ dv$$

where $\tilde{p}_{\Sigma_2,m}(u_2,v,r)=\sum_{|\alpha_2|=M_2} a_{\alpha_2}(0,v,r)\ u_2^{\alpha_2}$.

PROOF. In this proposition if a function $f(z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\ z<-\frac{2n}{m}\}$ and has at most a simple pole at $z=-\frac{2n}{m}$ as the first singularity, we say that $f(z)$ is negligible and write $f(z)\equiv 0$. That $\mathrm{Tr}(P^z)$ is holomorphic in $\{z\,;\ \mathscr{R}_e\ z<-\frac{2n}{m}\}$ follows from Proposition 5.1. Outside Σ_2, by using the symbol $(p_m+r^{m-\mathrm{Min}(M_1,M_2)/2})^z$, we see that the corresponding integral is negligible. Thus we may consider $I(z)=$

$$\int_{r\geq 1,\,|u_2|\leq 1} h(u_2,v,r)\{\tilde{p}_{\Sigma_2}(u_2,v,r)+r^{m-M_1/2}(|u_2|^2+r^{-1})^{M_2/2}\}^z\ du_2\ dv\ dr.$$

However by the way as the preceding arguments we have $I(z)=$

$$\int_1^\infty r^{N(m,M_2)z+N(2n,d_2)-1}dr\times$$

$$\int_{1\leq|u_2|\leq r^{1/2}} h(0, v, 1)\{\sum_{|\alpha_2|=M_2} a_{\alpha_2}(0, v, 1)\ u_2^{\alpha_2}\}^z\, du_2\, dv$$
$$=A_6'(z)\int_1^{\infty} r^{N(m, M_2)z+N(2n, d_2)-1}\, dr \int_1^{r^{1/2}} t^{M_2z+d_2-1}\, dt$$

where $A_6'(z)=$

$$\int_{(\Sigma_2\cap S^*\boldsymbol{R}^{2n})\times S^*\boldsymbol{R}^{d_2}} h(0, v, 1)\{\sum_{|\alpha_2|=M_2} a_{\alpha_2}(0, v, 1)\omega_2^{\alpha_2}\}^z\, d\omega_2\, dv.$$

Thus we have

$$I(z)\equiv\frac{A_6'(z)}{M_2z+d_2}\int_1^{\infty} r^{N(m, M_2)z+N(2n, d_2)-1}\ (r^{(M_2z+d_2)/2}-1)\ dr.$$

This completes the proof.

§ 6. The asymptotic behavior of eigenvalues of P

Let $P(x, D)\in OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$. In this section we assume that $P(x, D)$ satisfies (1. 3), (1. 4) and (H. 1)~(H. 6). As in § 4, define an unbounded self-adjoint operator P in $L^2(\boldsymbol{R}^n)$. Then P has the spectrum consist only of eigenvalues of finite multiplicity. By (H. 6), we can write the sequence of eigenvalues: $0<\lambda_1\leq\lambda_2\ldots$, $\lim_{k\to\infty}\lambda_k=+\infty$ with repetition according to multiplicity. Let $N(\lambda)$ be the counting function, i. e., $N(\lambda)=\sum_{\lambda_k\leq\lambda} 1$. Then we have

THEOREM 6. 1. *Let $P(x, D)\in OPL^{m, M_1, M_2}(\Sigma_1, \Sigma_2)$. Assume that* (1. 3), (1. 4) *and* (H. 1)~(H. 6) *hold.*

(I) *If* $\frac{d_1}{M_1}\geq\frac{d_2}{M_2}>\frac{2n}{m}$, *then we have* $N(\lambda)=A_1\lambda^{2n/m}+o(\lambda^{2n/m})$, $\lambda\to+\infty$.

(II) *If* $\frac{d_1}{M_1}>\frac{d_2}{M_2}=\frac{2n}{m}$, *then we have*

$$N(\lambda)=\frac{A_6}{n(2m-M_2)}\lambda^{2n/m}(\log\lambda)+o(\lambda^{2n/m}\log\lambda),\ \lambda\to+\infty.$$

(III) *If* $\frac{d_1}{M_1}>\frac{4n-d_2}{2m-M_2}>\frac{2n}{m}$, *then we have*

$$N(\lambda)=\frac{2A_3}{4n-d_2}\lambda^{(4n-d_2)/(2m-M_2)}+o(\lambda^{(4n-d_2)/(2m-M_2)}),\ \lambda\to+\infty.$$

(IV) *If* $\frac{d_1}{M_1}=\frac{4n-d_2}{2m-M_2}>\frac{2n}{m}$, *then we have* $N(\lambda)=$

$$\frac{2M_1A_5}{(M_2d_1-M_1d_2)(2m-M_1-M_2)(4n-d_1-d_2)}\lambda^{(4n-d_2)/(2m-M_2)}(\log\lambda)+$$

$o(\lambda^{(4n-d_2)/(2m-M_2)}\log\lambda)$, $\lambda\to+\infty$.

(V) *If* $\frac{4n-d_2}{2m-M_2}>\frac{2n}{m}, \frac{d_1}{M_1}$, *then we have*

$N(\lambda)=\frac{2A_2}{4n-d_1-d_2}\lambda^{(4n-d_1-d_2)/(2m-M_1-M_2)}+o(\lambda^{(4n-d_1-d_2)/(2m-M_1-M_2)})$,

$\lambda\to+\infty$.

(VI) *If* $\frac{d_1}{M_1}=\frac{d_2}{M_2}=\frac{2n}{m}$, *then we have* $N(\lambda)=$

$\frac{(4m-M_1-M_2)A_4}{4n(2m-M_1)(2m-M_2)(2m-M_1-M_2)}\lambda^{2n/m}(\log\lambda)^2+$

$o(\lambda^{2n/m}(\log\lambda)^2)$, $\lambda\to+\infty$.

Here $A_1\sim A_6$ *are defined by* (5.2), (5.3), (5.4), (5.5), (5.7) and (5.8).

REMARK 6.2. Since we see easily that $\frac{2n}{m}>\frac{d_2}{M_2}$ if and only if $\frac{4n-d_2}{2m-M_2}>\frac{2n}{m}$, taking (1.4) into consideration, this theorem covers all the cases.

For the proof, we use the following extended Ikehara's Tauberian theorem.

PROPOSITION 6.3. ([2; *Proposition* 5.3]) *Let* $\sum_{k=1}^{\infty}\lambda_k^z$ *be convergent for* $\mathscr{R}_e\, z<s_0(<0)$, *hence holomorphic. Assume that there exist real numbers* A_1, $A_2, \ldots, A_p$ *such that*

$$\sum_{k=1}^{\infty}\lambda_k^z-\sum_{j=1}^{p}\frac{A_j}{(z-s_0)^j}$$

is continuous on $\{z\,;\,\mathscr{R}_e\, z\le s_0\}$. *Then we have*

$$N(\lambda)=\frac{(-1)^{p-1}A_p}{(p-1)!\,s_0}\lambda^{-s_0}(\log\lambda)^{p-1}+o(\lambda^{-s_0}(\log\lambda)^{p-1}),\ \lambda\to+\infty.$$

End of the proof of Theorem 6.1

It is well known that if $\mathscr{R}_e\, z<0$ and $|z|$ is large, $\mathrm{Tr}(P^z)=\sum_{k=1}^{\infty}\lambda_k^z$. For example, we consider the case (VI): $\frac{d_1}{M_1}=\frac{d_2}{M_2}=\frac{2n}{m}$. By Proposition 5.5, $\sum_{k=1}^{\infty}\lambda_k^z$ has a triple pole at $z=-\frac{2n}{m}$ as the first singularity with the coefficient of $(z+\frac{2n}{m})^{-3}$ equal to $A_4'=-\frac{(4m-M_1-M_2)A_4}{m(2m-M_1)(2m-M_2)(2m-M_1-M_2)}$. Thus by Proposition 6.2, we have

$$N(\lambda)=\frac{-m\,A_4'}{4n}\lambda^{2n/m}(\log\lambda)^2+o(\lambda^{2n/m}(\log\lambda)^2),\ \lambda\to+\infty.$$

Since the other case are proved similarly, we omit them.

EXAMPLE 6.4. (1) Let $P(x, D)=(D_{x_1}^2+x_1^2)^2(D_{x_2}^2+x_2^2)^2(|D_x|^2+|x|^2)^2+\mu(D_{x_1}^2+D_{x_2}^2+x_1^2+x_2^2)^2(|D_x|^2+|x|^2)^3+\nu(|D_x|^2+|x|^2)^4$
on $\boldsymbol{R}^3$ for any positive numbers μ and ν. Then we can put
$\Sigma_1=\{x_1=\xi_1=0\}$, $\Sigma_2=\{x_2=\xi_2=0\}$. Since $M_1=M_2=4$, $d_1=d_2=2$, $m=12$ and $n=3$, we have the case (VI), i. e.,

$$N(\lambda)=\frac{1}{3840}\lambda^{1/2}(\log\lambda)^2+o(\lambda^{1/2}(\log\lambda)^2),\ \lambda\to+\infty.$$

(2) Let $P(x, D)=\frac{1}{2}(x_3^2+D_{x_3}^2)^2[(x_1^2+x_2^2+D_{x_1}^2)^2(|D_x|^2+|x|^2)^3+(|D_x|^2+|x|^2)^3(x_1^2+x_2^2+D_{x_1}^2)^2]+\frac{1}{2}[(x_1^2+x_2^2+D_{x_1}^2)^2(|D_x|^2+|x|^2)^4+(|D_x|^2+|x|^2)^4(x_1^2+x_2^2+D_{x_1}^2)^2]+(x_3^2+D_{x_3}^2)^2(|D_x|^2+|x|^2)^4+\mu(|D_x|^2+|x|^2)^5$

on $\boldsymbol{R}^5$ for any positive number μ. Then we can put $\Sigma_1=\{x_1=x_2=\xi_1=0\}$, $\Sigma_2=\{x_3=\xi_3=0\}$. Since $M_1=M_2=4$, $d_1=3$, $d_2=2$, $m=14$ and $n=5$, we have the case (IV), i. e.,

$$N(\lambda)=\frac{\pi}{625}\lambda^{3/4}\log\lambda+o(\lambda^{3/4}\log\lambda),\ \lambda\to+\infty.$$

(3) Let $P(x, D)=\frac{1}{2}[D_{x_1}^2 D_{x_2}^2(|x|^2+|D_x|^2)^3+(|x|^2+|D_x|^2)^3 D_{x_1}^2 D_{x_2}^2]+\mu(D_{x_1}^2+D_{x_2}^2)(|x|^2+|D_x|^2)^{7/2}+\mu(|x|^2+|D_x|^2)^{7/2}(D_{x_1}^2+D_{x_2}^2)+\nu(|x|^2+|D_x|^2)^4$

on $\boldsymbol{R}^2$ for any positive numbers μ and ν. Then we can put
$\Sigma_1=\{\xi_1=0\}$, $\Sigma_2=\{\xi_2=0\}$. Since $M_1=M_2=2$, $d_1=d_2=1$, $m=10$ and $n=2$, we have the case (I), i. e.,

$$N(\lambda)=\frac{5\{\Gamma(1/10)\}^2}{8\pi\Gamma(1/5)}\lambda^{2/5}+o(\lambda^{2/5}),\ \lambda\to+\infty.$$

Finally we give a generalization.

REMARK 6.5. We can also define a symbol class which is an extension of Definition 1.1. Let $\Sigma_1, \Sigma_2, \dots, \Sigma_p$ be closed conic submanifolds of codimension $d_1, d_2, \dots, d_p$ in $\boldsymbol{R}^{2n}\backslash 0$ and m a real number and moreover $M_1, M_2, \dots, M_p$ non-negative integers.

Then $OPL^{m, M_1, M_2, \dots, M_p}(\Sigma_1, \Sigma_2, \dots, \Sigma_p)$ is a set of all pseudo-differential operators $P(x, D)$ on $\boldsymbol{R}^n$ whose symbol $p(x, \xi)$ satisfies (1.1) and

$$(6.2)' \quad \frac{|p_{m-j/2}(x, \xi)|}{r(x, \xi)^{m-j/2}} \leq C \sum_{\substack{k_1+\ldots+k_p=j \\ k_i \leq M_i}} d_{\Sigma_1}^{M_1-k_1} \ldots d_{\Sigma_p}^{M_p-k_p},$$

for $j=0, 1, \ldots, M_1+M_2+\ldots+M_p$. Here

$$d_{\Sigma_i} = \inf_{(x', \xi') \in \Sigma_i} \left(|x' - \frac{x}{r}| + |\xi' - \frac{\xi}{r}|\right), \quad i=1, 2, \ldots, p.$$

As in Definition 1.1, we say that $P(x, D)$ is regularly degenerate if p satisfies

$$(6.3)' \quad \frac{|p_m(x, \xi)|}{r(x, \xi)^m} \geq C\, d_{\Sigma_1}^{M_1} \ldots d_{\Sigma_p}^{M_p}.$$

We assume (H. 1)$\sim$(H. 6). Here (H. 2), (H. 3) and (H. 4) are revised according to this case. Then in the particular case: $\frac{d_1}{M_1} = \frac{d_2}{M_2} = \ldots = \frac{d_p}{M_p} = \frac{2n}{m}$, we have for some constant A

$$N(\lambda) = A\, \lambda^{2n/m} (\log \lambda)^{p-1} + o(\lambda^{2n/m} (\log \lambda)^{p-1}), \quad \lambda \to +\infty.$$

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