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On realization of conformally-projectively flat statistical manifolds and the divergences

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Abstract. We give a necessary and sufficient condition for a statistical manifold to be realized by a nondegenerate centroaffine immersion of codimension two. We also show that if a statistical manifold is realized as above, then so is the dual statistical manifold. Moreover, we construct a canonical contrast function on a simply connected conformally-projectively flat statistical manifold, using a geometric method.

Key words: centroaffine immersions, statistical manifolds, conformally flatness, divergences.

Introduction

An n-dimensional manifold M with a torsion-free affine connection ∇ and a pseudo-Riemannian metric h is called a *statistical manifold* if ∇h is symmetric. From the statistical point of view, Amari suggested that the following embedding problem in his writings: Find the condition when a statistical manifold can be realized in affine spaces. Kurose [7] determined the statistical manifolds realized by affine immersions of codimension one. The main purpose of this paper is to find the condition when statistical manifolds are realized by centroaffine immersions of codimension two.

The notion of centroaffine immersions of codimension two was introduced by Walter [11], and applied to the hypersurface theory in the real projective space $\mathbf{P}^{n+1}$ by Nomizu and Sasaki [9]. To state the condition for embedding problem, we introduce in Section 2 the notion of the conformally-projectively flatness of statistical manifolds, which is a generalization of the 1-conformally flatness defined by Kurose [6] in affine geometry, the projective flatness in projective geometry and the conformally flatness in conformal geometry. In section 3, we prove:

Main theorem *Let* $\{f, \xi\} : M \to \mathbf{R}^{n+2}$ *be a nondegenerate equiaffine centroaffine immersion, and denote by* ∇ *the induced connection and by* h *the affine fundamental form. Then* (M, ∇, h) *is a conformally-projectively*

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flat statistical manifold.

Conversely, suppose that (M, ∇, h) is a simply connected conformally-projectively flat statistical manifold of dimension n. Then there exists a nondegenerate equiaffine centroaffine immersion $\{f, \xi\} : M \to \mathbf{R}^{n+2}$ with induced connection ∇ and affine fundamental form h. Provided that $n \geq 3$, such an immersion is uniquely determined up to an affine transformation of $\mathbf{R}^{n+2}$.

This theorem can be regarded as a generalization of Radon's theorem for centroaffine immersion of codimension two (cf. Dillen, Nomizu and Vrancken [3]). We note that Abe [1] studied Amari's embedding problem in a different formulation.

For a given statistical manifold, we always have its dual statistical manifold (see Section 2). In Section 4, for a realizable statistical manifold, we study the embedding problem of its dual. Kurose [6] showed that the dual statistical manifold is realized by an affine immersoin of codimension one if and only if it has constant curvature. In contrast to the codimension one case, we show that if a statistical manifold is realized by a centroaffine immersion of codimension two, then so is the dual.

In Section 5, as an application of the main theorem, we give a geometric way to construct a contrast function which induces a given conformally-projectively flat statistical manifold. We note that our result is a generalization of Nagaoka and Amari's construction for flat statistical manifolds (cf. [2]) and Kurose's construction for 1-conformally flat statistical manifolds (cf. [8]).

1. Preliminaries

In this section, we recall the notion of centroaffine immersions of codimension two. For more details, see Nomizu and Sasaki [9].

We assume that all the objects are smooth throughout this paper.

Let M be an n-dimensional manifold and f an immersion of M into $\mathbf{R}^{n+2}$. Let D be the standard flat affine connection of $\mathbf{R}^{n+2}$ and η the radial vector field of $\mathbf{R}^{n+2} - \{0\}$. ($\eta = \sum_{i=1}^{n+2} x^i \partial/\partial x^i$, where $\{x^1, \ldots, x^{n+2}\}$ is an affine coordinate system.)

Definition 1.1 An immersion $f : M \to \mathbf{R}^{n+2}$ is called a *centroaffine immersion of codimension two* if there exists, at least locally, a vector field

ξ along f such that, at each point $x \in M$, the tangent space $T_{f(x)}\mathbf{R}^{n+2}$ is decomposed as the direct sum of the span $\mathbf{R}\{\eta_{f(x)}\}$, the tangent space $f_*(T_xM)$ and the span $\mathbf{R}\{\xi_x\}$:

$$T_{f(x)}\mathbf{R}^{n+2} = \mathbf{R}\{\eta_{f(x)}\} \oplus f_*(T_xM) \oplus \mathbf{R}\{\xi_x\},$$

where $\mathbf{R}\{\eta\}$ and $\mathbf{R}\{\xi\}$ mean 1-dimensional subspaces spanned by η and ξ, respectively. We call ξ a *transversal vector field.*

According to this decomposition, the vector fields D_Xf_*Y and $D_X\xi$, where X and Y are vector fields on M, have the following expressions:

$$D_Xf_*Y = T(X,Y)\eta + f_*(\nabla_XY) + h(X,Y)\xi, \tag{1.1}$$

$$D_X\xi = \mu(X)\eta - f_*(SX) + \tau(X)\xi, \tag{1.2}$$

where ∇ is a torsion-free affine connection, T, h are symmetric (0,2)-tensor fields, μ, τ are 1-forms, and S is a (1,1)-tensor field on M. The torsion-free affine connection ∇, the symmetric $(0,2)$-tensor field h and 1-form τ are called the *induced connection*, the *affine fundamental form* and the *transversal connection form*, respectively.

Since the connection D is flat, we have fundamental equations for centroaffine immersions of codimension two.

Gauss:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY - T(Y,Z)X + T(X,Z)Y, \tag{1.3}$$

Codazzi:

$$\begin{aligned}(\nabla_XT)(Y,Z) + \mu(X)h(Y,Z) &= (\nabla_YT)(X,Z) + \mu(Y)h(X,Z),\\ (\nabla_Xh)(Y,Z) + \tau(X)h(Y,Z) &= (\nabla_Yh)(X,Z) + \tau(Y)h(X,Z), \\ (\nabla_XS)(Y) - \tau(X)SY + \mu(X)Y &= (\nabla_YS)(X) - \tau(Y)SX + \mu(Y)X,\end{aligned} \tag{1.4}$$

Ricci:

$$\begin{aligned}T(X,SY) - T(Y,SX) &= (\nabla_X\mu)(Y) - (\nabla_Y\mu)(X)\\ &\quad + \tau(Y)\mu(X) - \tau(X)\mu(Y),\\ h(X,SY) - h(Y,SX) &= (\nabla_X\tau)(Y) - (\nabla_Y\tau)(X).\end{aligned}$$

The objects ∇, T, h, S, τ and μ depend on the choice of ξ. We shall examine the dependence on the change of transversal vector field for later use.

Lemma 1.2 *Let a be a function, ϕ a nonzero function and U a vector field on M. Suppose that we change a transversal vector field ξ for*

$$\tilde{\xi} = \phi^{-1}(\xi + a\eta + f_*U).$$

Then the induced connection, the affine fundamental form and the transversal connection form change as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y - h(X,Y)U, \tag{1.5}$$

$$\tilde{h}(X,Y) = \phi h(X,Y), \tag{1.6}$$

$$\tilde{\tau}(X) = \tau(X) - X(\log\phi) + h(X,U). \tag{1.7}$$

Equation (1.6) means that the conformal class of h is independent of the choice of ξ. If h is nondegenerate everywhere, we say that the immersion f is *nondegenerate*. When h is nondegenerate, we can take a transversal vector field ξ such that τ vanishes because of equation (1.7). We say that ξ (or that the pair of $\{f,\xi\}$) is *equiaffine* if τ vanishes. In this case, ∇h is symmetric because of equation (1.4). As we shall see in the next section, this implies that the triplet (M,∇,h) is a statistical manifold.

We take a positive function ψ on M, and change an immersion f for $g = \psi f$. We can consider the relationship between f and g.

Lemma 1.3 *We obtain the following formulas with respect to the immersion $\{f,\xi\}$ and $\{g,\xi\}$:*

$$\bar{\nabla}_X Y = \nabla_X Y + d(\log\psi)(Y)X + d(\log\psi)(X)Y, \tag{1.8}$$

$$\bar{h}(X,Y) = \psi h(X,Y), \tag{1.9}$$

$$\bar{\tau}(X) = \tau(X). \tag{1.10}$$

The equation (1.9) shows that the conformal class of h is preserved and the equation (1.10) implies that, if $\{f,\xi\}$ is equiaffine, then so is $\{g,\xi\}$.

Consider two immersions $f^i : M \to \mathbf{R}^{n+2} - \{0\}$, $(i = 1,2)$, with transversal vector fields ξ^i. We say that f^1 and f^2 are *affinely equivalent* if $f^1 = Af^2$ for a general linear transformation A in $GL(n+2,\mathbf{R})$. The following lemma shows the rigidity of centroaffine immersions of codimension two. (cf. [9, Proposition 5.5])

Lemma 1.4 *Let $n \geq 3$. Suppose $\nabla := \nabla^1 = \nabla^2$ and $h := h^1 = h^2$. If h*

is nondegenerate, then f^1 and f^2 are affinely equivalent.

2. Statistical manifolds

Let ∇ be a torsion-free affine connection and h a pseudo-Riemannian metric on M. We call (M, ∇, h) a *statistical manifold* if ∇h is symmetric. For a statistical manifold (M, ∇, h), we can define another torsion-free affine connection ∇^* by

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z),$$

where X, Y and Z are arbitrary vector fields on M. It is straightforward to show that (M, ∇^*, h) is also a statistical manifold. We say that ∇^* is the *dual connection* of ∇ with respect to h and (M, ∇^*, h) the *dual statistical manifold* of (M, ∇, h).

We note that statistical manifolds play a very important role of the geometric theory of statistics, because each family of probability density functions has the statistical manifolds structure which is naturally determined by the family. (See Amari [2].)

We recall the notion of α-conformality formulated by Kurose [7]. For a given $\alpha \in \mathbf{R}$, two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are said to be *α-conformally equivalent* if there exists a positive function ϕ on M such that

$$\begin{aligned} \tilde{h}(X, Y) &= \phi h(X, Y), \\ h(\tilde{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) - \frac{1+\alpha}{2} d(\log\phi)(Z)\, h(X, Y) \\ &\quad + \frac{1-\alpha}{2}\{d(\log\phi)(X)\, h(Y, Z) + d(\log\phi)(Y)\, h(X, Z)\}. \end{aligned}$$

We remark that two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are α-conformally equivalent, if and only if the dual statistical manifolds (M, ∇^*, h) and $(M, \tilde{\nabla}^*, \tilde{h})$ are $(-\alpha)$-conformally equivalent.

We say that a statistical manifold (M, ∇, h) is *flat* if the affine connection ∇ is flat, that is, the curvature tensor R of ∇ vanishes. We say that a statistical manifold (M, ∇, h) is *α-conformally flat* if (M, ∇, h) is α-conformally equivalent to a flat statistical manifold in a neighbourhood of an arbitrary point of M.

Suppose that $\{f, \xi\}$ is a nondegenerate equiaffine centroaffine immersion of codimension two, then the triplet (M, ∇, h) is a statistical manifold, where ∇ is the induced connection and h the affine fundamental form.

The following propositions follow easily from Lemmas 1.2 and 1.3.

Proposition 2.1 *Assume that a statistical manifold (M, ∇, h) is realized in $\mathbf{R}^{n+2}$ by a centroaffine immersion $\{f, \xi\}$. If we take a transversal vector field $\tilde{\xi} := \phi^{-1}\{\xi + a\eta + \phi^{-1} f_* \mathrm{grad}_h \phi\}$, where a is a function, ϕ a positive function on M and $\mathrm{grad}_h \phi$ the gradient vector field of ϕ with respect to h. Then the statistical manifold $(M, \tilde{\nabla}, \tilde{h})$ realized by $\{f, \tilde{\xi}\}$ is 1-conformally equivalent to (M, ∇, h).*

Proposition 2.2 *Assume that a statistical manifold (M, ∇, h) is realized in $\mathbf{R}^{n+2}$ by a centroaffine immersion $\{f, \xi\}$. If we change an immersion f for $g = \psi f$, where ψ is a positive function on M. Then the statistical manifold $(M, \bar{\nabla}, \bar{h})$ realized by $\{g, \xi\}$ is (-1)-conformally equivalent to (M, ∇, h).*

In order to discuss statistical manifolds that are obtained by centroaffine immersions of codimension two, we need to define more generalized conformally equivalence relation.

Definition 2.3 Two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are said to be *conformally-projectively equivalent* if there exist two positive functions ϕ and ψ on M such that

$$\tilde{h}(X, Y) = \phi\psi\, h(X, Y), \tag{2.1}$$

$$h(\tilde{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - d(\log\phi)(Z)\, h(X, Y) + d(\log\psi)(X)\, h(Y, Z) + d(\log\psi)(Y)\, h(X, Z). \tag{2.2}$$

We remark that if $\phi = \psi$ in equation (2.1) then it is a well known conformally equivalence relation in Euclidean differential geometry, if ϕ is constant then equation (2.2) implies that ∇ and $\tilde{\nabla}$ are projectively equivalent, and if ψ is constant then ∇ and $\tilde{\nabla}$ are dual-projectively equivalent (cf. [5]).

Proposition 2.4 *Two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are conformally-projectively equivalent if and only if the dual statistical manifolds (M, ∇^*, h) and $(M, \tilde{\nabla}^*, \tilde{h})$ are also conformally-projectively equivalent.*

Proof. Suppose that two statistical manifolds (M, ∇, h) and $(M, \tilde{\nabla}, \tilde{h})$ are conformally-projectively equivalent. By definition, there exist positive functions ϕ and ψ on M. Hence we have

$$\tilde{h}(Y, \tilde{\nabla}^*_X Z) = X\tilde{h}(Y, Z) - \tilde{h}(\tilde{\nabla}_X Y, Z)$$

$$
\begin{aligned}
&= X(\psi)\phi h(Y,Z) + \psi X(\phi) h(Y,Z) + \tilde{h}(\nabla_X Y, Z) \\
&\quad + \tilde{h}(Y, \nabla^*_X Z) - \tilde{h}(\tilde{\nabla}_X Y, Z) \\
&= d(\log\psi)(X)\tilde{h}(Y,Z) + d(\log\phi)(X)\tilde{h}(Y,Z) \\
&\quad + \tilde{h}(\nabla_X Y, Z) + \tilde{h}(Y, \nabla^*_X Z) \\
&\quad - \{\tilde{h}(\nabla_X Y, Z) - d(\log\phi)(Z)\tilde{h}(X,Y) \\
&\quad + d(\log\psi)(X)\tilde{h}(Y,Z) + d(\log\psi)(Y)\tilde{h}(X,Z)\} \\
&= \tilde{h}(Y, \nabla^*_X Z) - d(\log\psi)(Y)\tilde{h}(X,Z) \\
&\quad + d(\log\phi)(X)\tilde{h}(Y,Z) + d(\log\phi)(Z)\tilde{h}(X,Y).
\end{aligned}
$$

By exchanging Y and Z, and dividing $\phi\psi$, we obtain

$$
\begin{aligned}
h(\tilde{\nabla}^*_X Y, Z) &= h(\tilde{\nabla}_X Y, Z) - d(\log\psi)(Z)\, h(X,Y) \\
&\quad + d(\log\phi)(X)\, h(Y,Z) + d(\log\phi)(Y)\, h(X,Z).
\end{aligned}
$$

$\tilde{h}(X,Y) = \psi\phi h(X,Y)$ is automatically satisfied by the assumption. □

We shall now define conformally-projectively flatness of statistical manofold.

Definition 2.5 A statistical manifold (M,∇,h) is said to be *conformally-projectively flat* if (M,∇,h) is conformally-projectively equivalent to a flat statistical manifold in a neighbourhood of an arbitrary point of M.

3. Proof of main theorem

Proof of main theorem. Suppose that $\{f,\xi\} : M \to \mathbf{R}^{n+2}$ is a nondegenerate equiaffine centroaffine immersion. By equation (1.4), the triplet (M,∇,h), where ∇ is the induced connection and h the affine fundamental form, is a statistical manifold. For arbitrary point $p \in M$, there exists a positive function ψ defined in some neighbourhood U_p of p such that $g = \psi f$ is contained in the affine hyperplane spanned by $f_*(T_pM)$ and ξ_p. By Lemma 1.3, we have

$$\bar{\nabla}_X Y = \nabla_X Y + d(\log\psi)(Y)X + d(\log\psi)(X)Y, \tag{3.1}$$

$$\bar{h} = \psi h. \tag{3.2}$$

We take a transversal vector field $\tilde{\xi}_x$ which is equal to ξ_p everywhere on U_p. Since $\eta_{g(x)}$, $g_*(T_xM)$ and ξ_x are linearly independent, we have a positive function ϕ, a function a and a tangent vector field V on U_p such that $\phi\tilde{\xi}_x = \xi_x + a\eta_{g(x)} + g_*V$. The distribution g is contained in the affine hyperplane spanned by $g_*(T_pM)$ and ξ_p, we obtain $\tilde{T} = 0$, and from Lemma 1.2, we have

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y - \bar{h}(X,Y)V, \tag{3.3}$$

$$\tilde{h} = \phi\bar{h}. \tag{3.4}$$

Since the transversal vector field $\tilde{\xi}$ is parallel around p, then $D_X\xi = 0$. Hence we obtain

$$\tilde{\mu} = 0, \quad \tilde{S} = 0 \quad \text{and} \quad \tilde{\tau} = 0.$$

From equation (1.1), $\tilde{\nabla}$ is a flat affine connection since $\tilde{S} = 0$ and $\tilde{T} = 0$. Then the triplet $(U_p, \tilde{\nabla}, \tilde{h})$ is a flat statistical manifold.

Since ξ_x and $\tilde{\xi}_x$ are equiaffine transversal vector fields, we have

$$\bar{h}(X,V) = X(\log\phi) = d(\log\phi)(X),$$

from equation (1.7) in Lemma 1.2. Hence, we obtain

$$V = \frac{1}{\phi\psi}\mathrm{grad}_h\phi, \tag{3.5}$$

where $\mathrm{grad}_h\phi$ is the gradient vector field of ϕ with respect to h.

By equations (3.1)–(3.5), we have

$$\begin{aligned} \tilde{h} &= \phi\psi h, \\ h(\tilde{\nabla}_X Y, Z) &= h(\nabla_X Y, Z) - d(\log\phi)(Z)\,h(X,Y) \\ &\quad + d(\log\psi)(X)\,h(Y,Z) + d(\log\psi)(Y)\,h(X,Z). \end{aligned}$$

Therefore, the flat statistical manifold $(U_p, \tilde{\nabla}, \tilde{h})$ induced by the centroaffine immersion $\{g, \tilde{\xi}\}$ is conformally-projectively equivalent to the original $(U_p, \nabla|_{U_p}, h|_{U_p})$. This implies that (M, ∇, h) is a conformally-projectively flat statistical manifold.

We prove the converse. Suppose that (M, ∇, h) is a simply connected conformally-projectively flat statistical manifold. For an arbitrary point $p \in M$, there exist a neighbourhood U_p of p, a flat statistical manifold $(U_p, \tilde{\nabla}, \tilde{h})$ and positive functions ϕ and ψ on U_p such that $(U_p, \nabla|_{U_p}, h|_{U_p})$ is conformally-projectively equivalent to $(U_p, \tilde{\nabla}, \tilde{h})$.

Since $(U_p, \tilde{\nabla}, \tilde{h})$ is flat, it can be realized in an affine hypreplane of $\mathbf{R}^{n+2}$ by a graph immersion $\{g, \tilde{\xi}\}$ (See [6]). By translating it if necessary, we may assume that the affine hyperplane does not go through the origin.

We change an immersion g for $f := \psi^{-1}g$ and take a transversal vector field $\xi := \phi\tilde{\xi} - \phi^{-1}f_*\mathrm{grad}_h\phi$. From Lemmas 1.2 and 1.3, the centroaffine immersion $\{f, \xi\}$ realizes the statistical manifold $(U_p, \nabla|_{U_p}, h|_{U_p})$ into $\mathbf{R}^{n+2} - \{0\}$.

Hence, (M, ∇, h) can be realized in $\mathbf{R}^{n+2}$ since such immersions are unique by Lemma 1.4 and M is simply connected. □

We remark that our main theorem implies that the universal cover of an arbitrary conformally-projectively flat statistical manifold can be realized by a centoaffine immersion of codimension two.

4. Dual maps

We study the embedding problem for the dual statistical manifold. In [6], Kurose proved the following theorem.

Theorem 4.1 *Let (M, ∇, h) be a statistical manifold realized in $\mathbf{R}^{n+1}$ by an affine immersion. If (M, ∇, h) has constant curvature, then the dual statistical manifold (M, ∇^*, h) is also realized in the dual affine space $\mathbf{R}_{n+1}$.*

We generalize Kurose's theorem for centroaffine immersions of codimension two.

We recall the definition of the dual map of a given centroaffine immersion of codimension two. Let $\{f, \xi\} : M \to \mathbf{R}^{n+2}$ be a centroaffine immersion and the objects ∇, T, h, μ and τ defined by equations (1.1) and (1.2). Let $\mathbf{R}_{n+2}$ be the dual vector space of $\mathbf{R}^{n+2}$, η^* the radial vector field of $\mathbf{R}_{n+2} - \{0\}$, and $\langle\ ,\ \rangle$ the pairing of $\mathbf{R}_{n+2}$ and $\mathbf{R}^{n+2}$. We assume that h is nondegenerate. For $\{f, \xi\}$, we define two maps v and $w : M \to \mathbf{R}_{n+2}$ as follows: for each $x \in M$, $v(x)$ and $w(x)$ satisfy

$$\begin{aligned} \langle v(x), \xi_x\rangle = 1, \quad \langle w(x), \xi_x\rangle = 0, \\ \langle v(x), \eta_{f(x)}\rangle = 0, \quad \langle w(x), \eta_{f(x)}\rangle = 1, \\ \langle v(x), f_*X_x\rangle = 0, \quad \langle w(x), f_*X_x\rangle = 0, \end{aligned} \tag{4.1}$$

for an arbitrary vector field X on M. The derivatives of the maps v and w are given as follows:

$$\begin{aligned} \langle v_*X, \xi\rangle = -\tau(X), \quad \langle w_*X, \xi\rangle = -\mu(X), \\ \langle v_*X, \eta\rangle = 0, \quad \langle w_*X, \eta\rangle = 0, \end{aligned}$$

$$\langle v_*X, f_*Y\rangle = -h(X,Y), \quad \langle w_*X, f_*Y\rangle = -T(X,Y).$$

Since h is nondegenerate, v is an immersion. By the definitions of v and w, $v(x)$ and $w(x)$ are linearly independent at each point x of M. Hence, $\{v, w\}$ is a centroaffine immersion of M into $\mathbf{R}_{n+2}$. We call the pair $\{v, w\}$ the *dual map* of $\{f, \xi\}$.

For the dual map $\{v, w\}$, the objects ∇^*, T^*, h^*, S^*, μ^* and τ^* are defined by

$$\begin{aligned} D_X v_* Y &= T^*(X,Y)\eta^* + v_*(\nabla^*_X Y) + h^*(X,Y)w, \\ D_X w &= \mu^*(X)\eta^* - v_*(S^*X) + \tau^*(X)w. \end{aligned}$$

By definition, we can easily prove the following lemma.

Lemma 4.2 *The following equations hold.*

$$h^*(X,Y) = h(X,Y), \tag{4.2}$$

$$Zh(X,Y) = h(\nabla_Z X, Y) + h(X, \nabla^*_Z Y) + \tau(Y)h(X,Z), \tag{4.3}$$

$$\tau^*(X) = 0. \tag{4.4}$$

Proof. See Nomizu and Sasaki [9].

If a transversal vector field ξ is equiaffine, then equation (4.3) implies that two connections ∇ and ∇^* are mutually dual with respect to h. □

Theorem 4.3 *If a statistical manifold is realized in $\mathbf{R}^{n+2}$ by a centroaffine immersion $\{f, \xi\}$, then the dual statistical manifold is realized in $\mathbf{R}_{n+2}$ by the dual map $\{v, w\}$ of $\{f, \xi\}$.*

Proof. Suppose that a statistical manifold (M, ∇, h) is realized in $\mathbf{R}^{n+2}$ by a centroaffine immersion $\{f, \xi\}$. Since $\{f, \xi\}$ is nondegenerate and equiaffine, we have the dual map $\{v, w\}$ of $\{f, \xi\}$. Equations (4.2)–(4.4) imply that the dual statistical manifold (M, ∇^*, h) is realized by the dual map. □

Let (M, ∇, h) be a simply connected conformally-projectively flat statistical manifold of dimension $n \geq 3$. By Proposition 2.4, the dual statistical manifold (M, ∇^*, h) is also conformally-projectively flat. By the main theorem, we have the unique centroaffine immersions $\{f, \xi\} : (M, \nabla, h) \to \mathbf{R}^{n+2}$ and $\{v, w\} : (M, \nabla^*, h) \to \mathbf{R}^{n+2}$. We can then regard $\{v, w\}$ as the dual map of $\{f, \xi\}$ identifying $\mathbf{R}^{n+2}$ with $\mathbf{R}_{n+2}$.

5. Geometric divergences

Let M be an n-dimensional manifold and ρ a function on $M \times M$. Identifying the tangent space $T_{(p,q)}(M \times M)$ with the direct sum of $T_pM \oplus T_qM$, we use the following notation:

$$\rho[X_1 \dots X_i \,|\, Y_1 \dots Y_j](p) := (X_1, 0) \dots (X_i, 0)(0, Y_1) \dots (0, Y_j)\rho|_{(p,p)},$$

where $p \in M$ and $X_1, \dots, X_i, Y_1, \dots, Y_j$ $(i, j \geq 0)$ are arbitrary vector fields on M. We call ρ a *contrast function* of M if

1) $\rho(p,p) = 0$ for an arbitrary point $p \in M$,
2) $h(X,Y) := -\rho[X|Y]$ is a pseudo-Riemannian metric on M.

(See Eguchi [4].)

For a contrast function ρ of M, we can define a torsion-free affine connection ∇ on M as follows:

$$h(\nabla_X Y, Z) = -\rho[XY|Z],$$

where X, Y and Z are arbitrary vector fields on M. It is easy to show that the triplet (M, ∇, h) is a statistical manifold. We call (M, ∇, h) the statistical manifold induced by the contrast function ρ.

Conversely Kurose [8] showed that simply connected 1-conformally flat statistical manifolds are induced by contrast functions called *geometric divergences*, which are constructed from affine immersions of codimension one and their dual maps. In this section, we construct such functions for the larger class of statistical manifolds.

Let (M, ∇, h) be a simply connected conformally-projectively flat statistical manifold of dimension n. Let $\{f, \xi\}$ be a nondegenerate equiaffine centroaffine immersion of (M, ∇, h) into $\mathbf{R}^{n+2}$ and $\{v, w\}$ the dual map of $\{f, \xi\}$.

Definition 5.1 We define a function ρ on $M \times M$ for $\{f, \xi\}$ by

$$\rho(p,q) := \langle v(q), f(p) - f(q) \rangle,$$

where p and q are arbitrary points in M. We call the function ρ the *geometric divergence* of $\{f, \xi\}$.

If we change an immersion f for $g := \psi f$ and a transversal vector field ξ for $\tilde{\xi} := \phi^{-1}\{\xi + a\eta + \phi^{-1} f_* \mathrm{grad}_h \phi\}$, where a is a function, ϕ and ψ are positive functions on M and $\mathrm{grad}_h\phi$ is the gradient vector field of ϕ with respect to h, then the geometric divergence $\tilde{\rho}$ of $\{g, \tilde{\xi}\}$ is given by

$$\tilde{\rho}(p, q) = \psi(p)\phi(q)\rho(p, q)$$

for (p, q) in $M \times M$.

The geometric divergence induces a statistical manifold (M, ∇, h). In fact, we have the following proposition.

Proposition 5.2 *The geometric divergence ρ equals zero at each diagonal point in $M \times M$ and*

$$\begin{aligned} &\rho[X|] = 0, \\ &\rho[X|Y] = -h(X, Y), \\ &\rho[XY|Z] = -h(\nabla_X Y, Z). \end{aligned}$$

Proof. By the definition of geometric divergences, we have

$$\begin{aligned} (X, 0)\rho(p, q) &= (X, 0)\langle v(q), f(p) - f(q)\rangle \\ &= \langle v(q), f_* X_p\rangle, \\ (X, 0)(0, Y)\rho(p, q) &= \langle v_* Y_q, f_* X_p\rangle, \\ (X, 0)(Y, 0)(0, Z)\rho(p, q) &= \langle v_* Z_q, T(X, Y)\eta_{f(p)} \\ &\qquad + f_*(\nabla_X Y)_p + h(X, Y)\xi_p\rangle. \end{aligned}$$

Setting $p = q$, we obtain the required equalities from the definition of the conormal map and the equations of (4.2)–(4.2). □

As an application of the main theorem, we obtain the following.

Theorem 5.3 *Suppose that (M, ∇, h) is a simply connected conformally-projectively flat statistical manifold and $\dim M \geq 3$. Then there exists a contrast function ρ which induces given (M, ∇, h).*

Proof. By the main theorem, there exists a nondegenerate equiaffine centroaffine immersion $\{f, \xi\}$ which realizes the given statistical manifold (M, ∇, h). Let ρ be the geometric divergence of $\{f, \xi\}$. From Proposition 5.2, the function ρ is a contrast function which induces the given statistical manifold (M, ∇, h). □

We remark that the geometric divergence is uniquely determined by a statistical manifold, because the realization of that is unique. The function $\rho(p, \cdot)$ for fixed p in $\mathbf{R}^{n+2}$ is known as the affine support function of $\{f, \xi\}$ from the point p, but the affine support function is an extrinsic object.

If a given statistical manifold is 1-conformally flat, then the geometric divergence obtained by Theorem 5.3 coincides with Kurose's geometric divergence in [8] up to the third order.

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